Physics 515, Electrodynamics III
Department of Physics, University of Washington
Spring quarter 2020
April 15, 2020, 11am
On-line lecture

Administrative:
1. Office hours today after class at
URL: https://washington.zoom/us/j/712804010

Lecture: **J. Chapter 10: Scattering & diffraction.**
1. J. Chapter 10.1: Diffraction & scattering in the long-wavelength limit. The multipole approximation.
a. Polarized incident wave.
b. Un-polarized incident wave.
c. Example: Scattering on small dielectric sphere.
d. Example: Scattering on small conducting sphere: The issue of what conducting means in a permittivity/permeability limit.
2. J. Chapters 10.1.D, 10.2.C-D: Scattering from a random volume distribution of scatterers. The problem with Rayleigh's argument. The "source" of the scattered wave. Critical opalescence.

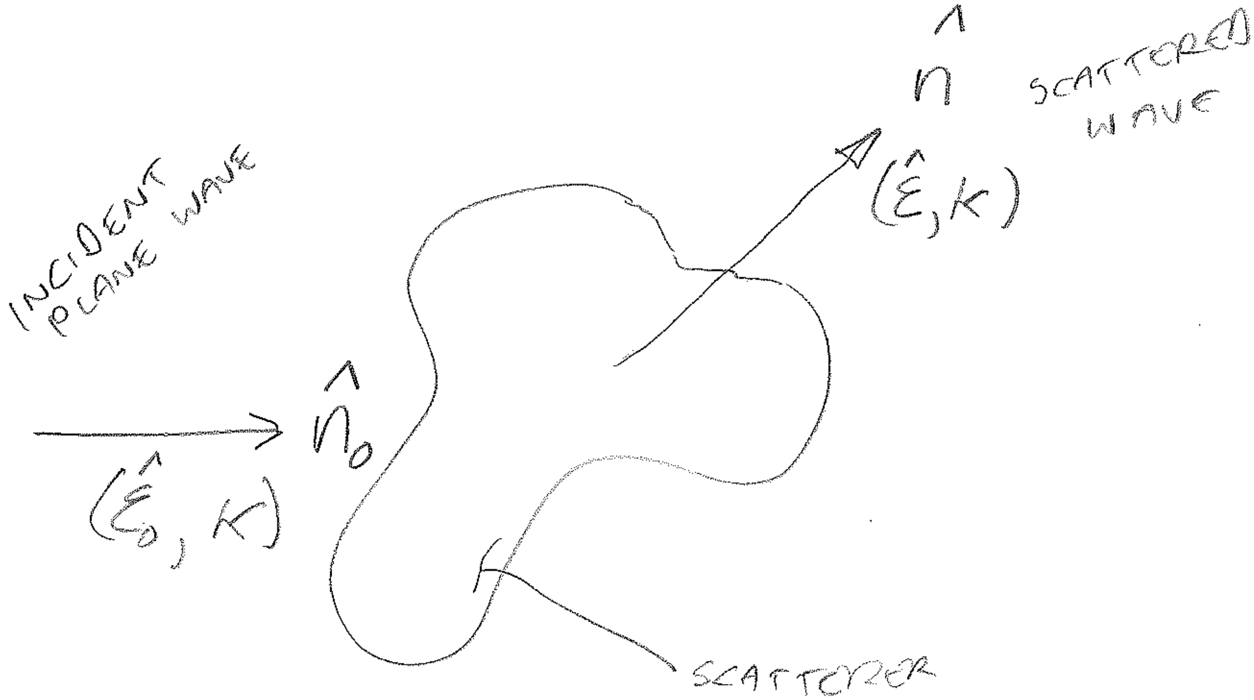
J. C. 10.1

DIFFRACTION IN THE LONG-WAVELENGTH LIMIT.

SIZE OF SCATTERER $\ll \lambda$.

BECAUSE THE SIZE OF THE SCATTER $\ll \lambda$, THE IDEA OF THIS SECTION IS TO THINK OF THE INCIDENT WAVE INDUCING ELECTRIC AND MAGNETIC MOMENTS THAT ARE COHERENT OVER THE SCATTERER; IN THE $\ll \lambda$ LIMIT, WE CAN USE THE TECHNIQUES OF STATICS TO EVALUATE THE MOMENTS.

HERE'S THE SCATTERING SITUATION



THE INCIDENT POLARIZED WAVE

$$\left.\begin{aligned} \vec{E}_0 &= \hat{\varepsilon}_0 E_0 e^{ik\hat{n}_0 \cdot \hat{r}} \\ \vec{H}_0 &= \frac{1}{Z_0} \hat{n}_0 \times \vec{E} \end{aligned}\right\} \times e^{i\omega t} \text{ AND } k = \omega/c$$

IN THIS QUASI-STATIC REGIME (COHERENT POLARIZATION OVER THE SCATTERER), WE REDUCE THE COMPLEXITY TO DIPOLE MOMENTS $\vec{p}$ AND $\vec{m}$ WITH ACCOMPANYING DIPOLE RADIATION.

THE RESULTING FAR (RADIATION) FIELDS ARE J. EQN. 9.19 AND 9.36:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r} (\hat{n} \times \vec{p}) \times \hat{n}$$

PLUS

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r} (\hat{n} \times \vec{m})$$

$$\vec{H} = \frac{1}{Z_0} \hat{n} \times \vec{E}$$

PROCEED TO POYNTING-ANALYSIS OF THE INCIDENT AND SCATTERED FIELDS, LEADING TO A CROSS-SECTION.

RECALL: THE TIME-AVERAGE POYNTING VECTOR IS THE INTENSITY, WITH UNITS POWER/AREA.

THE TOTAL SCATTERED POWER IS COMPUTED BY EVALUATING THE POYNTING VECTOR AT r WEIGHTED BY $r^2 d\Omega$: $\vec{S}_{SCATT}$ HAS UNITS $\frac{\text{POWER}}{\text{AREA} \cdot \text{SOLID-ANGLE}}$

HENCE,

$$\frac{d\sigma_{sc}}{d\Omega}(\hat{n},\hat{\varepsilon};\hat{n}_0,\hat{\varepsilon}_0) = \frac{r^2 \frac{1}{2Z_0}|\hat{\varepsilon}^* \cdot \vec{E}|^2}{\frac{1}{2}Z_0|\hat{\varepsilon}_0^* \cdot \vec{E}_0|^2}$$

- COMMENTS: PLANE-WAVE EXPRESSION FOR $|\vec{S}|$: J. EQN. 7.13.
- WE WANT TO PICK OUT A PARTICULAR SET OF POLARIZATIONS $\hat{\varepsilon}_0$ AND $\hat{\varepsilon}$, HENCE THE DOT-PRODUCTS.
- THE COMPLEX-CONJUGATION ALLOWS FOR, e.g., CIRCULAR POLARIZATION. SEE J. EQN. 7.23.

WE CAN INSERT $\vec{E}(\vec{p},\vec{m})$

$$\frac{d\sigma_{sc}}{d\Omega}(\hat{n},\hat{\varepsilon};\hat{n}_0,\hat{\varepsilon}_0) = \frac{1}{(4\pi\varepsilon_0)^2}\frac{1}{E_0^2}k^4 \times \left|\hat{\varepsilon}^* \cdot \vec{p} + (\hat{n}\times\hat{\varepsilon}^*)\cdot\vec{m}\frac{1}{c}\right|^2 \quad \left(\text{J. EQN. } 10.4\right)$$

NOTICE WE RE-DISCOVERED THE $k^4 \sim \omega^4$ DEPENDENCE IN THE LONG-WAVELENGTH LIMIT (RAYLEIGH)

Q: THOUGHT EXPERIMENT. HOW WOULD YOU MAKE A QUADRUPOLE-DOMINANT SCATTERER.

Q: WOULD THE QUADRUPOLE SCATTERER OBEY ω^4?

<u>EXAMPLE</u> SCATTERING FROM A SMALL DIELECTRIC SPHERE.

REACQUAINT-YOURSELF WITH THE STATIC CASE (J. EQN. 4.54)

$$\Phi_{IN}(r) = -\frac{3}{\varepsilon/\varepsilon_0 + 2} E_0$$

$$\Phi_{OUT}(r) = +\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} E_0 \frac{R^3}{r^2} \cos\theta$$

+ APPLIED POTENTIAL

WE ALSO RECALL THE POTENTIAL FROM A DIPOLE

$$\Phi_{DIP} = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2} \quad \text{(J. EQN. 4.12)}$$

HENCE

$$\vec{p} = 4\pi\varepsilon_0 \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} R^3 \vec{E}_0$$

$$\vec{m} = 0$$

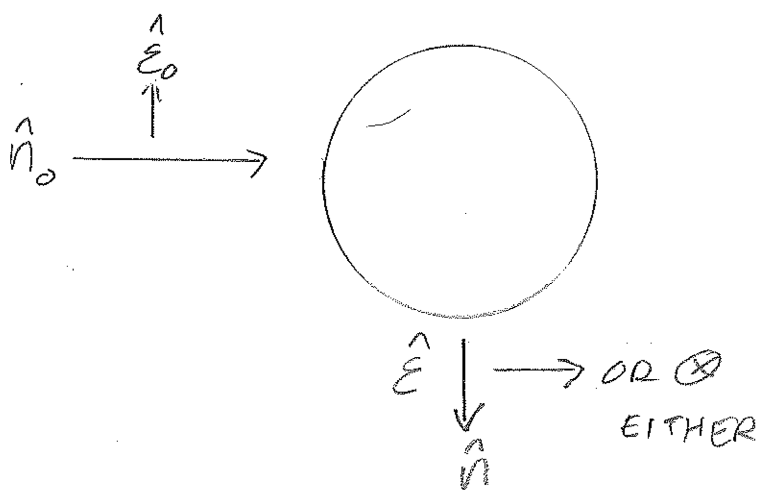
LEADING TO

$$\frac{d\sigma_{sc}}{d\Omega} = k^4 R^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \left| \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 \right|^2$$

NOTICE $|\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0|^2$:

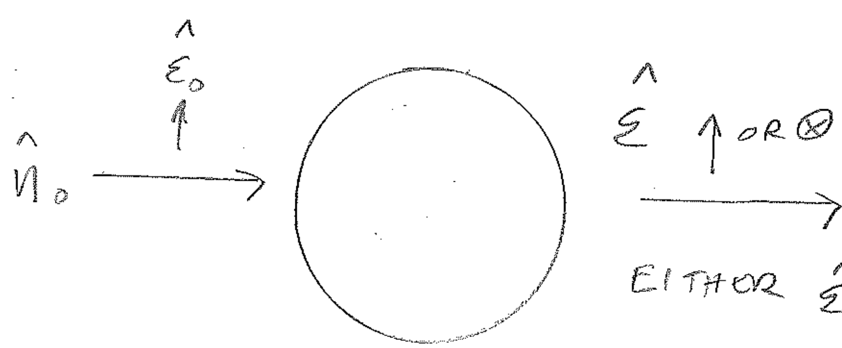
- THE SCATTERED WAVE HAS THE CHARACTER OF DIPOLE RADIATION.

AT $\theta = 0, \pi$:



EITHER: $\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 = 0$

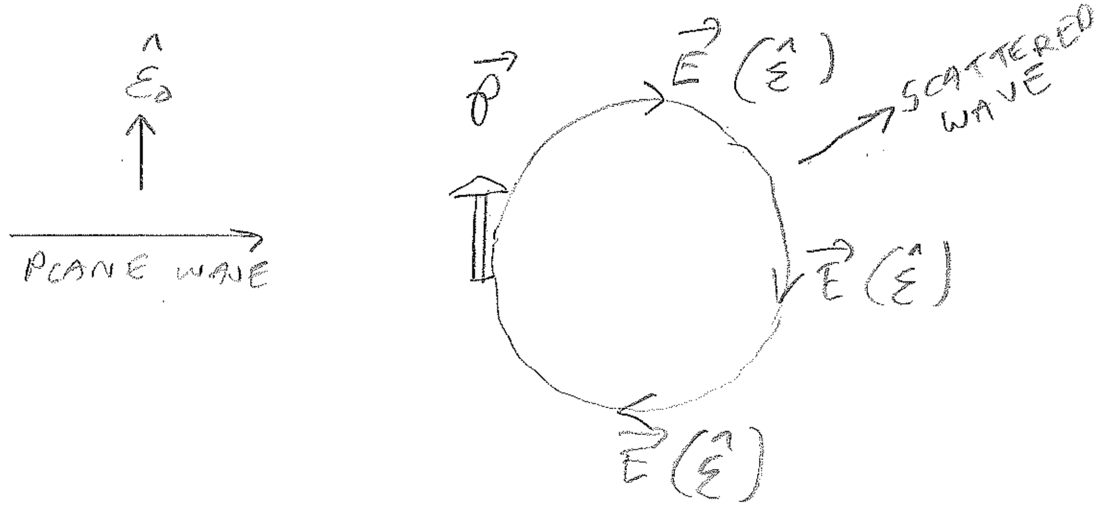
AT $\theta = \pi/2$



EITHOR $\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 = 0$ OR 1

"THE SCATTERED RADIATION IS POLARIZED IN THE PLANE DEFINED BY $\vec{p}$ (THAT IS, $\hat{\varepsilon}_0$) AND $\hat{n}$."

ANOTHER WAY TO VIEW THIS



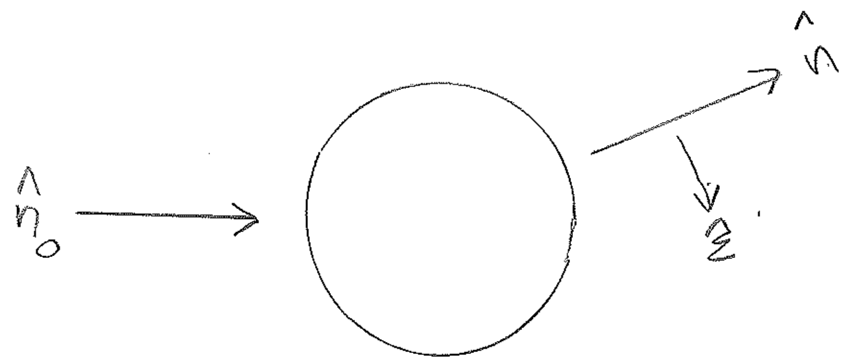
$$E \sim |\vec{\mathcal{E}}_0 \cdot \hat{\mathcal{E}}|$$

MAXIMUM AT EQUATOR
(FOR "CORRECT" POLARIZATION)
MINIMUM AT POLES.

LEADS TO THE FAMILIAR $\sin^2\theta$ CROSS-SECTION.

SOMETIMES THE PROBLEM IS PRESENTED DIFFERENTLY: SUPPOSE THE INCIDENT RADIATION IS UNPOLARIZED. WHAT'S THE CROSS-SECTION FOR A SPECIFIC STATE OF SCATTERED POLARIZATION?

- THE SCATTERING PLANE



$\hat{n}_0$ AND $\hat{n}$ DEFINE A "SCATTERING PLANE".
$\hat{\varepsilon}$ CAN BE "$\perp$" TO THE PLANE OR IN THE PLANE ("$\parallel$").

WITH CORRESPONDING CROSS-SECTIONS

$$\frac{d\sigma_\perp}{d\Omega} \quad \text{AND} \quad \frac{d\sigma_\parallel}{d\Omega}$$ (J. EQN'S 10.7)

(YOU AVERAGE OVER THE TWO INCIDENT POLARIZATION DIRECTIONS.)

THE "AMOUNT OF POLARIZATION" IS

$$\Pi(\theta) = \frac{\frac{d\sigma_\perp}{d\Omega} - \frac{d\sigma_\parallel}{d\Omega}}{\frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega}} \quad \text{(J. EQN. 10.8).}$$

SPECIAL CASE: SCATTERING OFF A SMALL DIELECTRIC SPHERE.

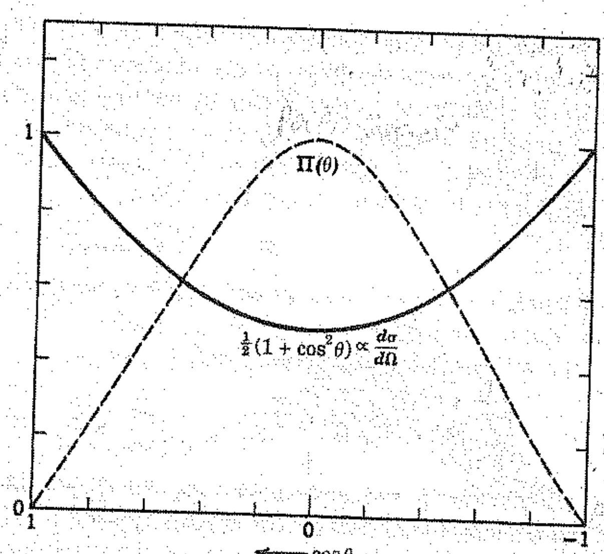


Figure 10.2 Differential scattering cross section (10.10) and the polarization of scattered radiation (10.9) for a small dielectric sphere (dipole approximation).

ANOTHER J.C 10.1 EXAMPLE.

SCATTERING ON A SMALL CONDUCTING SPHERE.

FOR THE TRANSITION OF $\vec{p}$ TO A CONDUCTOR, TREAT THE DIELECTRIC AS INFINITELY POLARIZABLE:

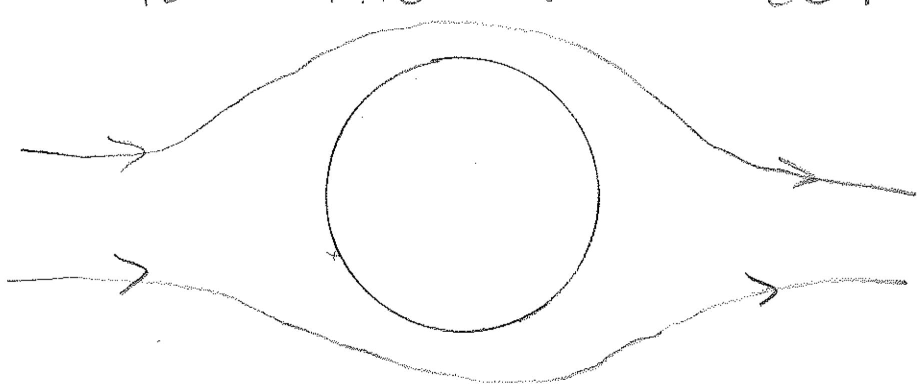
$$\vec{p} = 4\pi\varepsilon_0 \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} R^3 \vec{E}_0 \longrightarrow 4\pi\varepsilon_0 R^3 \vec{E}_0$$

THIS HAS LINES OF $\vec{E}$ NORMAL TO THE SURFACE.

SUPPOSE INSTEAD WE TAKE THE LIMIT OF $\varepsilon \rightarrow 0$. THIS GIVES

$$\vec{p} = -2\pi\varepsilon_0 R^3 \vec{E}_0$$

THIS HAS LINES OF $\vec{E}$ PARALLEL TO THE SURFACE.



THIS IS THE SHAPE OF FIELD LINES OF $\vec{H}$ AROUND A PERMEABLE SPHERE WITH $\mu \rightarrow 0$.

J. § 5.11. (FOR THIS CASE).

WE HAVE $\vec{p}$ FOR $\varepsilon \to 0$,

SO FROM DUALITY WE HAVE

$\vec{m}$ FOR $\mu \to 0$.

THE $\mu \to 0$ LIMIT TAKES US TO A CONDUCTOR

$$\vec{m} = -2\pi R^3 \vec{H}_0$$

NOTICE $\vec{m} \perp \vec{p}$:

THIS COMES FROM $\vec{E}_0 \perp \vec{H}_0$.

OR, YOU COULD START WITH J. EQN. 5.115, THE MAGNETIC DIPOLE MOMENT OF A PERMEABLE SPHERE IN A UNIFORM FIELD:

$$\vec{M} = \frac{3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)\vec{B}_0 \quad \text{(UNIFORM)}.$$

$$\vec{m} = \frac{4}{3}\pi R^3 \frac{3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)\vec{B}_0$$

IN A PERFECT, LOSSLESS, CONDUCTOR,
$\vec{B} = 0$ FROM FARADAY'S LAW.

BUT $\vec{H}$ IS FINITE SINCE $\oint H \cdot d\ell = 0$.

FROM $\vec{B} = \mu \vec{H}$, A PERFECT CONDUCTOR HAS $\mu \to 0$.

HENCE $\vec{m} = -2\pi R^3 \vec{H}_0$.

EVERYTHING FOLLOWS FROM THIS:

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}; \hat{n}_0, \hat{\varepsilon}_0)$$

$$\frac{d\sigma_\perp}{d\Omega}, \quad \frac{d\sigma_\parallel}{d\Omega}, \quad \Pi(\theta).$$

INCLUDING $\vec{\sigma}$! (J. EQN.'S 10.14 - 17).

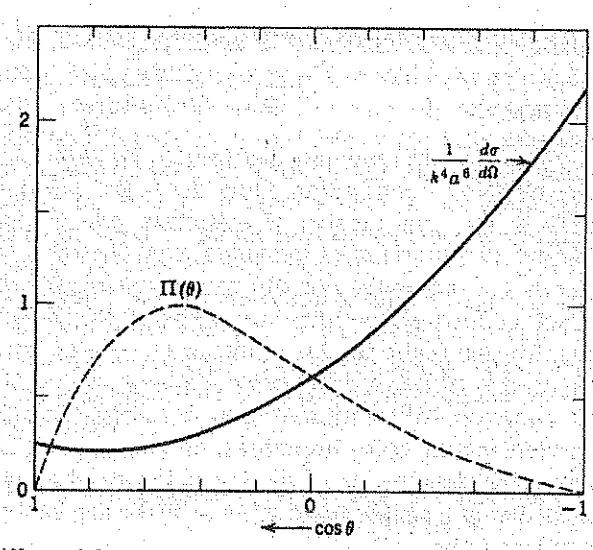


Figure 10.3 Differential scattering cross section (10.16) and polarization of scattered radiation (10.17) for a small perfectly conducting sphere (electric and magnetic dipole approximation).

SCATTERING FROM A VOLUME DISTRIBUTION OF SCATTERERS. J. 10.1 D, J. 10.2C - D.

WITH MULTIPLE SCATTERS, THE COMPLETELY-CORRECT APPROACH IS TO SUPERIMPOSE OUTGOING SCATTERED WAVELETS INCLUDING THEIR APPROPRIATE PHASES.

IF THE POSITIONS OF THE SCATTERS ARE RANDOM, THE PHASE DIFFERENCES ARE AS OFTEN POSITIVE AS NEGATIVE SO AS TO "AVERAGE-AWAY" INTERFERENCE (EXCEPT IN THE FORWARD DIRECTION); THIS LEADS TO A SUM OF INTENSITIES.

NOW CONSIDER A DIFFERENT PROBLEM, SAY BRAGG DIFFRACTION ON A "REGULAR" 3D LATTICE. THERE ARE CERTAIN DIRECTIONS OF MAXIMUM INTENSITY WHERE CONTRIBUTIONS FROM NEIGHBORING SCATTERERS ARE IN PHASE.

NOW THINK OF A 1D DIFFRACTION GRATING. IT DOES NOT PRODUCE A DIFFRACTION PATTERN IF THE GRATING SPACING $\ll \lambda_0$. THE 3D ANALOGY IS: IF $\lambda_0 \gg$ THE 3D LATTICE SPACING, THE CONDITION FOR A DIFFRACTION MAXIMUM CANNOT BE MET (EXCEPT IN THE FORWARD DIRECTION).

NOW, RETURN TO MULTIPLE SCATTERS IN THE LONG-WAVELENGTH LIMIT! EXAMINE A CUBE OF SIDE $\lambda_0/2$ CONTAINING MANY AIR MOLECULES RANDOMLY DISTRIBUTED.

TO AN OBSERVER VIEWING THE SCATTERERS AT NEARLY RIGHT-ANGLES TO THE INCIDENT BEAM, THE RADIATION FROM ONE PARTICULAR CUBE WILL, ON AVERAGE, BE OUT OF PHASE WITH ANOTHER CUBE ALONG THE LINE OF SIGHT. THIS LEADS TO NET ZERO SCATTERING INTO ANY ANGLE (EXCEPT IN THE FORWARD DIRECTION).

BUT THIS IS THE SITUATION FOR ATMOSPHERIC SCATTERING OF LIGHT! SO WE NEED TO RETURN TO THE QUESTION "WHY IS THE SKY BLUE?".

RAYLEIGH SCATTERING, REVISITED. FOR UNPOLARIZED LIGHT AND A SINGLE SCATTERER (J. EQN. 10.10)

$$\frac{d\sigma}{d\Omega} = K^4 R^6 \frac{|\epsilon/\epsilon_0 - 1|}{|\epsilon/\epsilon_0 + 2|} \frac{1}{2}\left(1 + \cos^2\theta\right)$$

RAYLEIGH ASSUMED N AIR MOLECULES LEADS TO $\times N$ SCATTERS IN A DIRECTION WITH NO INTERFERENCE! THIS ASSUMPTION VIOLATES COMMON SENSE.

IT TOOK UNTIL THE EARLY 1900'S FOR THE ANSWER (SMOLUKOWSKI 1908, EINSTEIN 1910) TO RAYLEIGH'S QUESTION FROM 1871.

BRIEFLY: THE NUMBER OF SCATTERERS PER CUBE IS CONSTANT ONLY ON AVERAGE, THE FLUCTUATIONS IN THIS NUMBER ARE RESPONSIBLE FOR SCATTERING.

EXPLAINATION. WITH N_j THE NUMBER OF AIR MOLECULES IN CUBE j, δN_j THE DEVIATION IN NUMBER FROM THE MEAN, AND ϕ_j THE PHASE OF SCATTERED RADIATION; THE SCATTERED INTENSITY IS PROPORTIONAL TO

$$\left| \sum_j e^{i\phi_j} \delta N_j \right|^2$$

$$= \sum_j (\delta N_j)^2 + \sum_{j \neq K} e^{i(\phi_j - \phi_K)} \delta N_j \, \delta N_K .$$

NOTICE $\left| \sum_j e^{i\phi_j} \delta N_j \right|^2$

- DOES NOT INCLUDE $\bar{N}$ (THE MEAN NUMBER IN A CELL... FROM THE INTRODUCTORY DISCUSSION)!
- IS AN INCOHERENT SUM; THE INTERFERENCE TERMS ARE ABSENT SINCE POSITIONS OF SCATTERERS ARE RANDOM.

AN OBSERVER SEES THIS ON AVERAGE; THERE ARE VERY MANY CUBES. FOR DILUTE GASSES, THE NEIGHBORING CUBES DON'T MUCH "TALK" TO EACH OTHER AND THE $\{\delta N_j\}$ ARE INDEPENDENT.

FURTHER

a) RECALL BY DEFINITION (ALMOST)

$\langle \delta N_j \rangle = 0.$

b) FROM ELEMENTARY STATISTICS OF GAUSSIAN RANDOM PROCESSES

$$\langle (\delta N_j)^2 \rangle = N_j$$

BY a), THE SECOND TERM IS ZERO.

BY b), THE TOTAL SCATTERING IS

$$\sum_j (\delta N_j)^2 = \sum N_j = N,$$

WITH N THE TOTAL NUMBER OF AIR MOLECULES. THIS IS WHAT RAYLEIGH FOUND FOR N INCOHERENT SCATTERERS, BUT THE SOURCE IS DENSITY FLUCTUATIONS IN THE ATMOSPHERE.

SEE FIG. 10.4, ATMOSPHERIC TRANSPARENCY VS. ω; A WONDERFUL FIGURE.

J. C. 10.2 D.

COMMENT ON PHASE TRANSITIONS.

NEAR CERTAIN PHASE TRANSITIONS, THE LATENT HEAT $\rightarrow 0$ AND THERE ARE HUGE DENSITY PERTURBATIONS.

SO $\frac{d\sigma}{d\Omega}$ BECOMES HUGE NEAR CERTAIN CRITICAL POINTS.

I'M MOSTLY AWARE OF THIS FOR ^4He AROUND 2.2K (THE NORMAL/SUPERFLUID TRANSITION). THE LIQUID BATH TAKES ON A MILKY-WHITE OPACITY:

CRITICAL OPALESCENCE.