Physics 515, Electrodynamics III
Department of Physics, University of Washington
Spring quarter 2020
May 15, 2020, 11am
On-line lecture

Administrative:

1. HW#6 due now (with some exceptions).
2. You should be getting your homework back; if not let me know.

Lecture:

J. Chapter 12: Dynamics of relativistic particles … .
1. J. C. 12.10 Stress tensor(s) and conservation laws.

STRESS TENSOR(S) AND CONSERVATION LAWS
J.D. 12.10

OVERARCHING IDEA: A SYSTEM OF CHARGED PARTICLES MOVING IN AN ELECTROMAGNETIC FIELD HAS KINETIC ENERGY OF THE PARTICLES PLUS "FIELD ENERGY".

RECALL THE VACUUM FIELD ENERGY

$$\mathcal{H} = \frac{1}{8\pi}(E^2 + B^2) \quad (CGS)$$

RECALL THE POYNTING THEOREM (ENERGY CONSERVATION)

$$\frac{d}{dt}\mathcal{H} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E}$$

WITH $-\vec{J}\cdot\vec{E}$ THE ENERGY REMOVED FROM THE PARTICLES BY THE FIELD, AND $\vec{S}$ THE POYNTING VECTOR

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} \quad (CGS).$$

WE'D LIKE TO ESTABLISH A MINKOWSKI TENSOR $T^{\mu\nu}$ THAT EMBEDS THE ABOVE FRAME-DEPENDENT EXPRESSIONS. $T^{\mu\nu}$ IS THE COVARIANT GENERALIZATION OF THE MAXWELL STRESS TENSOR T_{ij}.

START WITH THE CANONICAL EXPRESSION FOR THE HAMILTONIAN $\mathcal{H}$ IN TERMS OF THE CANONICAL MOMENTUM P_i (OR Π_i) CORRESPONDING TO DEGREE-OF-FREEDOM i:

$$\mathcal{H} = \sum_i \Pi_i \dot{\phi}_i - \mathcal{L}$$

$$= \sum_i \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi_i}{\partial t}\right)} \dot{\phi}_i - \mathcal{L}.$$

VIA OUR USUAL TECHNIQUE OF GENERATING COVARIANT EXPRESSIONS FROM FRAME-DEPENDENT EXPRESSIONS; THE SUBLETY IS THE COVARIANT EXPRESSION SHOULD BE A RANK-2 TENSOR:

$$\frac{\partial}{\partial t}\phi_i \longrightarrow \frac{\partial}{\partial x_\mu}\phi_i$$

$$\mathcal{L} \longrightarrow g^{\mu\nu}\mathcal{L} \quad \text{GIVING}$$

$$T^{\mu\nu} = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial^\nu \phi_i - g^{\mu\nu}\mathcal{L}$$

J. EQN. 12.103

THE EXPRESSION WE JUST FOUND IS GENERIC; WE APPLY IT TO THE ELECTROMAGNETIC FIELD

$$\mathcal{L}_{EM} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \quad \text{(CGS)}$$

$$\phi_k \rightarrow A^\alpha, \quad \text{GIVING}$$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}_{EM}}{\partial(\partial_\mu A^\alpha)} \partial^\nu A^\alpha - g^{\mu\nu} \mathcal{L}_{EM}$$

THIS IS THE "CANONICAL STRESS TENSOR".

Q: IS THIS SYMMETRIC OR ANTI-SYMMETRIC?

A: WE WILL SEE

COMMENT: RECALL $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\alpha)}$ IS VERY CLOSELY RELATED TO THE FIELD-STRENGTH TENSOR J: EQN. 12.87.

WHAT'S THE "STRUCTURE" OF $T^{\mu\nu}$?

FIELD MOMENTUM DENSITY

$$T^{\mu\nu} = \begin{pmatrix} \begin{pmatrix} 3 \times 3 \\ \text{MAXWELL} \\ \text{STRESS} \\ \text{TENSOR} \end{pmatrix} & \begin{pmatrix} \\ \\ \end{pmatrix} \\ \begin{pmatrix} \text{FIELD} \\ \text{MOMENTUM DENSITY} \end{pmatrix} & \begin{pmatrix} \mathcal{H} \end{pmatrix} \end{pmatrix}$$

WE WILL SEE THIS EXPLICITLY.

EXAMPLE: INDIVIDUAL COMPONENTS OF $T^{\mu\nu}$

WE CONTINUE FROM THE PREVIOUS PAGE

$$\frac{\partial \mathcal{L}_{EM}}{\partial(\partial_\mu A^\alpha)} \partial^\nu A^\alpha = -\frac{1}{4\pi} g^{\mu\varepsilon} F_{\varepsilon\alpha} \partial^\nu A^\alpha$$

JACKSON EQN. 12.104.

LET'S LOOK AT THE COMPONENTS OF $T^{\mu\nu}$.

EXAMPLE (USING JACKSONS $\mu = 0, 1, 2, 3$

AND $g^{\mu\nu} = \begin{pmatrix} + & & & \\ & - & & \\ & & - & \\ & & & - \end{pmatrix}$:

$$T^{00} = -\frac{1}{4\pi} g^{0\varepsilon} F_{\varepsilon\alpha} \partial^0 A^\alpha - g^{00} \underbrace{\mathcal{L}_{EM}}_{\frac{1}{8\pi}(E^2 - B^2)}$$

COMMENTS

- $g^{00} = 1$; $g^{01} = g^{02} = g^{03} = 0$,

$\longrightarrow$ ONLY $\varepsilon = 0$ SURVIVES.

- $F_{00} = 0$; $F_{01} \sim E_x$, $F_{02} \sim E_y$, $F_{03} \sim E_z$.
- $A_1 = A_x$, $A_2 = A_y$, $A_3 = A_z$.
- $\partial^0 \sim \partial_t$

$$T^{00} = -\frac{1}{4\pi} \left\{ E_x \partial_t A_x + E_y \partial_t A_y + E_z \partial_t A_z \right\} - \frac{1}{8\pi}(E^2 - B^2)$$

$$= -\frac{1}{4\pi} \left\{ \vec{E} \cdot \frac{\partial}{\partial t} \vec{A} \right\} - \frac{1}{8\pi}(\ldots$$

$$= -\frac{1}{4\pi} \left\{ \vec{E} \cdot (\vec{E} - \vec{\nabla} \Phi) \right\} - \frac{1}{8\pi}(\ldots$$

$$= \frac{1}{8\pi}(E^2 + B^2) + \frac{1}{4\pi} \vec{E} \cdot \vec{\nabla} \Phi$$

THE SECOND TERM IS A TOTAL DIVERGENCE (Q: WHY?) IF THE SOURCES OF $\vec{E}$ AND $\vec{B}$ ARE LOCALIZED, THIS CAN BE IGNORED IN THE INTEGRAL ENERGY BALANCE, SO:

$$T^{00} = \frac{1}{8\pi}\left(E^2 + B^2\right) + \text{TOTAL DIVERGENCE}$$

$$T^{0K} = \frac{1}{4\pi}\left[\vec{E}\times\vec{B}\right]_K + \text{TOTAL DIVERGENCE}$$

$$T^{JK} = \frac{1}{4\pi}\left\{E_j E_k + B_j B_k - \frac{1}{2}\delta_{jk}\left(E^2 + B^2\right)\right\}$$

+ A MESS THAT VANISHES OVER THE INTEGRATED SPHERE AT ∞

THAT IS

$$T^{00} = \mathcal{H}$$

$$T^{0K} = -\frac{1}{c} S_K$$

$$T^{JK} = \left[JK \text{ COMPONENT OF THE MAXWELL STRESS TENSOR}\right]$$

JACKSON EQN'S 12.105

COVARIANT FORCES

INSPIRED BY FORCES $\rho\vec{E}$ AND $\vec{J}\times\vec{B}$,
CONSIDER A MINKOWSKI VECTOR

$$K_\mu = \frac{1}{c} F_{\mu\alpha} J^\alpha \qquad \left\{ \begin{array}{l} \text{VIA } \partial_\alpha F^{\alpha\mu} = \frac{4\pi}{c} J^\mu \\ \text{A MAXWELL EQN,} \end{array} \right\}$$

$$\frac{1}{4\pi} F_{\mu\alpha}\, \partial_\beta F^{\alpha\beta}$$

EVEALUATE COMPONENTS OF K_μ IN
A SPECIFIC FRAME

$$K_0 = \frac{1}{c} F_{0\alpha} J^\alpha$$

$$= \frac{1}{c}\left\{ \cancelto{0}{F_{00}} J^0 + F_{01} J^1 + F_{02} J^2 + F_{03} J^3 \right\}$$

$$= \frac{1}{c}\left\{ E_x J_x + E_y J^y + E_z J_z \right\}$$

$$= \frac{1}{c} \vec{E}\cdot\vec{J} \qquad \text{CGS}$$

SIMILARLY

$$K_i \doteq \rho\left\{ \vec{E} + \frac{1}{c}\vec{v}\times\vec{B} \right\}$$

THE "SPATIAL" PART IS THE ELECTROMAGNETIC FORCE PER UNIT VOLUME, OR THE RATE OF CHANGE OF THE MECHANICAL MOMENTUM PER UNIT VOLUME.

THE "TIME" PART IS THE RATE THE FIELDS DO WORK ON SOURCES PER UNIT VOLUME.

K_μ IS THE "MINKOWSKI FORCE DENSITY."

RECALL: EUCLIDIAN FORCES ARE OBTAINED FROM THE MAXWELL STRESS TENSOR VIA

$$\frac{d f_i}{dV} = \frac{\partial T_{ij}}{\partial x_j}$$

IT'S THEREFORE SENSIBLE TO THINK THE COVARIANT GENERALIZATION IS

$$K^\mu = \partial_\alpha T^{\mu\alpha}.$$

A MORE RIGOROUS DERIVATION IS DIFFICULT. THE PATH IS:

- TAKE $T^{\mu\nu}$ IN THE FORM OF J. EQN 12.113;
- TAKE THE TENSOR DIVERGENCE OF $T^{\mu\nu}$;
- NOTICE THE ANTI-SYMMETRY OF $\partial_\nu F^{\mu\alpha} F_\alpha{}^\nu$
- APPLY THE BIANCHI IDENTITY JACKSON EQN 11.143 (JOHN WHEELER TOLD ME ITS "EASY" TO VISUALIZE THE BIANCHI IDENTITY: "IT'S THE BOUNDARY OF A BOUNDARY").
- NOTICE CANCELLATIONS.

THIS FORM OF K_μ GIVES US

$$K_0 = \partial^\nu T_{0\nu}$$

$$= \partial^0 T_{00} + \partial^k T_{0k}$$

WITH

$$K_0 = \frac{1}{c} \vec{E} \cdot \vec{J}$$

$$T_{0k} = -\frac{1}{4\pi} \left[\vec{E} \times \vec{B} \right]_k$$

$$T_{00} = \frac{1}{8\pi} \left\{ E^2 + B^2 \right\},$$ WE HAVE

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial \mathcal{H}}{\partial t} = -\vec{E} \cdot \vec{J} \,!$$

THE "TIME" COMPONENT OF THE MINKOWSKI FORCE IS POYNTING'S THEOREM.

FOR THE "SPATIAL" COMPONENTS OF K_μ:

$$K_j = \partial^\mu T_{j\mu} = \underbrace{\partial^k T_{jk}}_{\substack{\text{EUCLIDIAN} \\ \text{DIVERGENCE} \\ \text{OF MAXWELL} \\ \text{STRESS TENSOR}}} + \partial^0 \underbrace{T_{j0}}_{-\frac{1}{c}[\vec{S}]_j}$$

$$= \frac{\partial T_{ij}}{\partial x_i} - \frac{1}{c^2}\left[\frac{\partial \vec{S}}{\partial t}\right]_j$$

THAT IS

$$\vec{K} = \underset{\text{OF}}{\overset{\text{TENSOR}}{\text{DIVERGENCE}}}\ \Pi - \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t}$$

WITH Π THE MAXWELL STRESS TENSOR

NOW INTEGRATE $\vec{K} + \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t}$ OVER SOME VOLUME

$$\iiint \left(\vec{K} + \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t}\right) dV$$

$$= \iiint \underset{\text{OF}}{\overset{\text{TENSOR}}{\text{DIVERGENCE}}}\ \Pi \, dV$$

THE $\iiint \vec{K}\, dV$ TERM IS THE TOTAL FORCE ON THE VOLUME, WHICH IS THE TIME DERIVATIVE OF THE MECHANICAL MOMENTUM $\vec{P}_{MECH}$.

RECALL $\frac{1}{c^2}\vec{S} = \frac{1}{4\pi c}(\vec{E}\times\vec{B})$ (CGS)

IS THE FIELD MOMENTUM DENSITY. THE TOTAL FIELD MOMENTUM WITHIN THE VOLUME $\vec{P}_{FIELD}$ IS $\iiint \frac{1}{c^2}\vec{S}\, dV$.

WE CAN THEREFORE WRITE

$$\frac{d}{dt}\left(\vec{P}_{MECH} + \vec{P}_{FIELDS}\right) = \iiint \text{TENSOR DIVERGENCE OF } \overleftrightarrow{T}\, dV$$

APPLYING THE TENSOR DIVERGENCE THEOREM:

$$\frac{d}{dt}\left(\vec{P}_{MECH} + \vec{P}_{FIELDS}\right) = \oint\!\!\oint \text{TENSOR DIVERGENCE OF } \overleftrightarrow{T}\cdot\hat{n}\, da$$

SUPPOSE THE SYSTEM IS ISOLATED AND THE FIELDS ARE SPATIALLY BOUNDED: TAKE A SURFACE OUTSIDE THE FIELDS. THEN

$$\frac{d}{dt}\left\{ \vec{P}_{\text{MECH}} + \vec{P}_{\text{FIELDS}} \right\} = 0.$$

THIS IS MOMENTUM CONSERVATION FOR AN ISOLATED ELECTROMECHANICAL SYSTEM.

SO $T^{\mu\nu}$ CONTAINS

- EUCLIDIAN MAXWELL STRESS TENSOR;
- MOMENTUM DENSITY;
- FIELD ENERGY DENSITY

IN A COVARIANT FORM.

FUN (AND SAD) EXAMPLE: CHAMELEONS.

ANN NELSON CONCEIVED A HYPOTHETICAL PARTICLE CALLED THE "CHAMELEON". THE MASS TERM IN ITS LAGRANGIAN IS

$$M^2 T^{\mu\nu} T_{\mu\nu}$$

THIS HAS AMAZING PROPERTIES.

RECALL THE FORM OF $T^{\mu\nu}$

$$T^{\mu\nu}_{EM} = \begin{array}{cc} \left(\begin{array}{c} 3 \times 3 \\ \text{MAXWELL} \\ \text{STRESS TENSOR} \end{array}\right) & \left(\begin{array}{c} \vdash \end{array}\right) \\ \left(\begin{array}{c} \text{EUCLIDIAN} \\ \text{MOMENTUM} \\ \text{DENSITY} \end{array}\right) & \left(\mathcal{H} \right) \end{array}$$

EUCLIDIAN MOMENTUM DENSITY

THIS HAS OBVIOUS GENERALIZATION TO

$$\begin{array}{cc} \left(\begin{array}{c} \text{PRESSURE} \\ \text{TENSOR} \end{array}\right) & \left(\begin{array}{c} \vdash \end{array}\right) \\ \left(\begin{array}{c} \text{EUCLIDIAN} \\ \text{MOMENTUM DENSITY} \end{array}\right) & \left(\quad \right) \end{array}$$

EUCLIDIAN MOMENTUM DENSITY

AS YOU TRANSITION TO GENERAL RELATIVITY, YOU'LL FIND "STRESSES" ($T^{\mu\nu}$) ARE THE SOURCE OF SPACE-TIME CURVATURE. GENERAL RELATIVITY STRESS $T^{\mu\nu}$ COMES FROM ALL SOURCES OF STRESS.

BACK TO CHAMELEONS: A MASS TERM $M_0^2 T^{\mu\nu} T_{\mu\nu}$, e.g., GIVES A DIFFERENT MASS DEPENDING ON THE PARTICLE'S ENVIRONMENT. ANN INTRODUCED THIS TO PROVIDE A PARTICLE EXPLAINATION FOR DARK ENERGY THAT EVADES PARTICLE EXCLUSION BOUNDS.

ERIC ADELBERGER'S "TORSION-BALANCE" EXPERIMENTS HAVE SOME OF THE BEST LIMITS ON PARTICLE DARK ENERGY.

Q: HOW WOULD THE CHAMELEON EVADE TORSION-BALANCE EXPERIMENTS?

ISSUE: THE CANONICAL STRESS TENSOR $T^{\mu\nu}$ IS NOT SYMMETRIC AS DEFINED (SEE J. PAGE 608).

EXAMPLE:

$$T^{0i} = \frac{1}{4\pi}(\vec{E}\times\vec{B})_i + -\frac{1}{4\pi}\vec{\nabla}\cdot(A_i\vec{E})$$

J. EQN. 2.105(b)

$$T^{i0} = \frac{1}{4\pi}(\vec{E}\times\vec{B})_i + \frac{1}{4\pi}\left[(\vec{\nabla}\times\Phi\vec{B})_i - \frac{\partial}{\partial t}(\Phi E_i)\right]$$

BUT WE NEED A SYMMETRIC $T^{\mu\nu}$ FOR CERTAIN OPERATIONS.

e.g., ANGULAR MOMENTUM DENSITY (JACKSON EQN. 12.109); $M^{\mu\nu\varepsilon}$ WON'T HAVE CONSERVATION OF ANGULAR MOMENTUM $\partial_\mu M^{\mu\nu\varepsilon} = 0$ (J. EQN. 12.109) UNLESS $T^{\mu\nu}$ IS SYMMETRIC.

JACKSON CALLS THE SYMMETRIZED FORM OF $T^{\mu\nu}$: $\Theta^{\mu\nu}$ (MOST ALSO CALL IT $T^{\mu\nu}$: CONFUSING).

CONSTRUCTING $\Theta^{\mu\nu}$ IS NOT TRIVIAL; IT NEEDS TO BE SYMMETRIC, TRACELESS, GAUGE INVARIANT, AND CONTAIN THE ESSENTIAL ELEMENTS OF THE STRESS TENSOR $T^{\mu\nu}$. WHAT WE'RE SAYING IS OUR USUAL PROCEDURE OF PRODUCING COVARIANT EXPRESSIONS FROM FRAME-DEPENDENT EXPRESSIONS DIDN'T QUITE WORK IN THIS CASE.

FOR THE PROCEDURE, SEE THE DISCUSSION OF JACKSON EQN.S 12.111–114.

BASICALLY: START WITH $T^{\mu\nu}$; SEPARATE INTO SYMMETRIC AND ANTI-SYMMETRIC PARTS; SUBTRACT THE LATTER LEAVING $\Theta^{\mu\nu}$; EVALUATE COMPONENTS OF $\Theta^{\mu\nu}$ ENSURING YOU RECOVER $\mathcal{H}$, $\vec{S}$ AND T^{ij} (J. EQNS. 12.114)

WE HAVE $\partial_\mu \Theta^{\mu\nu}$ (AS $\partial_\mu T^{\mu\nu}$) ~~EMBEDDING ENERGY CONSERVATION~~ (THAT IS, POYNTING'S THEOREM) AND MOMENTUM CONSERVATION. ANGULAR MOMENTUM IS CONSERVED (J. PROBLEM 12.19).

$\Theta^{\mu\nu}$ HAS THE SAME STRUCTURE AS $T^{\mu\nu}$;

$$\Theta^{\mu\nu} = \begin{pmatrix} \begin{pmatrix} 3\times 3 \\ \text{MAXWELL} \\ \text{STRESS TENSOR} \end{pmatrix} & \begin{pmatrix} \text{FIELD} \\ \text{MOMENTUM} \end{pmatrix} \\ \begin{pmatrix} \text{FIELD} \\ \text{MOMENTUM} \end{pmatrix} & \left(H \right) \end{pmatrix}$$