Physics 515, Electrodynamics III
Department of Physics, University of Washington
Spring quarter 2020
April 29, 2020, 11am
On-line lecture

Administrative

1. The midterm exam will be posted Friday, May 1, at 3 pm PDT. The exam is due via email Monday, May 4, at 11am. The exam is open book, you may use Jackson. See course web site for exam information. See the exam for email submission instructions.
2. You should be getting your homework back; if not let me know.
3. Office hours Wednesday after class at URL https://washington.zoom/us/j/712804010

Lecture

Chapter 11: Special theory of relativity.
1. J. C. 11.1: Gedanken experiments. Length contraction and time dilation. Postulates of special relativity.
2. J. C. 11.2 etc.: Historical development.
3. J. C. 11.3: Lorentz tranformations. Introduction to 4-vectors. Proper time. Spacetime (Minkowski) diagram.
4. J. C. 11.4 Velocity addition; Lorentz transformations have group properties.
5. Miscellaneous material: Some properties of tensors. Manipulation of tensors. Some sensor identities. Tensor concept of a conservation law.

J. C. 11: REVIEW OF SPECIAL RELATIVITY. SECTIONS 11.1 — 4

PATH FORWARD: LEAVE MAXWELL ELECTRODYNAMICS UNCHANGED; MODIFY NEWTONIAN MECHANICS. THIS PATH WAS CHOSEN BY EXPERIMENT.

EINSTEIN'S POSTULATES 1905

- LAWS OF ELECTRODYNAMICS (INCLUDING THE SPEED OF LIGHT) AND LAWS OF MECHANICS ARE THE SAME IN ALL INERTIAL FRAMES.

- NO EXPERIMENT CAN DETECT "ABSOLUTE MOTION"; NO INERTIAL FRAME HAS SPECIAL PROPERTIES.

"AUXILIARY PRINCIPLES":

- "INVARIANCE OF THE SENSE OF TIME"; TIED TO ENTROPY OF COMPLEX SYSTEMS.
- "INVARIANCE OF PROPER QUANTITIES"; ALL OBSERVERS AT REST OBSERVE THE SAME LENGTH AND TIME.

CONSTRUCT THE LORENTZ TRANSFORMATIONS VIA 3 GEDANKEN EXPERIMENTS.

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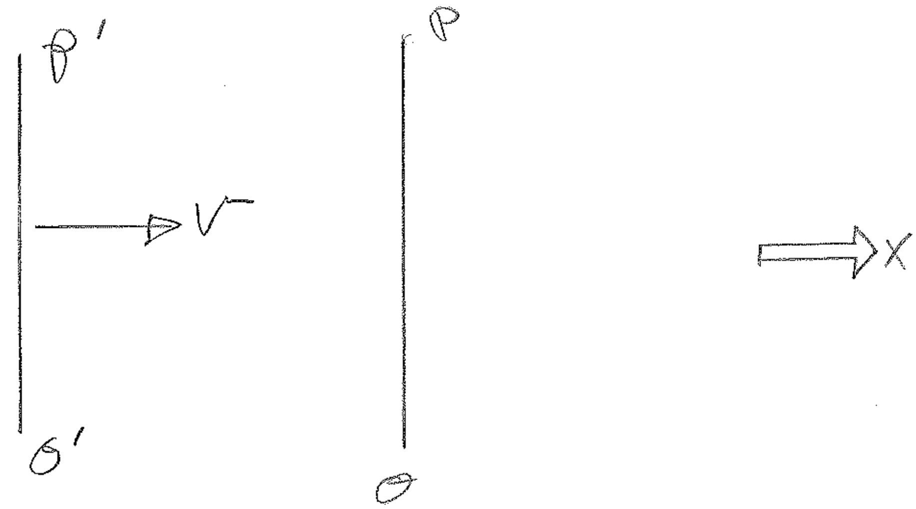
1. "PERPENDICULAR" STICKS.



OP = O'P' BY CONSTRUCTION.

$y' = y, \quad z' = z$

TRANSVERSE LENGTHS UNCHANGED.

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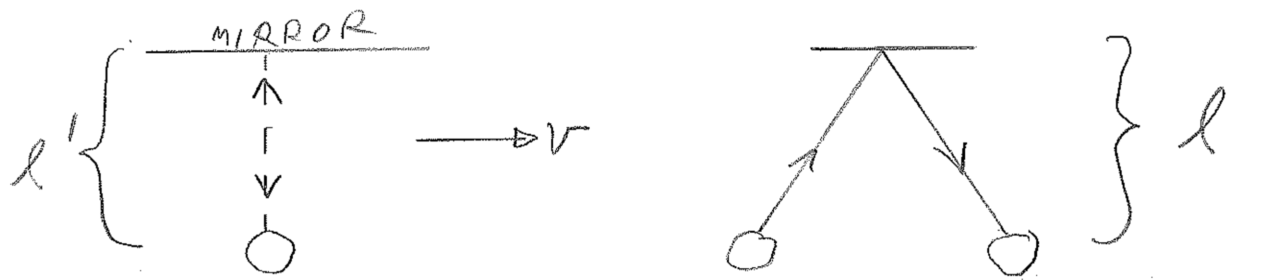
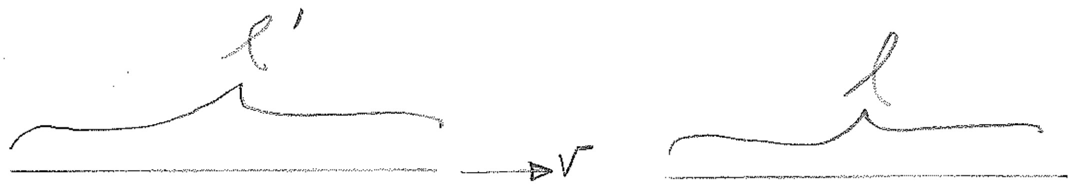
2. "LIGHT CLOCKS"



$$\Delta t = \Delta t' \frac{1}{\sqrt{1-\beta^2}} \; ; \; \beta = \frac{v}{c} \; ; \; \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

"MOVING CLOCKS RUN SLOW".

3. "PARALLEL" STICKS



$$\ell = \ell' \sqrt{1-\beta^2} \qquad \left\{ = \frac{\ell'}{\gamma} \right\}$$

THESE ARE COMBINED INTO THE LORENTZ TRANSFORMATIONS;

WITH $\beta = v/c$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$$x' = \frac{1}{\gamma}\left\{x - \beta c t\right\}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{1}{\gamma}\left\{t - \beta x/c\right\}$$

FOR RELATIVE MOTION ALONG $\hat{x}$

BE CARFUL ABOUT $-\beta$ OR $+\beta$; YOU SHOULD HAVE THE OVERALL RELATIVE MINUS SIGN.

EXERCISE: SHOW

$$x^2 + y^2 + z^2 - c^2t^2$$
$$= x'^2 + y'^2 + z'^2 - c^2t'^2.$$

THAT IS, THE INTERVAL IS INVARIANT UNDER A LORENTZ TRANSFORMATION.

THEN FROM THIS, ARGUE THAT A LIGHT SIGNAL PROPAGATED IN ALL DIRECTIONS IN THE UNPRIMED FRAME (AT SPEED C) IS A LIGHT SIGNAL PROPAGATED IN ALL DIRECTIONS IN THE PRIMED FRAME (AT SPEED c); THIS IS IN ACCORD WITH THE FIRST POSTULATE.

THE SPACE-TIME INTERVAL IS INVARIANT

$$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-c^2(t_2-t_1)^2$$

AND THE DIFFERENTIAL INTERVAL IS LIKWISE INVARIANT

$$ds^2=-dx^2-dy^2-dz^2+c^2dt^2$$

(THE – – – + "SIGNATURE IS A CONVENTION, WITH CAUTION ABOUT SIGNS).

EVENT: A POINT IN LORENTZ SPACE (x, y, z, t).

AN EVENT IN ONE FRAME MAY BE A DIFFERENT EVENT (THAT IS, THE COORDINATES ARE DIFFERENT) AS VIED IN ANOTHER FRAME.

TIMELIKE, SPACELIKE AND LIGHTLIKE INTERVALS.

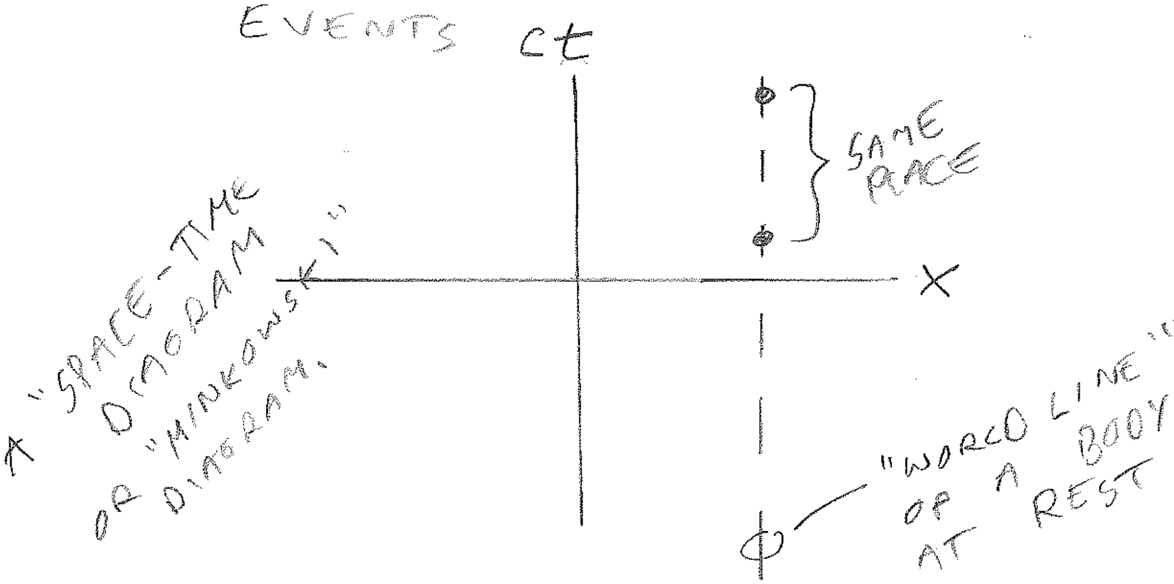
- IF IN SOME FRAME

$$\Delta x'^2 + \Delta y'^2 + \Delta z'^2 < c^2 \Delta t'^2$$

THEN THERE EXISTS A FRAME WHERE THE TWO EVENTS OCCUR AT THE SAME PLACE ($x = x'$, $y = y'$, $z = z'$), SO

$$\Delta s = c\,\Delta t.$$

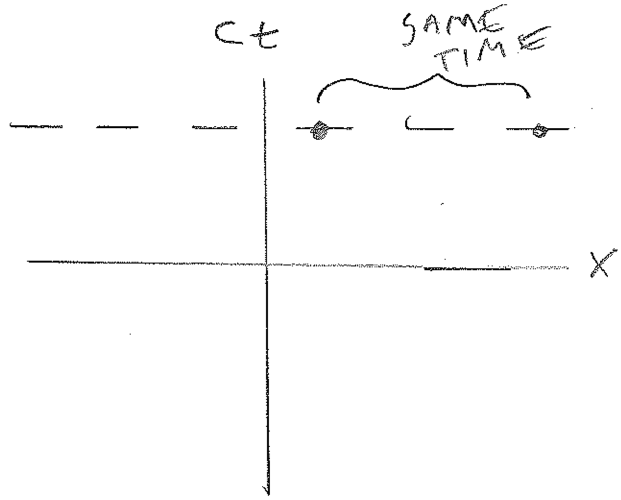
WE SAY THERE IS A "TIMELIKE" SEPARATION WITH $c\Delta t$ THE "PROPER TIME" INTERVAL BETWEEN EVENTS



- IF IN SOME FRAME

$$\Delta x'^2 + \Delta y'^2 + \Delta z'^2 > c^2 \Delta t'^2$$

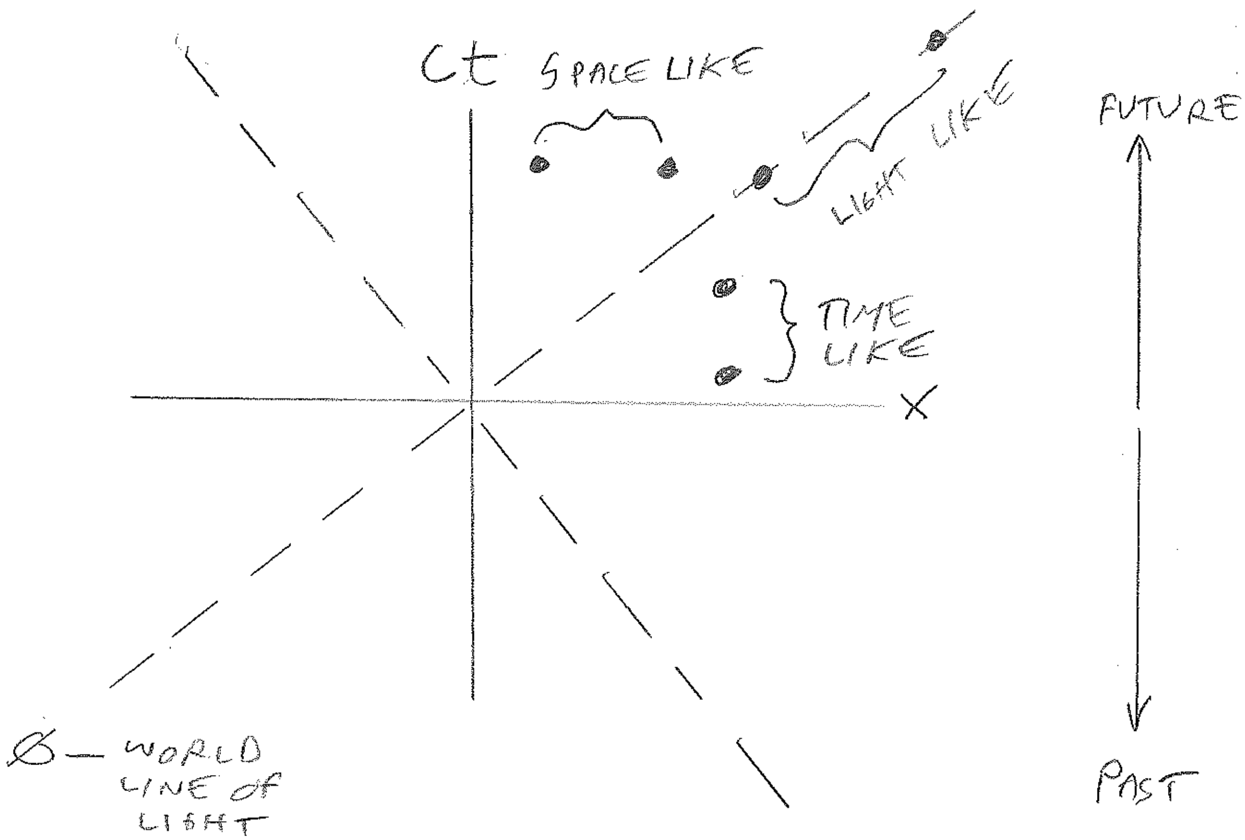
THEN THERE EXISTS A FRAME WHERE $\Delta t = 0$ AND THE EVENTS ARE SEPARATED BY A "SPACE LIKE" SEPARATION.



- IF IN SOME FRAME

$$\Delta x'^2 + \Delta y'^2 + \Delta z'^2 = c^2 \Delta t'^2$$

THEN THIS RELATION HOLDS FOR ALL FRAMES; A "LIGHT LIKE" SEPARATION



NO LORENTZ TRANSFORMATION CAN CHANGE TIMELIKE, SPACELIKE, LIGHTLIKE INTO EACH OTHER.

BACK TO "PROPER TIME".

CONSIDER A FRAME WHERE AN OBJECT IS AT REST.

THEN $|d\vec{x}| = 0$ AND $dS = c\,d\tau$.

$d\tau$ IS A LORENTZ INVARIANT CALLED THE "PROPER TIME"

FROM THE LORENTZ TRANSFORMATION

$$\Delta t' = \frac{1}{\gamma}\left\{\Delta t - \beta \Delta x/c\right\}$$ WE HAVE

$$d\tau = \frac{dt}{\gamma}$$

Q: DOES THIS SHOW "TIME DILATION"?

VELOCITY ADDITION.

$$v_x' = \frac{v_x - v}{1 - \frac{v_x v}{c^2}}$$

$$v_y' = v_y \frac{1}{\gamma} \frac{1}{1 - \frac{v_x v}{c^2}}$$

$$v_z' = v_z \frac{1}{\gamma} \frac{1}{1 - \frac{v_x v}{c^2}}$$

EXAMPLE:

IN Σ; $v_x = \frac{dx}{dt}$.

IN Σ': $dx' = \gamma\left\{dx - v\,dt\right\}$

$$dt' = \gamma\left\{dt - \frac{v}{c^2}dx\right\}$$

$$v_x' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx}$$

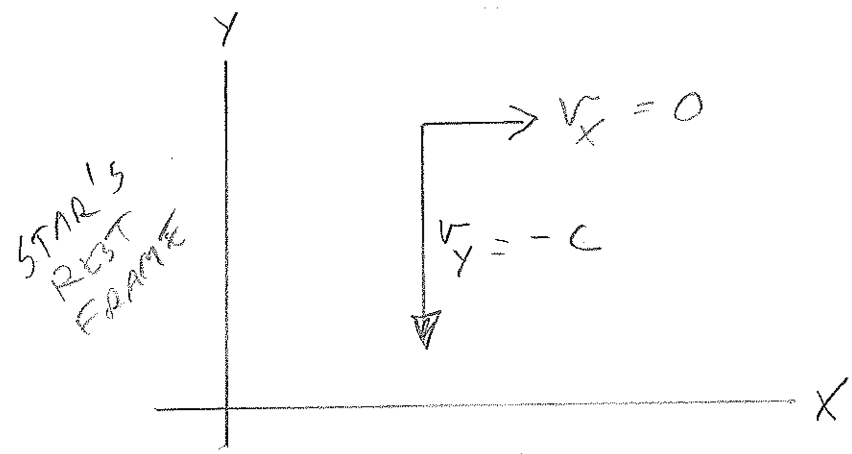
$$= \frac{dx/dt - v}{1 - v/c^2\, dx/dt}$$

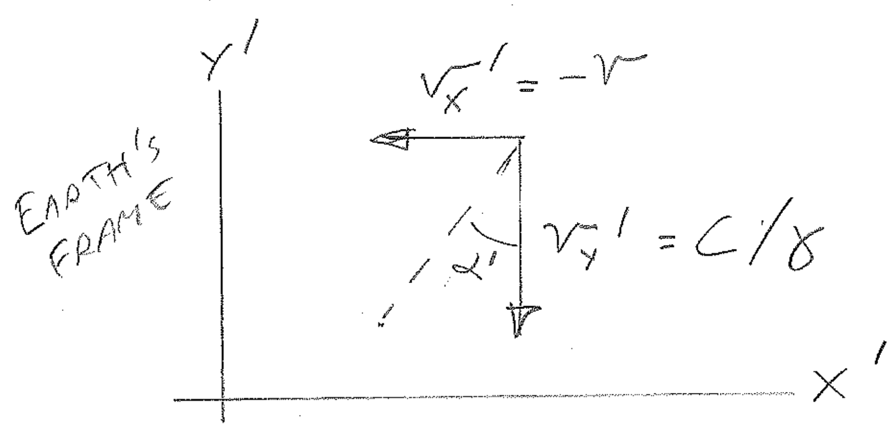
$$= \frac{v_x - v}{1 - \frac{v_x v}{c^2}}$$

N.B. THE VELOCITY-ADDITION FORMULAE SUGGEST LORENTZ TRANSFORMATIONS FORM A GROUP.

EXAMPLES OF VELOCITY ADDITION

- MOVING MEDIA (SEE HOMEWORK).
- STELLAR ABERRATION. A RAY OF STARLIGHT APPROACHES EARTH IN A DIRECTION PERPENDICULAR TO THE EARTH'S VELOCITY:





$$\tan\alpha' = \frac{v_x'}{v_y'} = \beta\gamma$$

4-VECTORS.

CAVEAT! "ORDER" AND "SIGNITURE" CONVENTIONS. ALSO $\alpha = 0, 1, 2, 3,$ OR $\alpha = 1, 2, 3, 4$)

A 4-VECTOR:

$$X^{\alpha} = (X^1, X^2, X^3, X^4); \quad X^4 = ct.$$

CAUTION X^{α} IS NOT THE SAME AS X_{α}: WE'LL GET TO THIS.

A TENSOR Q^{α}_{β} CAN TRANSFORM A 4-VECTOR INTO ANOTHER 4-VECTOR

CAUTION Q^{α}_{β} IS NOT THE SAME AS $Q^{\alpha\beta}$ OR $Q_{\alpha\beta}$.

WITH $X'^{\alpha} = Q^{\alpha}_{\beta} X^{\beta}$

WITH "SUMMATION NOTATION"

$$X'^{\alpha} = \sum_{\beta=1}^{4} Q^{\alpha}_{\beta} X^{\beta},$$

IT HAPPENS THAT $(Q^{\alpha}_{\beta})^{-1}$ IS THE SAME AS Q^{α}_{β} BY CHANING $-\beta\delta \rightarrow \beta\delta$ (AS EXPECTED FOR THE "LORENTZ GROUP").

$$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

FOR A FRAME MOVING ALONG $\hat{x}$.

IF SOME PHYSICAL RELATION CAN BE WRITTEN AS A 4-VECTOR EQUATION WHERE COMPONENTS TRANSFORM IN THIS WAY, THEN WE SAY THE EQUATION IS IN "LORENTZ COVARIANT" FORM (OR SIMPLY "LORENTZ COVARIANT").

CAUTION: THIS IS NOT THE SAME AS SAYING A 4-VECTOR IS A "COVARIANT 4-VECTOR"

IN OTHER WORDS, IF $A'^{\alpha} = \Phi^{\alpha}_{\beta} A^{\beta}$, THEN A' IS LORENTZ COVARIANT.

A_{α} IS A "COVARIANT 4-VECTOR",
A^{α} IS A "CONTRAVARIANT 4-VECTOR".

(IT IS INDEED CONFUSING "COVARIANT" HAS TWO MEANINGS.)

((I SUPPOSE YOU COULD SAY $\vec{F} = m \frac{d^2\vec{x}}{dt^2}$ IS IN "EUCLIDIAN INVARIANT FORM!"... BUT NO ONE DOES.)

WE ARE NOT GOING BEYOND SPECIAL RELATIVITY ("FLAT SPACETIME").
IN SPECIAL RELATIVITY THE COMPONENTS OF Q^{α}_{β} ARE CONSTANTS IN A FRAME AND INDEPENDENT OF COORDINATES.

IN GENERAL RELATIVITY THE COMPONENTS OF Q^{α}_{β} CAN BE FUNCTIONS OF COORDINATES.
THE ANALOG OF $X'^{\alpha} = Q^{\alpha}_{\beta} X^{\beta}$ IS

$$dX'^{\alpha} = \left(\frac{dX'^{\alpha}}{dX^{\beta}}\right) dX^{\beta}$$

INDEED

$\frac{1}{dX^{\beta}} X^{\beta}$ "CONTRACTS".

THIS IS CONFUSING SO IT'S SOMETIMES WRITTEN $d_{\beta} X^{\beta}$.

SOME MISCELLANEOUS COMMENTS:

1. THE "DISPLACEMENT VECTOR" X^α HAS AN ANALOG

$$X^\alpha = (x_1, x_2, x_3, ct)$$
$$= (x, y, z, ct)$$

... HOW CONFUSING.

USUALLY, BUT NOT ALWAYS, (SEE ABOVE) X^α (OR X_α) IS A COMPONENT OF A 4-VECTOR;
X_i IS A COMPONENT OF A 3-VECTOR.

COMMENTS ...

2. THE INVARIANT INTERVAL OF $\mathbb{X}$ IS

$$s^2 = -x^2 - y^2 - z^2 + c^2t^2.$$

WE COULD HAVE, IF WE WISHED, DEFINED

$$\mathbb{X} = (ix, iy, iz, ct) \quad \text{SO}$$

$$X_\mu X_\mu = -x^2 - y^2 - z^2 + c^2t^2.$$

... NO CONTRA- OR CO-VARIANT SEPARATION.

THIS IS FINE, I SUPPOSE. IT'S THE CONVENTION IN BJORKEN AND DRELL. BUT IT CAUSED ME ENORMOUS GRIEF WHEN I WAS A STUDENT TAKING QFT, AND IT'S A DEAD-END FOR GENERAL RELATIVITY.

INSTEAD, WE HAVE CONTRAVARIANT VECTOR X^α

$$X^\alpha = (-x, -y, -z, ct)$$

AND COVARIANT VECTOR X_α

$$X_\alpha = (x, y, z, ct).$$

WITH THE USUAL SUMMATION NOTATION

$$X^\alpha X_\alpha = -x^2 - y^2 - z^2 + c^2 t^2.$$

$X^\alpha X^\alpha$ (AND $X_\beta X_\beta$) ARE A TWO-INDEX OBJECTS THEN TAKING A TRACE.

THE CONFUSION ARISES BECAUSE WE HAVE 2 KINDS OF 4-VECTORS, WE WRITE X^α AND X_α TO MEAN THE VECTOR OBJECT AND NOT JUST THE COMPONENTS.

... CONFUSING.

WE NEED A WAY TO CONVERT X^α TO X_β AND VICE VERSA:

$$X_\alpha = g_{\alpha\beta} X^\beta$$

WITH $$g_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{pmatrix}$$

THE "METRIC TENSOR".

$g_{\alpha\beta}$ IS THE STARTING POINT TO UNCOVER AN AUTHOR'S CONVENTION

MISCELLANEOUS TENSOR RELATIONS.

- SCALAR

  ds^2 IS A TENSOR OF RANK 0.
  $(ds^2 = dx^\alpha dx_\alpha)$

- 4-VECTORS

  CONTRAVARIANT & COVARIANT 4-VECTORS ARE TENSORS OF RANK 1.

- RANK 2 TENSORS.

$$T'^{\mu\nu} = Q_\alpha^{\ \mu} Q_\beta^{\ \nu} T^{\alpha\beta}$$

A CONTRAVARIANT TENSOR OF RANK 2.

$$T'_{\mu\nu} = Q_\mu^{\ \alpha} Q_\nu^{\ \beta} T_{\alpha\beta}$$

A COVARIANT TENSOR OF RANK 2.

$$T'^{\ \nu}_{\mu} = Q_\mu^{\ \alpha} Q_\beta^{\ \nu} T_\alpha^{\ \beta}$$

A MIXED TENSOR OF RANK 2.

⋮

etc.

ONLY RARELY DO YOU RUN ACROSS TENSORS OF RANK > 2 IN ELECTRODYNAMICS. AN EXCEPTION IS THE LEVI-CIVITA TENSOR.

IN EUCLIDEAN SPACE WE FOUND A LEVI-CIVITA TENSOR IN THE CROSS PRODUCT

$$X_K = \varepsilon_{ijk} X_i X_j .$$

WHERE ε_{ijk} IS COMPLETELY ANTISYMMETRIC.

THE MINKOWSKI-SPACE ANALOG IS

$$\varepsilon_{\alpha\beta\mu\nu}$$

WHERE $\varepsilon_{\alpha\beta\mu\nu}$ IS COMPLETELY ANTISYMMETRIC.

- LINEARITY. THE SUM OF TWO TENSORS OF EQUAL RANK IS A TENSOR OF THE SAME RANK.

EQUATIONS HAVE TO BE "RANK CONSISTENT"!

$$Q_\alpha{}^\beta \stackrel{?}{=} A_\alpha B_\beta \quad \text{IS BAD.}$$

- SOME RULES OF TENSOR MANIPULATION:
  - A MIXED TENSOR HAVING THE SAME CONTRA- AND CO-VARIANT INDEX IS "CONTRACTED" INTO A TENSOR 2 RANKS LOWER

    e.g., $T_\alpha{}^\alpha$ IS A SCALAR.

  - THE "LINE ELEMENT" CAN BE WRITTEN

$$ds^2 = g_{\alpha\beta}\, dX^\alpha dX^\beta .$$

EXERCISE: SHOW $g_{\alpha\beta}$ IS A TENSOR.

THE DERIVATIVE

$\frac{\partial}{\partial X^\alpha}$ $\{\partial_\alpha\}$ ACTING ON

A TENSOR TRANSFORMS LIKE

A COVARIANT FACTOR. A

"GRADIENT" THEREFORE

INCREASES THE RANK BY 1.

IF T^0 IS A RANK-0 TENSOR, ITS GRADIENT $\partial_\mu T^0$ IS A COVARIANT VECTOR, AND A SMALL INCREMENT OF THE RANK-0 TENSOR IS

$$dT^0 = \partial_\mu T^0 \, dx^\mu .$$

THIS ALSO INTRODUCES A

4-DIVERGENCE $\partial_\alpha T^\alpha$

∴ IF A 4-VECTOR OBEYS $\partial_\alpha T^\alpha = 0$, AND IF THE COMPONENTS OF T^α DIFFER FROM 0. IN A FINITE SPATIAL REGION, THEN THE 3-VOLUME INTEGRAL

$$\iiint T^4 \, dV$$

IS AN INVARIANT.

(THIS WILL BE RELEVANT WHEN WE COMBINE CURRENT AND CHARGE INTO A 4-VECTOR.).

∴ SIMILARLY, IF $\partial_\alpha T^{\alpha\beta} = 0$, THEN $\iiint T^{4\alpha}$ IS A 4-VECTOR.

WE CALL THINGS LIKE

$$\partial_\alpha T^\alpha = 0 \quad \text{AND} \quad \partial_\alpha T^{\alpha\beta} = 0$$

"CONSERVATION LAWS".

THERE'S ALSO A MINKOWSKI ANALOG OF GAUSS' LAW

$$\iiiint \partial_\alpha T^\alpha \, d^4x = \oint\!\!\oint\!\!\oint T^\alpha \, da_\alpha$$