VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$	(permeability of free space)
$c = 3.00 \times 10^8\ \mathrm{m}/s$	(speed of light)
$e = 1.60 \times 10^{-19}\ \mathrm{C}$	(charge of the electron)
$m = 9.11 \times 10^{-31}\ \mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$\mathbf{F} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qQ}{r^2}\hat{r}$ Coulomb's Law $\quad \mathbf{F} = Q\mathbf{E} + Q\mathbf{v}\times\mathbf{B}$ Lorentz force

$\mathbf{E} = \dfrac{1}{4\pi\varepsilon_0}\iiint \dfrac{\rho(\mathbf{r}')}{r^2}\hat{r}\,d\tau'$ electric field from continuous charge distribution

$\oiint \mathbf{E} \bullet d\mathbf{a} = \frac{1}{\varepsilon_0} Q_{\text{encl}}$ Gauss' Law (integral) $\oint \mathbf{E} \bullet d\mathbf{l} = 0$ (statics)

$V(\mathbf{r}) = -\int_{\wp}^{\mathbf{r}} \mathbf{E} \bullet d\mathbf{l}$ Electrostatic potential, and $\mathbf{E} = -\nabla V$ (statics)

$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ Poisson's equation, and $\nabla^2 V = 0$ Laplace's equation (regions of no charge)

$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ and $V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(\mathbf{r}')}{r} d\tau'$ (setting reference point at infinity)

$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$ and $\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = \frac{1}{\varepsilon_0}\sigma$ boundary conditions

$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f$

$V(\mathbf{b}) - V(\mathbf{a}) = \frac{W}{Q}$ $W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)$ and $W = \frac{1}{2}\iiint \rho V \, d\tau$

$Q = CV$ $W = \frac{1}{2}CV^2$ $W = \frac{1}{2}Q^2 / C$ capacitors

$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho$ Poisson's equation $\nabla^2 V = 0$ Laplace's equation

$V(x,y) = \sum_{n=0}^{\infty}(A_n e^{+kx} + B_n e^{-kx})(C_n \sin ky + D_n \cos ky)$ solution to Laplace's equation in Cartesian coordinates in two dimensions

$\int_0^a \sin(n\pi \frac{y}{a}) \sin(n'\pi \frac{y}{a}) dy = \begin{cases} 0 & n \neq n' \\ \frac{a}{2} & n = n' \end{cases}$ orthogonality of sines

$\int_0^a \sin(n\pi \frac{y}{a}) dy = \begin{cases} 0 & \text{n even} \\ \frac{2a}{n\pi} & \text{n odd} \end{cases}$ $e^{+kx} + e^{-kx} = 2\cosh kx$

$$\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta = 2/3$$

$P_0(x)$	$= 1$
$P_1(x)$	$= x$
$P_2(x)$	$= (3x^2 - 1)/2$
$P_3(x)$	$= (5x^3 - 3x)/2$
$P_4(x)$	$= (35x^4 - 30x^2 + 3)/8$
$P_5(x)$	$= (63x^5 - 70x^3 + 15x)/8$

TABLE 3.1 Legendre Polynomials.

$$V(r,\theta) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}) P_\ell(\cos\theta)$$

solution to Laplace's equation in spherical coordinates with azimuthal symmetry

$$\int_0^{\pi} P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \, d\cos\theta = \begin{cases} 0 & \ell \neq \ell' \\ \dfrac{2}{2\ell+1} & \ell = \ell' \end{cases}$$ orthogonality of Legendre polynomials

$$V(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta$$ potential outside a neutral conducting sphere in uniform field

Law of cosines $c^2=a^2+b^2-2ab\cos\gamma$, with γ the angle opposite to side c.

$$\frac{1}{\mathfrak{R}} = \frac{1}{r} \sum_0^{\infty} \left(\frac{r'}{r} \right)^n P_n(\cos\alpha)$$ 1/r expansion in Legendre polynomials

$$V(r) = \frac{1}{4\pi\varepsilon_0} \sum_0^{\infty} \frac{1}{r^{n+1}} \iiint (r')^n P_n(\cos\alpha) \rho(\mathbf{r}') d\tau'$$ multipole expansion

$$\vec{p} = \iiint \vec{r}' \rho(\vec{r}') d\tau' \quad \vec{p} = \sum_1^n q_i \vec{r}_i' \quad V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \bullet \hat{r}}{r^2}$$ dipole moment

$$\vec{E}_{\text{dip}} = \frac{p}{4\pi\varepsilon_0 r^3} (2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta})$$

$$\mathbf{F}_{\text{mag}} = \int I \, d\mathbf{l} \times \mathbf{B}$$

$$\mathbf{K} = d\mathbf{I} / d\ell_\perp \quad \mathbf{J} = d\mathbf{I} / da_\perp$$ surface and volume currents

$$\nabla \bullet \mathbf{J} + \frac{d\rho}{dt} = 0$$ conserved current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{r}}{|\mathbf{r} - \mathbf{r}'|^2} \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K} \times \hat{r}}{|\mathbf{r} - \mathbf{r}'|^2} da' \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{\mathbf{J} \times \hat{r}}{|\mathbf{r} - \mathbf{r}'|^2} d\tau'$$

$$\oint \mathbf{B} \bullet d\mathbf{l} = \mu_0 I_{\text{enc}}$$

$\mathbf{B} = \nabla \times \mathbf{A}$ magnetic vector potential.

$\nabla \bullet \mathbf{A} = 0$ "Coulomb gauge" convention $\quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ (for Coulomb gauge)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau' \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}da' \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{I}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{l}'$$

$$\oint \mathbf{A} \bullet d\mathbf{l} = \Phi_m$$

$$B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0 \qquad B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K \qquad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 \mathbf{K} \times \hat{n}$$

$$\mathbf{A}_{\text{above}} - \mathbf{A}_{\text{below}} = 0 \qquad \frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$$

$$\mathbf{m} = I\iint \hat{n}\, da = I\mathbf{a}$$

$$\mathbf{m} = \frac{1}{2}\oint \mathbf{r} \times \mathbf{I}\, d\ell \quad \mathbf{m} = \frac{1}{2}\iint \mathbf{r} \times \mathbf{K}\, da \quad \mathbf{m} = \frac{1}{2}\iiint \mathbf{r} \times \mathbf{J}\, d\tau$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \hat{r}}{r^2}$$

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \quad \mathbf{F} = \nabla(\mathbf{m} \bullet \mathbf{B}) \quad U = -\mathbf{m} \bullet \mathbf{B}$$

$$\mathbf{J}_b = \nabla \times \mathbf{M} \quad \mathbf{K}_b = \mathbf{M} \times \hat{n}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \qquad \nabla \times \mathbf{H} = \mathbf{J}_f \qquad \oint \mathbf{H} \bullet d\mathbf{l} = I_{\text{f,encl}}$$

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}) \qquad \mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{n}$$

$$\mathbf{M} = \chi_m \mathbf{H} \qquad \mu = \mu_0(1+\chi_m) \qquad \mathbf{B} = \mu \mathbf{H}$$

$$\mathbf{J} = \sigma \mathbf{E} \quad V = IR \quad P = VI = I^2 R$$

$$\mathcal{E} = \oint \mathbf{E} \bullet d\ell \quad \mathcal{E} = -\frac{d\Phi_M}{dt}$$

$$\Phi_2 = M\, I_1 \quad \Phi = LI \quad \mathcal{E}_2 = -M\frac{dI_1}{dt} \quad \mathcal{E} = -L\frac{dI}{dt} \quad W = \frac{1}{2}LI^2$$

$$\nabla \bullet \left(\frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}\right) + \frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right) = -\mathbf{E} \bullet \mathbf{J}$$ Poynting's theorem (differential)

$$\frac{dW}{dt} = -\frac{d}{dt}\iiint\left(\frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right)d\tau - \oiint\left(\frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}\right) \bullet \hat{n}\, da$$ Poynting's thm (integral)

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy: $$U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$$

Momentum: $$\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$$

Poynting vector: $$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$