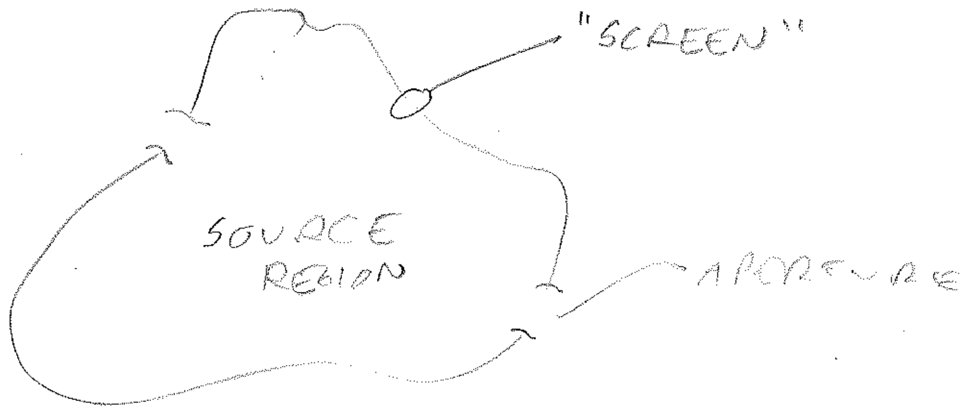
Physics 515, Electrodynamics III
Department of Physics, University of Washington
Spring quarter 2020
April 22, 2020, 11am
On-line lecture

Administrative:
1. Office hours today after class at URL
https://washington.zoom/us/j/712804010

Lecture: **J. Chapter 10: Scattering & diffraction.**
1. J. Chapter 10.5: Scalar Diffraction theory.
a. Examples: Finding planar Green's function; circular aperture in plane.
b. Complementary screens and Babinet's principle .
c. Example: Absorbing disk.
2. J. Chapter 10.11: Optical Theorem & optics.
a. Example: Absorbing disk.

J. §10.5 SCALAR DIFFRACTION THEORY



DIFFRACTION REGION

IN SCALAR DIFFRACTION THEORY, WE'RE MOSTLY CONCERNED WITH INTENSITIES (THE "DIFFRACTION PATTERN"). SO, WORK WITH THE SCALAR FIELD $\psi(\vec{r}, t)$ (USUALLY $E(\vec{r}, t)$).

WE ASSUME FAR FROM THE SOURCE REGION THERE ARE OUTGOING SPHERICAL WAVES (THE "RADIATION CONDITION").

WE MAKE THE HUYGENS - FRESNEL ASSUMPTION; EACH POINT OF THE "SURFACE" OF THE APERTURE IS A SOURCE OF OUTGOING SPHERICAL WAVES.

THIS LEADS TO J, EQN. 10.79; THE KIRCHHOFF INTEGRAL,

$$\psi(\vec{r}) = -\frac{1}{4\pi}\iint_{S_1} \frac{e^{ikR}}{R}$$

$$\times\, \hat{n}' \cdot \left\{ \vec{\nabla}'\psi + ik\left(1 + \frac{i}{kR}\right)\frac{\vec{R}}{r} \right\} da'$$

HERE $\vec{R} = \vec{r} - \vec{r}\,'$,

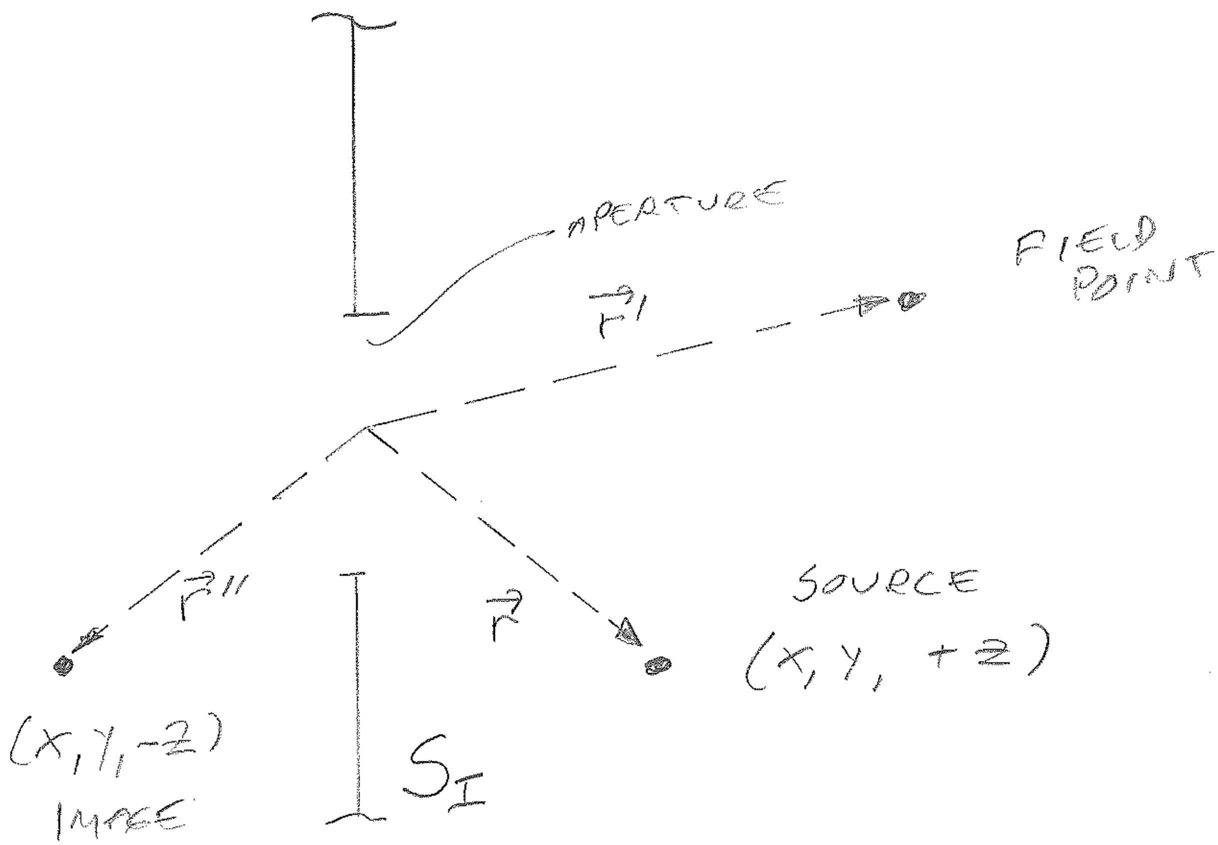
$\hat{n}'$ POINTS INTO THE DIFFRACTION REGION

S_1 IS THE DIFFRACTION SCREEN, INCLUDING APERTURES.

THEN APPLY THE KIRCHHOFF APPROXIMATION

- ψ (AND $\frac{\partial \psi}{\partial n}$) VANISH EVERYWHERE EXCEPT AT APERTURES;
- ψ (AND $\frac{\partial \psi}{\partial n}$) IN THE "SURFACE" OF THE APERTURE ARE THOSE OF ψ (AND $\frac{\partial \psi}{\partial n}$) WITHOUT THE DIFFRACTING SCREEN.

FINDING $G(\vec{r}, \vec{r}\,')$
FOR A CONDUCTING PLANE AT $z=0$
WITH DIRICHLET BOUNDARY CONDITIONS,



WE'VE FOUND THE GREEN'S FUNCTION
FOR THIS GEOMETRY BEFORE!

$$G(\vec{r}, \vec{r}\,') = \frac{1}{4\pi} \frac{e^{+ikR}}{R} - \frac{1}{4\pi} \frac{e^{-ikR'}}{R'}$$

$$R = |\vec{r} - \vec{r}\,'| = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

$$R' = |\vec{r} - \vec{r}\,''| = \left[(x-x')^2 + (y-y')^2 + (z+z')^2\right]^{1/2}$$

WE THEN DIRECTLY EVALUATE

$$\psi(\vec{r}) = \iint_{S_I} \psi(\vec{r}\,') \frac{d}{dn'} G(\vec{r}, \vec{r}\,') \, da'$$

RECALL $\frac{d}{dn'} \rightarrow \hat{n}' \cdot \vec{\nabla}'$; SO

$$\psi(\vec{r}) = \frac{ik}{2\pi} \iint_{S_I} \frac{e^{ikR}}{R} \left(1 + \frac{i}{kR}\right)$$

$$\times \; \hat{n}' \cdot \frac{\vec{R}}{r} \psi(\vec{r}\,') \, da'$$

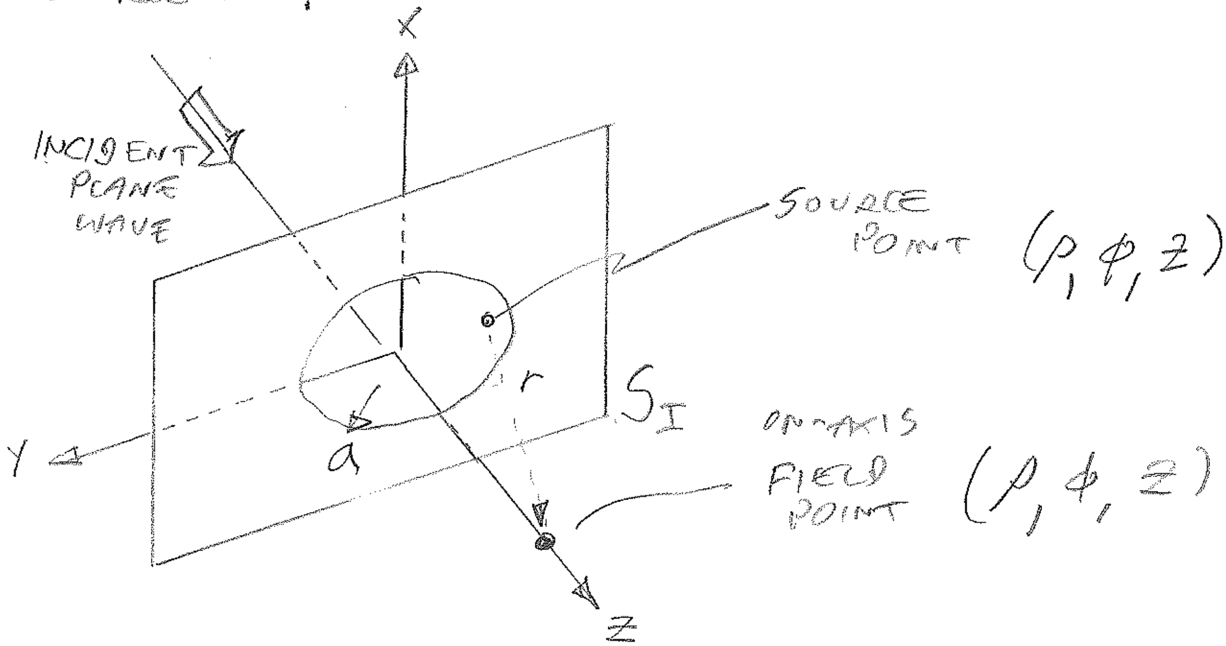
AS EXPECTED: KIRCHHOFF'S INTEGRAL (J. EQN. 10.85).

WE WORK IN THE "RADIATION ZONE" (FAR FROM THE APERTURE):

$$1 + \frac{i}{kR} \rightarrow 1$$

THIS IS THE ORIGINAL HUYGENS-FRESNEL FORMULATION.

EXAMPLE: CIRCULAR APERTURE IN AN ABSORBING ("BLACK") PLANAR SCREEN.



EVALUATE THE RADIATION-ZONE (FAR) FIELDS; $\Psi \to E$; ON-AXIS.

$$E(z) = \frac{ik}{2\pi} \iint_{S_I} \frac{e^{ikR}}{R} \, \hat{n}' \cdot \frac{\vec{R}}{R} \, E(\vec{r}\,') \, da'$$

$$= \frac{ik}{2\pi} E_0 \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \frac{e^{ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} \, \rho \, d\rho \, d\phi$$

SINCE $\rho^2 + z^2 = r^2$; $\rho\, d\rho = r\, dr$ SO

$$E(z) = ikE_0 \int_{r=z}^{r=\sqrt{a^2+z^2}} e^{ikR}\, dr$$

$$E(z) = \underbrace{E_0 e^{ik\sqrt{a^2+z^2}}}_{a} - \underbrace{E_0 e^{ikz}}_{b}$$

a DESCRIBES A PLANE WAVE TRAVELING FROM THE APERTURE RIM TO THE FIELD POINT.

b DESCRIBES A PLANE WAVE TRAVELING FROM THE APERTURE CENTER TO THE FIELD POINT.

FIND THE RESULTING ON-AXIS INTENSITY

$$I = \varepsilon_0 c \langle E^2 \rangle = \varepsilon_0 c \tfrac{1}{2} \mathrm{Re}\, E E^*$$

$$= \varepsilon_0 c \tfrac{1}{2} E_0^2 \left\{ 2 - 2\cos\left(k\sqrt{a^2+z^2} - kz\right) \right\}$$

RECALL $\dfrac{1-\cos\alpha}{2} = \sin^2 \dfrac{\alpha}{2}$, SO

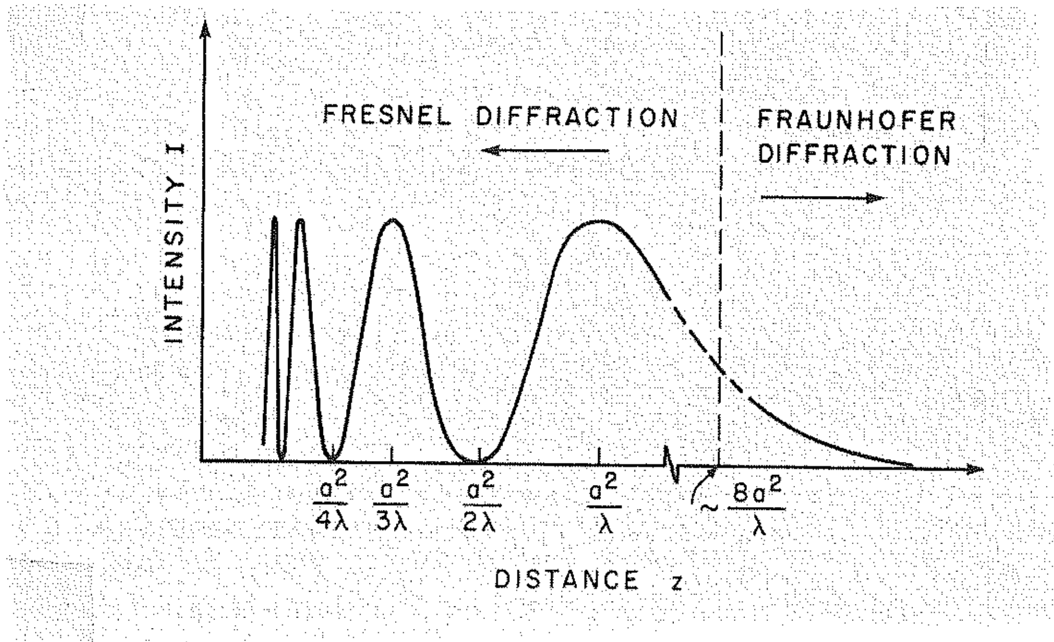
$$I = \varepsilon_0 c E_0^2 \sin^2 \left\{ \frac{k\sqrt{a^2+z^2} - kz}{2} \right\}$$

WE'LL CONSIDER FIELD POINTS IN THE "FRAUNHOFER REGION" $z \gg R$;

$$\sqrt{a^2+z^2} \longrightarrow z + \frac{a^2}{2z} + \cdots$$

$$I \approx \varepsilon_0 c E_0^2 \sin^2 \left\{ \frac{kz + k\frac{a^2}{2z} - kz}{2} \right\}$$

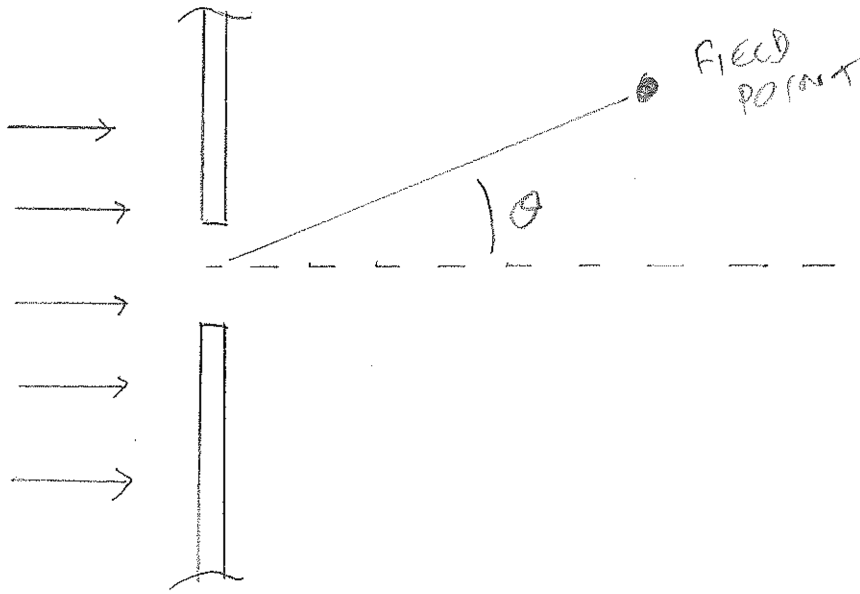
$$= \varepsilon_0 c E_0^2 \sin^2 \left\{ \frac{ka^2}{4z} \right\}$$



NOTICE BEYOND a^2/λ THERE IS NO MORE SUCCESSION OF MAXIMA & MINIMA: THERE'S A "DARK SPOT" ON-AXIS IN THE FRAUNHOFER REGION.

(WHAT ABOUT OFF-AXIS DIFFRACTION? THE GEOMETRY REQUIRED IS VERY AWKWARD. THERE'S AN APPROXIMATION TECHNIQUE "FRESNEL ZONES", NOT CALLED SUCH BY JACKSON, THAT CAN BE APPLIED. SEE JACKSON EQN 10.114 OR LANDAU & LIFSHITZ "... FIELDS" P. 157.)

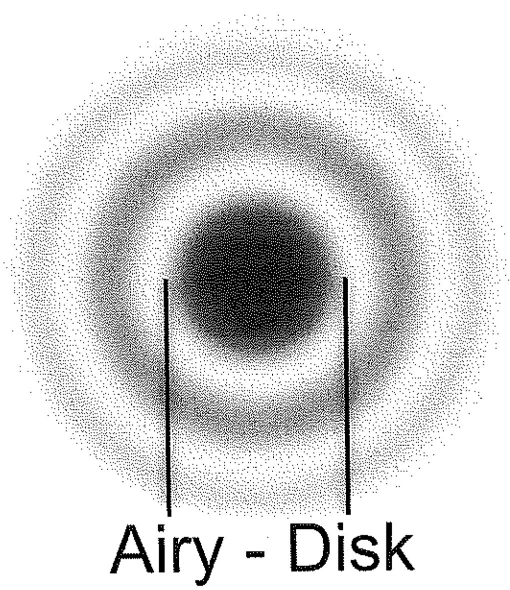
WE SIMPLY ASSERT



$$\frac{I(\theta)}{I(0)} = \left\{ \frac{2 J_1(\rho)}{\rho} \right\}^2 ; \quad \rho = K \sin\theta \frac{a}{2}$$

J. EQN. 10.114 IN THE FRAUNHOFER REGION

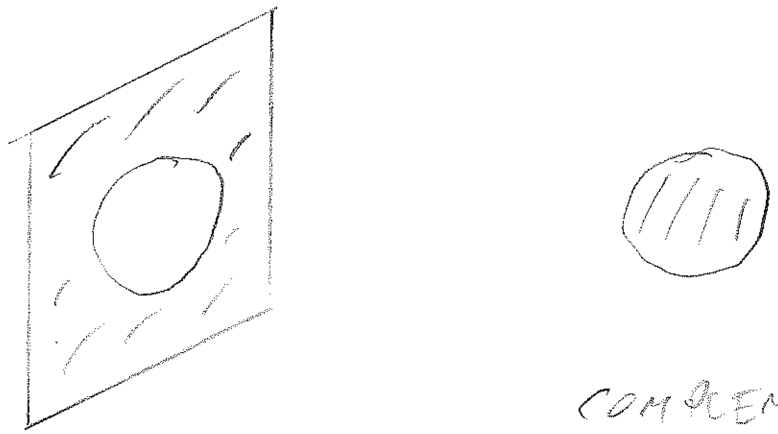
AN AIRY PATTERN



THE AIRY DISK HAS ANGULAR WIDTH $\sim 2.44 \, \lambda / R$

BABINET'S PRINCIPLE II.

BABINET'S PRINCIPLE AND COMPLEMENTARY SCREENS.

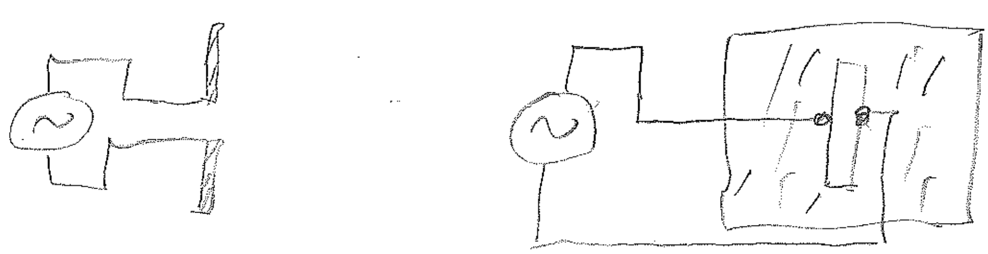


SCREEN

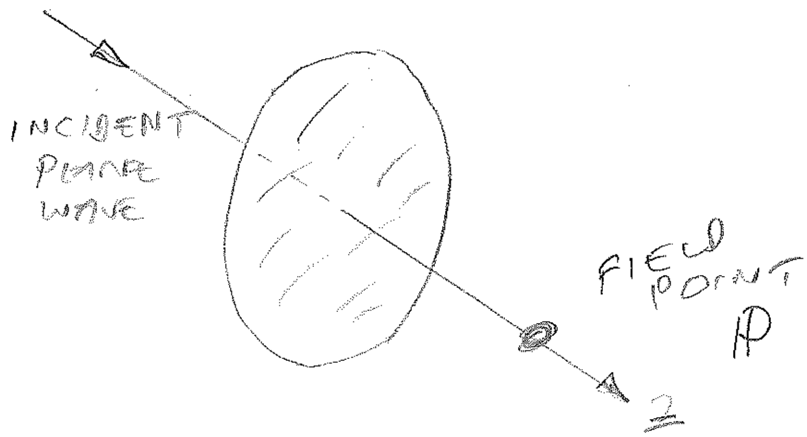
COMPLEMENTARY SCREEN

BABINET'S PRINCIPLE: "WHEN THE FIELD BEHIND A SCREEN IS ADDED TO THE FIELD OF A COMPLEMENTARY SCREEN, THAT SUM IS THAT OF THE FIELD WITH NO SCREEN".

THIS IS OBVIOUS AND VERY POWERFUL. Eg., I USED IT TO RELATE A LINEAR ELECTRIC DIPOLE ANTENNA TO A "SLOT" ANTENNA:



EXAMPLE: INTENSITY ALONG THE AXIS OF A CIRCULAR DISK



WE COULD DO THIS AB INITIO. WE'LL INSTEAD USE BABINET'S PRINCIPLE.

UNOBSTRUCTED WAVE AT $\mathbb{P}$

= WAVE AT $\mathbb{P}$ DUE TO APERTURE IN SCREEN + WAVE AT $\mathbb{P}$ DUE TO COMPLEMENTARY SCREEN.

$$\frac{ik}{2\pi}\iint E_0 \frac{e^{ikr}}{r}\,da$$ INTEGRATION OVER ENTIRE PLANE

$$= \frac{ik}{2\pi}\iint E_0 \frac{e^{ikr}}{r}\,da$$ INTEGRATION OVER APERTURE: DONE ALREADY

$$+ \frac{ik}{2\pi}\iint E_0 \frac{e^{ikr}}{r}\,da$$ INTEGRATION OVER PLANE BUT NOT APERTURE

THAT IS

$$E_0 e^{ikz}$$ NORMALIZATION FROM HOMEWORK.

$$= E_{APERTURE} + E_{DISK}.$$

RECALL $E_{APERTURE} = E_0 e^{ikz} - E_0 e^{ik\sqrt{a^2+z^2}}$

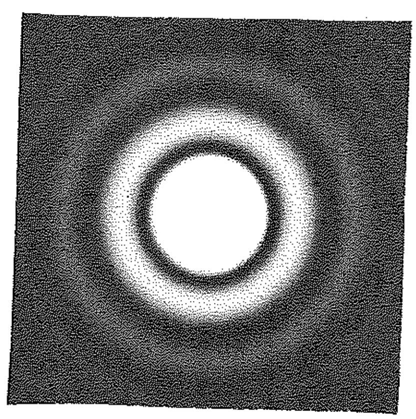
SO $$E_{DISK} = E_0 e^{ik\sqrt{a^2+z^2}}.$$

$$I = \varepsilon_0 c \langle E^2 \rangle = \varepsilon_0 c \tfrac{1}{2} \mathrm{Re}\, E E^*$$
$$= \tfrac{1}{2}\varepsilon_0 c E_0^2.$$

NOTICE I = CONSTANT ON-AXIS, INDEPENDENT OF z.

ALSO $I \neq 0$ ON-AXIS! THERE'S A BRIGHT SPOT; "POISSON'S SPOT".

THIS WAS HOTLY DEBATED AND THEN OBSERVED BY FRESNEL & ARAGO (1818) AND ALSO AMPÈRE.



POISSON'S SPOT

AIRY DISK CONTAINS ~84% OF THE LIGHT.

J. 10, 11 THE OPTICAL THEOREM.
AS PER OPTICS.

VIA EUGENE FEENBERG THESIS 1932.

RECALL FROM LAST QUARTER

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial}{\partial t} \vec{H} \qquad \vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial}{\partial t} \vec{E} + \vec{J}$$

1. $\cdot \vec{H}$ $\qquad\qquad \cdot \vec{E}$

2. SUBTRACT ONE FROM THE OTHER

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\frac{1}{2} \frac{\partial}{\partial t} (D^2 + H^2) - \vec{E} \cdot \vec{J}.$$

WE'LL EXTEND THIS BY STARTING WITH

$$\vec{\nabla} \times \vec{E}^* = -\mu \frac{\partial}{\partial t} \vec{H}^* \qquad \vec{\nabla} \times \vec{H}^* = \varepsilon \frac{\partial}{\partial t} \vec{E}^* + \vec{J}^*$$

1. $\cdot \vec{H}$ $\qquad\qquad \cdot \vec{E}$
2. $\vec{J} = \sigma \vec{E}$ & $\vec{J}^* = \sigma \vec{E}^*$
3. SUBTRACT ONE FROM THE OTHER.
4. ADD TO THE ORIGINAL
5. COMPLEX CONJUGATE.

$$\vec{\nabla} \cdot \left\{ \frac{\vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H}}{4} \right\} \qquad (a)$$

$$= -\frac{1}{4}\frac{\partial}{\partial t}\left\{ \varepsilon E^2 + \mu H^2 \right\} \qquad (b)$$

$$-\frac{1}{2}\sigma E^2 \qquad (c)$$

(a) IS THE (REAL) ENERGY-FLOW DENSITY;

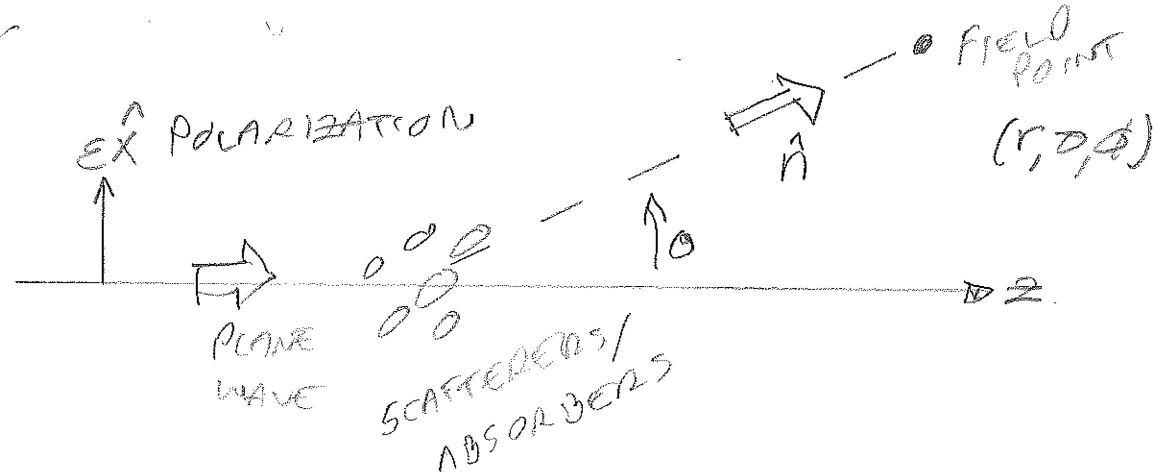
(b) IS THE VOLUME ENERGY DENSITY;

(c) IS THE RATE OF ENERGY ABSORBED PER UNIT VOLUME.

NOW INTEGRATE THIS OVER THE VOLUME OF A LARGE SPHERE CONTAINING SCATTERING OBJECTS. THE SPHERE IS SO LARGE THAT THE SCATTERED WAVES ARE SPHERICAL WAVES AT THE SURFACE.

BUT FIRST, INTRODUCE SPECIAL FORMS OF $\vec{E}$ AND $\vec{H}$.

START WITH AN INCIDENT PLANE WAVE TRAVELING ALONG $\hat{z}$ ENCOUNTERING SCATTERERS/ABSORBERS.



FOR FIELD POINTS AT LARGE DISTANCES

$$\vec{E} = E_0 \left\{ \hat{x} e^{ik\hat{z}\cdot\vec{r}} + \vec{F}(\theta,\phi) \frac{e^{ikr}}{kr} \right\}$$

$$\vec{H} = \frac{1}{Z_0} E_0 \left\{ \hat{z}\times\hat{x}\, e^{ik\hat{z}\cdot\vec{r}} + \hat{n}\times\vec{F}(\theta,\phi) \frac{e^{ikr}}{kr} \right\}$$

$\vec{F}(\theta,\phi)$ IS THE VECTOR AMPLITUDE OF THE SCATTERED ELECTRIC FIELD (MAY BE COMPLEX) RELATIVE TO E_0.

WE'LL NOW GET BACK TO THE VOLUME OF THE LARGE SPHERE.

WE SIMPLIFY BY ASSUMING STEADY-STATE! $\frac{\partial}{\partial t} \to 0$.

$$-\iiint \frac{1}{2} \sigma E^2 \, dv =$$

$$\frac{1}{4} \frac{1}{Z_0} E_0^{\,2} \oint\!\!\oint \Big\{$$

$$2\hat{z} + \left([\hat{x} \times (\hat{n} \times \vec{F}^*)] + [\vec{F}^* \times (\hat{z} \times \hat{x})] \right)$$

$$\times \frac{e^{ik r(1-\cos\theta)}}{kr}$$

$$+ \left([\vec{F} \times (\hat{z} \times \hat{x})] + [\hat{x} \times (\hat{n} \times \vec{F})] \right)$$

$$\times \frac{e^{ikr(1-\cos\theta)}}{kr}$$

$$+ \frac{1}{(kr)^2} \Big([\vec{F}^* \times (\hat{n} \times \vec{F})]$$

$$+ [\vec{F} \times (\hat{n} \times \vec{F}^*)] \Big) \Big\} \cdot d\vec{a}$$

THIS IS A MESS, BUT THE INTEGRAL OVER $2\hat{z}$ VANISHES OVER A CLOSED SPHERE.

NOTICE $e^{ikr(1-\cos\theta)}$ OSCILLATES RAPIDLY IN θ FOR LARGE kr UNLESS $\cos\theta \approx 1$ (FORWARD DIRECTION).

AND FOR $\theta \approx 0$,

$$\hat{x} \times (\hat{n} \times \vec{F}) + \vec{F} \times (\hat{z} \times \hat{x}) = 2\,\vec{F} \cdot \hat{x}.$$

HENCE, e.g., FOR THE INTEGRAL TERM LINEAR IN $\vec{F}$:

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left\{ \hat{x} \times (\hat{n} \times \vec{F}) + \vec{F}(\hat{z} \times \hat{x}) \right\} \times \frac{e^{ikr(1-\cos\theta)}}{kr}\, r^2 \, d\cos\theta \, d\phi$$

$$\approx \frac{-2}{ik^2} \int_{\phi=0}^{2\pi} \left(\vec{F} \cdot \hat{x}\right)_{\theta=0} d\phi = \frac{4\pi i}{k^2} \left(\vec{F} \cdot \hat{x}\right)_{\theta=0}.$$

THE ANALOGOUS PROCEDURE IS APPLIED TO THE INTEGRAL TERM LINEAR IN $\vec{F}^*$.

NOW FOR THE TERM CONTAINING $\vec{F} \ldots \times \vec{F}^*$.

FOR $\theta \approx 0$,

$$\vec{F} \times (\hat{n} \times \vec{F}^*) = \hat{n}(\vec{F} \cdot \vec{F}^*)$$

ALSO $\vec{F}^* \times (\hat{n} \times \vec{F}) = \hat{n}(\vec{F} \cdot \vec{F}^*)$. HENCE

$$\frac{1}{(kr)^2} \oiint \left\{ \left[\vec{F}^* \times (\hat{n} \times \vec{F})\right] + \left[\vec{F} \times (\hat{n} \times \vec{F}^*)\right] \right\} \cdot d\vec{a}$$

$$= \frac{2}{k^2} \int_{\phi\, 0}^{2\pi} |\vec{F}|^2 \sin\theta \, d\theta .$$

THE ENTIRE VOLUME INTEGRAL IS

$$\frac{1}{Z_0} E_0^2 \left[\frac{1}{k^2} \int_{\phi\,0}^{2\pi} \int_{\theta\,0}^{\pi} |\vec{F}|^2 \sin\theta \, d\theta \, d\phi \right. \quad \text{(a)}$$

$$\left. + \frac{2\pi i}{k^2} \hat{x} \cdot (\vec{F} - \vec{F}^*)_{\theta = 0} \right] \quad \text{(b)}$$

$$= - \iiint \sigma |\vec{E}|^2 \, dv \quad \text{(c)}$$

TERM (a) IS THE SCATTERED POWER. THE SCATTERING CROSS SECTION IS THE POWER SCATTERED PER UNIT SOLID ANGLE PER UNIT INCIDENT POWER PER UNIT AREA. BY EXAMINING (a):

$$\frac{d\sigma_{SCATT}}{d\Omega} = \frac{1}{K^2}\left|\vec{F}(\theta,\phi)\right|^2$$

TERM (C) IS THE ABSORBED POWER. THE ABSORPTION CROSS SECTION IS THE POWER ABSORBED PER UNIT AREA. BY EXAMINING (C)

$$\sigma_{ABS} = \frac{Z_0}{E_0^2}\iiint \sigma \left|\vec{E}\right|^2 dv.$$

THE TOTAL CROSS SECTION IS THE SUM OF THE SCATTERING AND ABSORPTION CROSS SECTION

$$\sigma_{TOTAL} = \frac{2\pi}{iK^2}\hat{x}\cdot\left(\vec{F}-\vec{F}^*\right)_{\theta=0}$$

$$= \frac{4\pi}{K^2}\,\mathrm{Im}\left(\vec{F}\right)_{\theta=0}.$$