Physics 515, Electrodynamics III
Department of Physics, University of Washington
Spring quarter 2020
May 13, 2020, 11am
On-line lecture

Administrative

1. Short lecture today (Dr's appointment).
2. You should be getting your graded homework back; if not let me know asap.
3. No office hours Wednesday after class at URL https://washington.zoom/us/j/712804010

Lecture

Chapter 12: Lagrangian formalism of electrodynamics.
1. J. C. 12.7: Lagrangian for the electromagnetic field.
2. J. C. 12.8: Proca Lagrangian; photon mass effects.
3. J. C. 12.10: Stress tensor(s) and conservation laws.

SO FAR: LAGRANGIAN FORMULATION OF ELECTRODYNAMICS: J.C. 12.

$$A = \int_{t_1}^{t_2} L \, dt$$

WITH $\frac{\delta A}{\delta x(t)} = 0 \rightarrow \frac{\delta L}{\delta x(t)} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}(t)} = 0.$

EXAMPLE: RELATIVISTIC FREE PARTICLE.

$$L = -m_0 c^2/\gamma.$$

THE $1/\gamma$ CAME FROM COVARIANCE.
THE $m_0 c^2$ CAME FROM IT BEING THE ONLY "ENERGY" SCALAR.
THE "—" COMES FROM A SENSIBLE LOW-ENERGY LIMIT

$$p = \frac{dL}{d\dot{x}} = \frac{d}{d\dot{x}}\left\{-\sqrt{1-\dot{x}^2/c^2}\, m_0 c^2\right\}$$

$$= \gamma m_0 \dot{x} \longrightarrow m\dot{x}.$$

$$E = H = \vec{p}\cdot\dot{\vec{x}} - L$$

$$= \gamma m_0 \dot{\vec{x}}^2 + \frac{1}{\gamma} m_0 c^2$$

$$= \frac{1}{\gamma} m_0 c^2 \left[\frac{\dot{x}^2}{c^2} + \left(1 + \frac{\dot{x}^2}{c^2}\right)\right]$$

$$= \gamma m_0 c^2 \longrightarrow m_0 c^2 + \frac{1}{2} m_0 \dot{x}^2$$

$$\frac{\delta L}{\partial x(t)} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}(t)} = 0;$$

$$-\frac{d}{dt}\frac{\delta L}{\delta \dot{x}(t)} = 0$$

$$-\frac{d}{dt}\frac{\delta}{\delta \dot{x}(t)}\left\{-\sqrt{1-\dot{x}^2/c^2}\, m_0 c^2\right\}$$

$$\frac{d}{dt}\left\{\gamma m_0 c^2\right\} = 0 \rightarrow \gamma = \text{CONSTANT}$$

$$\rightarrow \dot{x} = \text{CONSTANT}.$$

THE VELOCITY IS CONSTANT; THE PARTICLE MOVES IN A STRAIGHT LINE.

NOW INCORPORATE EXTERNAL POTENTIALS

$$A_\mu = (\vec{A}, \Phi).$$

THE COVARIANT INTERACTION IS

$$-eA_\mu dx^\mu \quad \left\{= -e(\Phi - \vec{v}\cdot\vec{A})\,dt\right\}.$$

THE SINGLE PARTICLE ACTION IS

$$A = \int_a^b \left[-m_0 \gamma c\, ds - e A_\mu dx^\mu \right]$$

$$= \int_{t_1}^{t_2} \left[-\frac{1}{\gamma} m_0 c^2 + e\vec{A} \cdot \vec{v} - e\Phi \right] dt$$

THE ASSOCIATED CANONICAL MOMENTUM IS

$$\vec{P} = \frac{\partial L}{\partial \dot{\vec{x}}} = \gamma m_0 \dot{\vec{x}} + e\vec{A}$$

THIS IS THE LAGRANGIAN FORMALISM FOR THE PARTICLE IN AN EXTERNAL POTENTIAL A_μ. WHAT ABOUT THE DYNAMICS OF THE ELECTROMAGNETIC FIELD?

THE ELECTROMAGNETIC FIELD IS CONTINUOUS, WE NEED THE LAGRANGIAN FORMALISM FOR CONTINUOUS SYSTEMS.

INSTEAD OF A DISCRETE SET OF EQUATIONS

$$\frac{\partial L}{\partial q_i(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i(t)} = 0,$$

WE HAVE A CONTINUOUS SET OF EQUATIONS:

$$i \longrightarrow x^\alpha$$

$$q_i \longrightarrow \phi(x) \,;\ \phi \text{ A CONTINUOUS FIELD.}$$

$$\dot{q}_i \longrightarrow \partial^\alpha \phi(x)$$

$$L = \sum_i L_i(q_i, \dot{q}_i)$$

$$\longrightarrow \iiint \mathcal{L}(\phi, \partial^\alpha \phi)\, d^3x .$$

THE EULER-LAGRANGE EQUATIONS

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

BECOME

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x_\beta}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x_\beta}\right)} = 0$$

OR $$\frac{\partial \mathcal{L}}{\partial \phi} - \partial^\beta \frac{\partial \mathcal{L}}{\partial \{\partial^\beta \phi\}} = 0.$$

BACK TO ELECTRODYNAMICS! WE SEEK THE LAGRANGIAN DESCRIBING THE FIELDS.

$$A = \iiiint \mathcal{L}\, d^3x\, dt$$

$$= \frac{1}{c} \iiiint \mathcal{L}\, d^4x$$

HERE $\mathcal{L}$ IS AN INVARIANT. SINCE d^4x IS AN INVARIANT (Q: WHY IS d^4x AN INVARIANT?).

RECALL ELECTROMAGNETIC INVARIANTS

$$F^{\mu\nu} F_{\mu\nu} = \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu}$$

$$= -2(E^2 - B^2)$$

AND $F^{\mu\nu} \tilde{F}_{\mu\nu} = -4\vec{E} \cdot \vec{B}$

WE'D LIKE A SCALAR (NOT PSEUDO-SCALAR), SO AN OBVIOUS CHOICE IS

$$\mathcal{L} \sim F^{\mu\nu} F_{\mu\nu}.$$

WE'VE ALREADY WRITTEN DOWN THE INTERACTION OF POINT SOURCES WITH POTENTIALS:

$$\mathcal{L} \sim e\vec{A}\cdot\vec{v} \text{ AND } e\Phi.$$

IN COVARIANT FORM

$$\mathcal{L} = -eA_\mu dx^\mu$$

IT SEEMS SENSIBLE FOR THE COVARIANT FORM OF THE INTERACTION TO HAVE FORM

$$\mathcal{L} \sim J_\mu A^\mu.$$

WE THEN HAVE THE FIELD LAGRANGIAN

$$\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{c}J_\mu A^\mu$$

(CGS)

WHERE DO CONSTANTS $-\frac{1}{16\pi}$ AND $\frac{1}{c}$ COME FROM.

$-\frac{1}{16\pi}$:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial (\dot{A}_i)} \dot{A}_i - \mathcal{L}$$

SUBLETY: IN THE VARIATION OF THE ACTION, ONLY THE POTENTIALS ARE ALLOWED TO VARY. THE CURRENT IS UNAFFECTED (WHY?).

YOU COULD EVALUATE THE CANONICAL MOMENTUM VIA JACKSON'S DISCUSSION IN EQN. 12.46, OR, YOU COULD IMAGINE FOR NOW SETTING $\Phi = 0$.

THEN $\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \ldots$

IS $\mathcal{L} = -\frac{1}{8\pi}(E^2 - B^2)$

$$= \frac{1}{8\pi}(\dot{A}^2 - B^2),$$

THE CONJUGATE MOMENTUM IS

$$\frac{\partial \mathcal{L}}{\partial (c \frac{\partial}{\partial_0} A_i)} \sim (\dot{A}_i)^2 \sim E^2$$

HENCE

$$\mathcal{H} = \frac{1}{8\pi}\left(E^2 + B^2\right)$$

AND THE CONSTANT $-\frac{1}{16\pi}$

MAKES SENSE.

$\frac{1}{c}$: NOW APPLY THE EULER-LAGRANGE EQUATION TO THE INTERACTION TERM:

$$\underbrace{\frac{\partial \mathcal{L}}{\partial A_\mu}}_{a} - \frac{\partial}{\partial x_\alpha} \underbrace{\frac{\partial \mathcal{L}}{\partial \left\{ \frac{\partial A_\mu}{\partial x_\alpha} \right\}}}_{b} = 0$$

TERM a $\sim J^\mu$

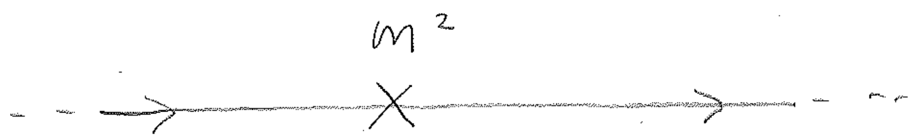
TERM b $\sim \frac{\partial}{\partial x_\alpha} F_\alpha{}^\mu$ (Q: WHY?)

THE EQUATIONS OF MOTION ARE THE INHOMOGENEOUS MAXWELL EQUATIONS

$$\frac{1}{4\pi}\partial^\alpha F_{\alpha\mu} = \frac{1}{c} J_\mu .$$

FAMOUS EXAMPLE: INCORPORATE PHOTON MASS ("PROCA LAGRANGIAN").

HERE IS THE PROCESS:



WE'RE LOOKING AT AN ENERGY, WITH NO INTERACTION, ON TOP OF THE KINETIC ENERGY.

SUCH TERMS HAVE FORM $m^2\phi^2$
WHY IS THIS?

A LANGRANGIAN OF FORM

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

HAS EULER-LAGRANGE EQUATION

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi; \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi,$$

SO THE EULER-LANGRANGIAN LEADS TO THE KLEIN-GORDON EQUATION

$$(\partial^\mu\partial_\mu + m^2)\phi = 0.$$

CAVEAT: JACKSON CHOOSES TO EXPRESS MASS m IN TERMS OF THE INVERSE COMPTON WAVELENGTH μ: (SEE P. 600)

$$\mu = mc/\hbar .$$

THE LAGRANGIAN READS

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{8\pi} \mu^2 A_\mu A^\mu - \frac{1}{c} J_\mu A^\mu ;$$

WE'LL JUSTIFY $\frac{1}{8\pi}$ LATER.

LIKE THE KLEIN-GORDON EXAMPLE, THE EQUATIONS-OF-MOTION (MAXWELL'S EQUATIONS ACQUIRE A MASS TERM

$$\partial^\beta F_{\beta\mu} + \mu^2 A_\mu = \frac{4\pi}{c} J_\mu$$

$$\left\{ \text{N.B.,} \quad \partial^\beta \tilde{F}_{\beta\mu} = 0 \text{ UNCHANGED} \right\}.$$

IT'S INTERESTING THE POTENTIALS A_μ NOW APPEAR IN MAXWELL'S EQUATIONS.

EXERCISE: SHOW

$$\partial^\beta F_{\beta\mu} + \mu^2 A_\mu = \frac{4\pi}{c} J_\mu$$

PLUS CONTINUITY $\partial_\beta J^\beta = 0$

YIELDS THE LORENTZ CONDITION.

EXERCISE: SHOW

$$\partial^\beta F_{\beta\mu} + \mu^2 A_\mu = \frac{4\pi}{c} J_\mu$$

CAN BE WRITTEN

$$\Box A_\mu + \mu^2 A_\mu = \frac{4\pi}{c} J_\mu$$

(JACKSON EQN. 12.93)

(HINT: APPLY CONTINUITY.)

SUPPOSE THE SYSTEM IS STATIC;

$$\Box \equiv -\partial_\alpha \partial^\alpha \longrightarrow -\nabla^2, \quad \text{AND}$$

$$\nabla^2 A_\mu - \mu^2 A_\mu = -\frac{4\pi}{c} J_\mu$$

FURTHER SUPPOSE THE SYSTEM CONSISTS OF A SINGLE POINT CHARGE e AT THE ORIGIN

$$\nabla^2 \Phi(r) - \mu^2 \Phi(r) = -\frac{4\pi}{c} e\, \delta(0),$$

THIS HAS SOLUTION

$$\Phi(r) = \frac{e}{r} e^{-\mu r}$$

A "YUKAWA POTENTIAL".

THE MASS TERM ALTERS THE $1/r$ POTENTIAL (AND THE $1/r^2$ FORCE LAW).

SEARCHING FOR DEVIATIONS FROM THE $1/r^2$ LAW CONSTRAINS μ.
(WE'LL RETURN TO THIS.)

EXERCISE: SHOW THAT THE MASS TERM $\frac{1}{8\pi}\mu^2 A_\mu A^\mu$ LEADS TO A MASSIVE-PARTICLE DISPERSION RELATION.

TAKE $\Box A_\mu + \mu^2 A_\mu = \frac{4\pi}{c} J_\mu$

AND CONSIDER A SOURCE-FREE REGION ($J_\mu = 0$) WITH HARMONIC SPATIAL AND TIME VARIATION:

$$\Box A_\mu + \mu^2 A_\mu = 0$$

$$\longrightarrow (-c^2K^2 + \omega^2)A_\mu + \mu^2 A_\mu = 0,$$

$$\omega^2 = c^2K^2 + \mu^2 c^2$$

(JACKSON EQN 12.95).

THIS IS THE DISPERSION RELATION WITH MASS.

ALSO RECALL IN ELECTROMAGNETISM $E \sim \omega$ AND $p \sim K$, SO THE DISPERSION RELATION CAN ALSO BE WRITTEN

$$E^2 = c^2 p^2 + m^2;$$

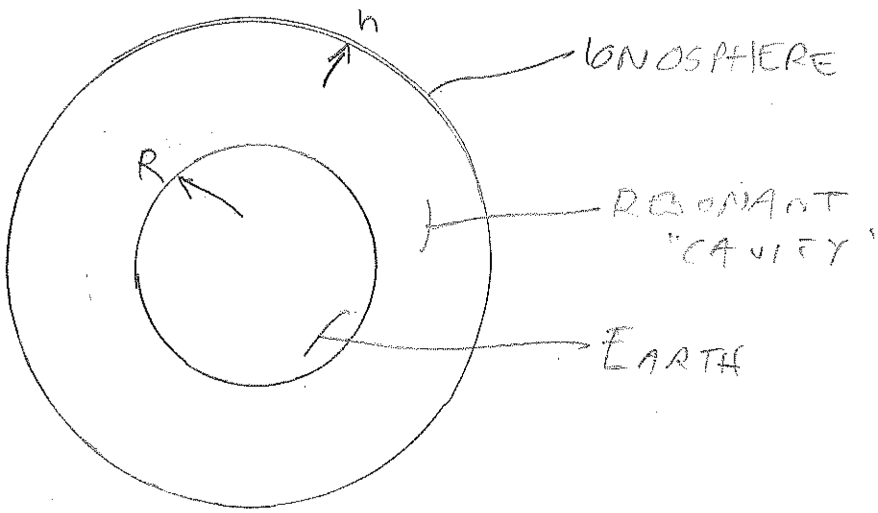
THE ENERGY-MOMENTUM RELATION FOR A PARTICLE WITH MASS.

THE BEST LIMIT ON PHOTON MASS IS THE ASYMPTOTIC DEVIATION OF THE EARTH'S MAGNETIC FIELD FROM $1/r^3$

$$m_\gamma \lesssim 10^{-20}\, m_e$$

(PROBABLY BETTER LIMIT NOW.)

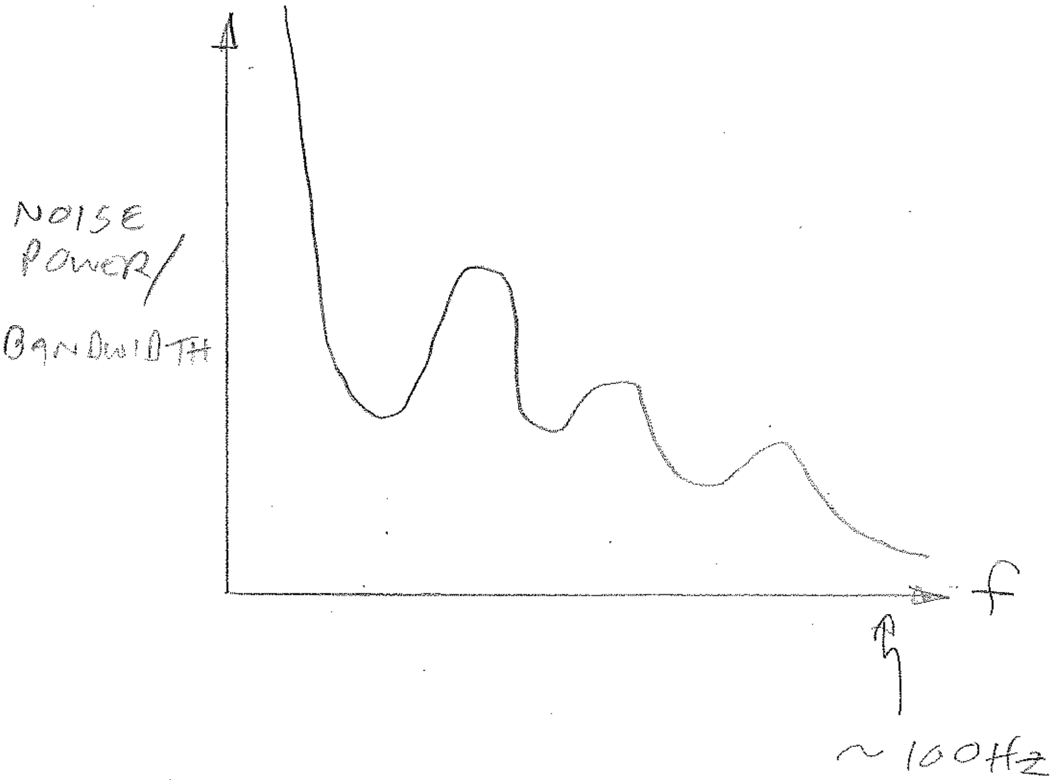
A LIMIT FROM "SCHUMANN RESONANCES" J. E. 8.7.



THIS SITUATION WAS EARLY DEDUCED FROM ANOMALIES IN RADIO COMMUNICATION

THIS IS A RARE INSTANCE OF A VERY-LOW FREQUENCY PURE RESONANT CAVITY.

SCHUMAN RESONANCES AND EXCESS NOISE.



SEE THE COMMENT J. P. 603 (JUST ABOVE J. EQ. 12.97).

THERE IS A MASS-DEPENDENT CHANGE IN FREQUENCY (FROM MASS-DISPERSION-RELATION); IN THE LOWEST MODE

$$\frac{\Delta f}{f} \approx \frac{\mu^2 c^2}{2\,\omega_0^2}\, g \qquad \text{J. EQN 12.97}$$

$$\text{WITH } g \approx 0.44\,\frac{h}{R}$$

THIS IS SENSITIVE TO 8 Hz SHIFT!

$$\longrightarrow m_\gamma \lesssim 10^{-18}\, m_e .$$