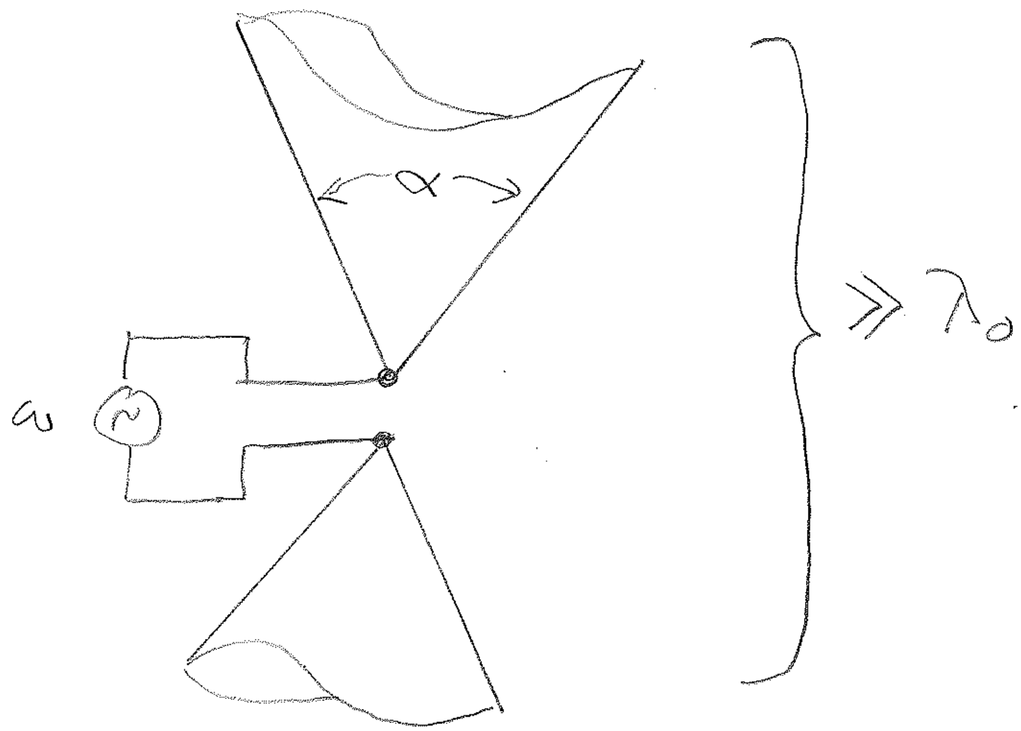
Physics 515, Electrodynamics III
Department of Physics, University of Washington
Spring quarter 2020
April 8, 2020, 11am
On-line lecture

Administrative
1. Homework 1 due Friday. (See submission details on homework assignment.)
2. Office hours today after class at URL: https://washington.zoom/us/j/712804010

Lecture: **Two special topics.**
1. The "bi-conical" antenna/transmission line. A solvable and useful antenna.
2. The spherical resonant cavity. Not done in Jackson (but see problems 98.22-23). Modes are spherical Bessel functions (Jackson eqn. 9.82).

A MORE SOPHISTICATED EXAMPLE.
BICONICAL ANTENNA.
(SEE, e.g., BALANIS, "ANTENNA THEORY", 1ST ED., CHAPT. 8, P. 323)



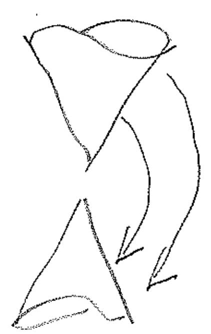
SOMETIMES ALSO CALLED A "TAPERED TRANSMISSION LINE".

APPLY A VOLTAGE, AS SHOWN. THIS CAUSES CURRENT TO FLOW DOWN THE CONES AND A VOLTAGE DEVELOPS BETWEEN THE CONES. HENCE, IT'S AN ANTENNA AS WELL AS A TEM WAVEGUIDE.

NOTICE THIS ANTENNA IS SCALE-INVARIANT; ALL PROPERTIES ARE THEREFORE SPECIFIED BY THE ANGLE α.

FOR EXAMPLE: WHAT'S THE FREQUENCY RESPONSE OF THIS ANTENNA... WHAT'S THE RESONANT FREQUENCY? FROM SCALE INVARIANCE, IT'S BROAD-BAND... NO "MAGIC" RESONANT FREQUENCY.

WHAT DOES THE ANTENNA $\vec{E}$-FIELD LOOK LIKE?

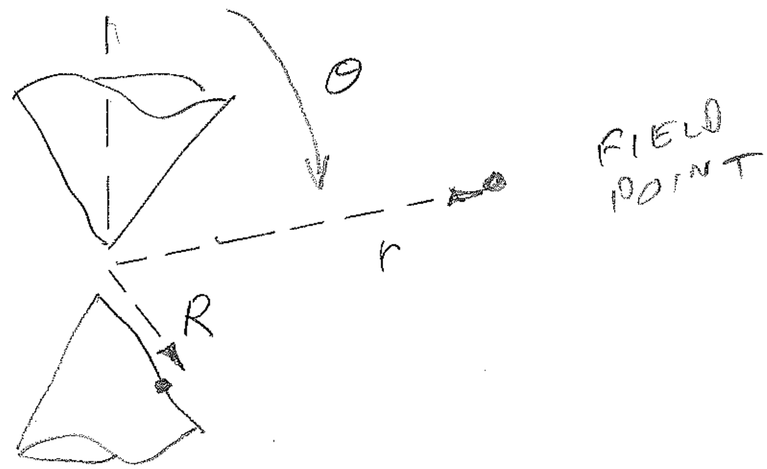


$\vec{E} \sim \hat{\theta}$ FROM SCALE-INVARIANCE AND AZIMUTHAL SYMMETRY.

(SEE, e.g., "MIS-ALIGNED" CAPACITOR-PLATE PROBLEM.

)

COORDINATE SYSTEM



LET'S ASK A QUESTION ABOUT THIS TRANSMISSION LINE / ANTENNA.

WHAT'S THE CHARACTERISTIC IMPEDANCE OF THE TRANSMISSION LINE $V_0/I_0 = \underline{Z}$? (ALSO THE RADIATION RESISTANCE, FOR THIS SYSTEM...)

FIND $\vec{H}$ (I). WHAT'S THE "CHARACTER" OF $\vec{H}$? FROM THE CHARACTER OF $\vec{E}$ WE EXPECT $\vec{H} \sim \hat{\phi}$.

WE COULD ESTABLISH THIS WITH RIGOR VIA FARADAY'S LAW

$$\vec{\nabla} \times \vec{E} = -\frac{d}{dt}\mu_0 \vec{H} = -i\omega\mu_0 \vec{H}.$$

RECALL THE CHARACTER OF $\vec{E}$:

$$\vec{E} \sim \hat{\theta}; \quad \frac{d\vec{E}}{d\phi} = 0.$$

$$\vec{\nabla} \times \vec{E} \sim \hat{\phi} \quad \text{ONLY.}$$

FIND $\vec{H}$ (II).

APPLY AMPÉRE'S LAW

$$\vec{\nabla} \times \mu_0 \vec{H} = \frac{1}{c^2}\frac{d}{dt}\vec{E} = \frac{1}{c^2} i\omega \vec{E}$$

BECAUSE $\vec{E} \sim \hat{\theta}$, $\vec{\nabla} \times \vec{H} \sim \hat{\theta}$.

HENCE

$$[\vec{\nabla} \times \vec{H}]_r = 0 = \frac{1}{r\sin\theta}\frac{d}{d\theta}(\sin\theta H_\phi)$$

$$\text{OR } \frac{d}{d\theta}(r\sin\theta H_\phi) = 0.$$

A WAY TO ENSURE THIS IS FOR $H_\phi = f(r)/\sin\theta$.

FOR H_ϕ TO BE AN OUTGOING TRAVELLING WAVE

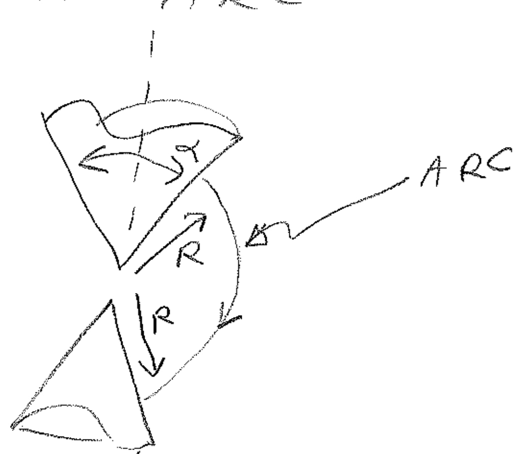
$$H_\phi = A_0 \frac{1}{\sin\theta}\frac{e^{i\frac{\omega}{c}r}}{r}$$

LIKELY MORE DETAILS ON HOMEWORK.

FIND $\vec{E}$.

FOR THIS TEM WAVEGUIDE, H_ϕ AND E_Θ ARE RELATED BY Z_0; $Z_0 = E_\Theta / H_\phi$.

FIND Z; FIND THE VOLTAGE DIFFERENCE ALONG THE CIRCULAR ARC



$$V(R) = \int_{\alpha/2}^{\pi-\alpha/2} \vec{E} \cdot d\vec{l} \quad \text{ALONG THE ARC.}$$

$$= \int_{\alpha/2}^{\pi-\alpha/2} \hat{\Theta} E_\Theta \cdot \hat{\Theta} \underbrace{r\, d\Theta}_{d\vec{l}} = \int_{\alpha/2}^{\pi-\alpha/2} E_\Theta R\, d\Theta$$

$$= \int_{\alpha/2}^{\pi-\alpha/2} Z_0 H_\phi R\, d\Theta$$

$$= Z_0 A_0 e^{-i\frac{\omega}{c}R} \int_{\alpha/2}^{\pi-\alpha/2} \frac{1}{\sin\Theta}\, d\Theta$$

$$V(R) = Z_0 A_0 e^{-i\frac{\omega}{c}R} \; 2 \ln\left\{\cot\frac{\alpha}{4}\right\}$$

$$I(R) = \int_0^{2\pi} K(R)\, R \sin\theta \, d\phi .$$

RECALL THE SIMPLE RELATION BETWEEN $\vec{H}_{\parallel}$ AND SURFACE CURRENTS (YOU MAY HAVE USED THIS FOR HW I; JACKSON EQN 8.14):

$$|\vec{K}_{EFF}| = |\vec{H}_{\parallel}|. \qquad \text{SO}$$

$$I(R) = \int_0^{2\pi} H_\phi \; R \sin\theta \, d\phi$$

$$= A_0 e^{-i\frac{\omega}{c}R} \cdot 2\pi .$$

THE CHARACTERISTIC IMPEDANCE OF THE TRANSMISSION LINE (OR, IN THIS CASE, ALSO THE ANTENNA RADIATION RESISTANCE) IS

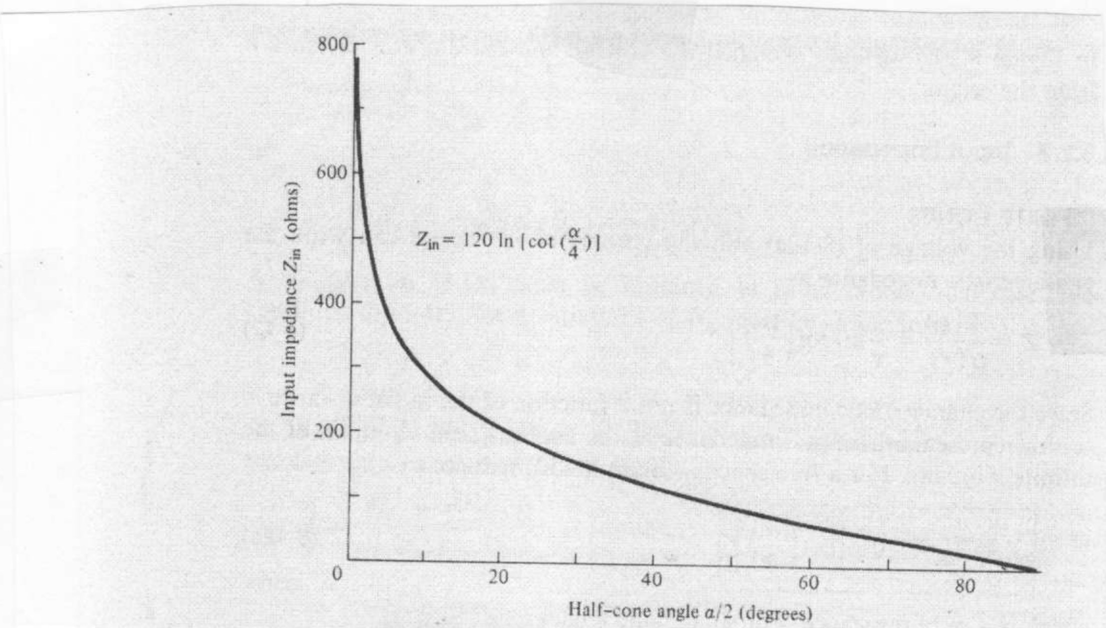
IS ...

$$Z = V(R)/I(R) = Z_0 \frac{1}{\pi} \ln\left\{\cot\frac{\alpha}{4}\right\}$$

- Z IS INDEPENDENT OF R. IT'S A PERFECTLY FINE TRANSMISSION LINE.

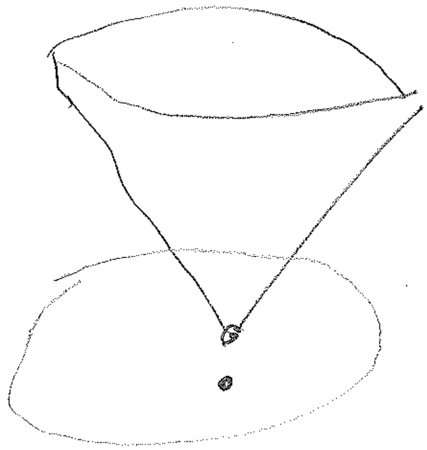
- Z IS (PURELY REAL FOR ANY ω AND) INDEPENDENT OF ω, AS EXPECTED FROM SCALE INVARIANCE.

FOR $\alpha \ll 1$, $\cot\frac{\alpha}{4} = \dfrac{1}{\tan\frac{\alpha}{4}} \approx \dfrac{4}{\alpha}$



BALANIS, "ANTENNA THEORY" 2ND ed., P. 328.

VARIANTS OF THE BI-CONICAL ANTENNA.



"DISCONE" ANTENNA

MAYBE YOU'LL SEE THIS ON HOMEWORK

UHF TV
"BOWTIE" ANTENNA.

SPHERICAL RESONATORS.

NOT FORMED BY PUTTING END-CAPS ON A SECTION OR WAVEGUIDE.

NOT COVERED BY JACKSON. (EXCEPT UNDER "APERTURES" PROBLEMS).

WE'LL ONLY CONSIDER THE MODES WITH AZIMUTHAL SYMMETRY. (THE ONLY PLACE I'VE SEEN THE GENERAL CASE IS STRATTON "ELECTROMAGNETIC THEORY".)

WHAT DO TM & TE MEAN IN THIS CASE?

- THE SPHERE IS THE TRANSVERSE SURFACE.
- "TM" HAS FIELD COMPONENTS

$$E_r, E_\theta; \quad H_\phi$$

- "TE" IS THE DUAL

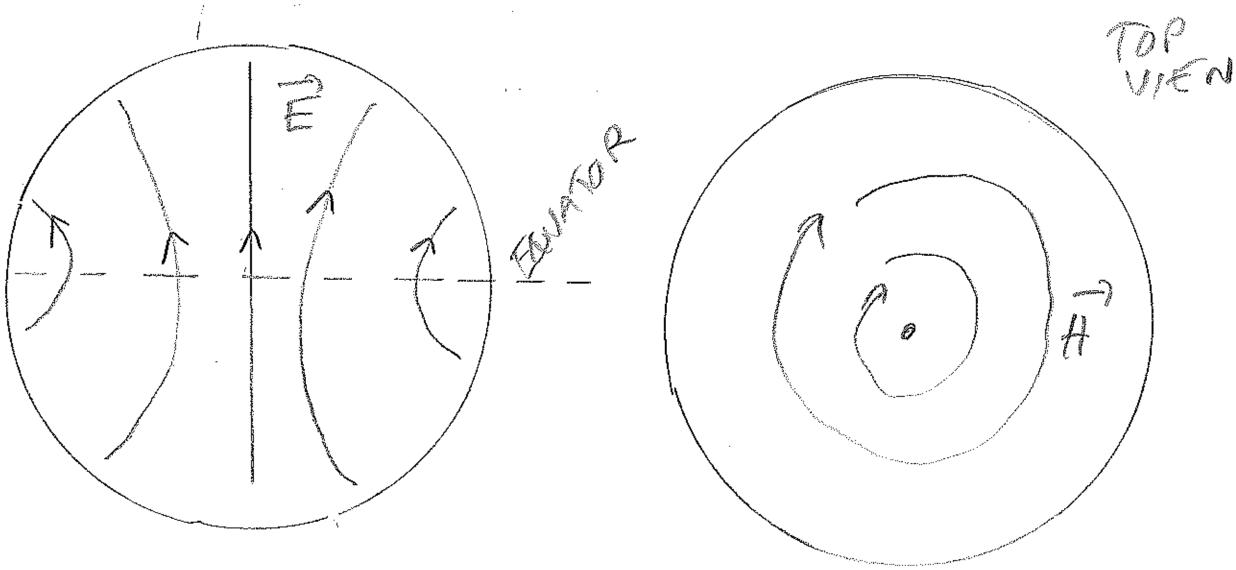
$$H_r, H_\theta; \quad E_\phi.$$

AZIMUTHAL SYMMETRY $\longrightarrow \frac{\partial}{\partial \phi} = 0.$

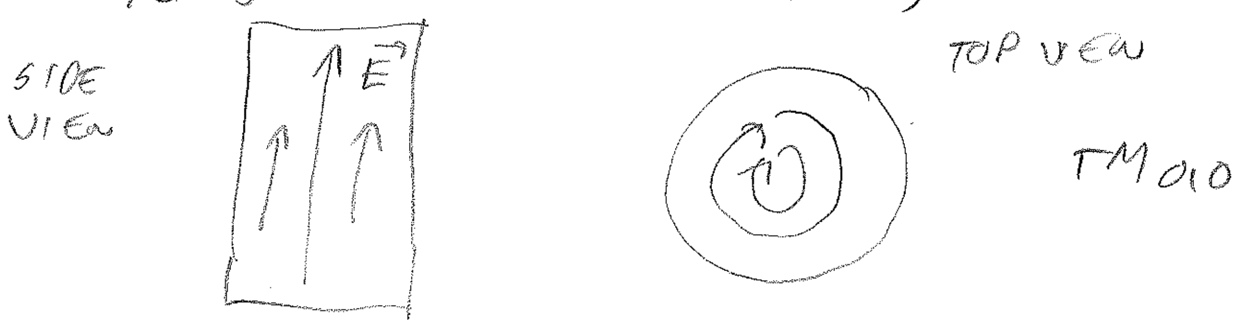
WE CAN'T DIRECTLY USE THE TRANSVERSE WAVE EQUATION. SO, GO BACK TO MAXWELL'S EQUATIONS; EVALUATE $\vec{\nabla} \times \vec{E}$ AND $\vec{\nabla} \times \vec{H}$.

"TM" SOLUTIONS: $E_r, E_\theta; H_\phi$.

e.g. TM_{010}



CALLED TM BECAUSE IT KIND OF LOOKS LIKE (FOR A CYLINDRICAL RESONATOR)



FARADAY'S & AMPÈRE'S LAW READ

$\vec{\nabla} \times \vec{E} = -i\omega\mu_0 \vec{H}$:

$$\left[\nabla \times E\right]_\phi = \frac{1}{r}\frac{d}{dr}(rE_\theta) - \frac{1}{r}\frac{d}{d\theta}E_r = -i\omega\mu_0 H_\phi \quad (1)$$

$\vec{\nabla} \times \vec{H} = i\omega\varepsilon_0 \vec{E}$:

$$= i\omega\varepsilon_0\left\{E_r \hat{r} + E_\theta \hat{\theta}\right\}$$

$$\left[\vec{\nabla} \times \vec{H}_\phi\right]_r = \frac{1}{r\sin\theta}\frac{d}{d\theta}(H_\phi \sin\theta) = i\omega\varepsilon_0 E_r \quad (2)$$

$$\left[\vec{\nabla} \times \vec{H}_\phi\right]_\theta = -\frac{1}{r}\frac{d}{dr}(rH_\phi) = i\omega\varepsilon_0 E_\theta \quad (3)$$

ACT ON (2) BY $\frac{d}{d\theta}$, SUBSTITUTE INTO (1).

ACT ON (3) $r\times$; ACT ON (3) BY $\frac{d}{dr}$, SUBSTITUTE INTO (1).

NOTICE THE ONLY FIELD QUANTITY NOW IN (1) IS H_ϕ. ALSO NOTICE IT'S A 2ND ORDER DIFFERENTIAL EQUATION.

1 BECOMES

$$\frac{d^2}{dr^2}(rH_\phi) + \frac{1}{r^2}\frac{d}{d\theta}\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}(rH_\phi \sin\theta)\right\} + \omega^2/c^2 \, rH_\phi = 0.$$

ALAS, WE NEED TO SOLVE THIS. FORTUNATELY, WE'VE SEEN IT BEFORE. IT'S THE WAVE EQUATION IN SPHERICAL COORDINATES (ALMOST).

SEPARATION-OF-VARIABLES

$$rH_\phi = \mathbb{R}(r)\Theta(\theta)$$

YIELDS $\Theta(\cos\theta) = P_n^1(\cos\theta)$

THE ASSOCIATED LEGENDRE POLYNOMIALS.

(THIS IS NOT OBVIOUS. SEE, e.g., RAMO, WHINNERY & VAN DUZER, "FIELDS & WAVES IN COMMUNICATION ELECTRONICS," 2nd ed., PP. 498–502.)

THEN REDEFINE $\mathbb{R}(r) = \frac{1}{\sqrt{r}} \mathbb{R}(r)$ IN THE $\mathbb{R}(r)$ SEPARATED EQUATION. THIS IS BESSEL'S EQUATION, BUT WITH $\frac{1}{2}$-INTEGER SOLUTIONS. (SEE JACKSON 9.82.)

$$\mathbb{R}\left(\tfrac{\omega}{c} r\right) = A_n \underbrace{J_{n+\frac{1}{2}}\left(\tfrac{\omega}{c} r\right)}_{\substack{\text{BESSEL FUNCTION}\\ \text{OF THE 1ST KIND,}\\ \frac{1}{2}\text{-INTEGER ORDER}}} + B_n \underbrace{N_{n+\frac{1}{2}}\left(\tfrac{\omega}{c} r\right)}_{\substack{\text{BESSEL FUNCTION}\\ \text{OF THE 2ND KIND,}\\ \frac{1}{2}\text{-INTEGER ORDER}}}$$

SOMETIMES WRITTEN

$$\mathbb{R}\left(\tfrac{\omega}{c} r\right) = A_n \underbrace{j_n\left(\tfrac{\omega}{c} r\right) + B_n\, n_n\left(\tfrac{\omega}{c} r\right)}_{\substack{\text{SPHERICAL BESSEL FUNCTIONS}\\ \text{OF THE 1ST \& 2ND KIND}}}$$

$j_n(x)$ IS REGULAR FOR $x \to 0$;
$n_n(x)$ IS IRREGULAR FOR $x \to 0$.

NOW WE HAVE TO "UNPACK" ALL OF THIS:

$$\mathbb{R}(r) = \frac{1}{\sqrt{r}} \mathbb{R}(r)$$

$$r H_\phi = \mathbb{R}(r) \Theta(\theta).$$

$$H_\phi(r,\theta) = A_n \frac{1}{\sqrt{r}} P_n^1(\cos\theta)$$

$$\times \left\{ \begin{array}{c} J_{n+\frac{1}{2}}\left(\frac{\omega}{c} r\right) \\ \text{AND/OR} \\ N_{n+\frac{1}{2}}\left(\frac{\omega}{c} r\right) \end{array} \right\}.$$

AND E_r AND E_θ THEN COME FROM AMPÉRE'S LAW.

<u>EXAMPLE</u> TM_{010} MODE

TM
010 ← θ
r φ

WE CAN USE THE EXPLICIT FORM OF $J_{1+\frac{1}{2}}$ (J. EQN. 9.87),

$$H_\phi = A \frac{1}{\sqrt{\frac{\pi}{2\omega/c}}} \frac{\sin\theta}{\frac{\omega}{c} r} \left\{ \frac{\sin\frac{\omega}{c} r}{\frac{\omega}{c} r} - \cos\frac{\omega}{c} r \right\}$$

E_θ COMES FROM AMPÉRE'S LAW

$$E_\theta \sim \frac{\left(\frac{\omega}{c} r\right)^2 - 1}{\frac{\omega}{c} r} \sin \frac{\omega}{c} r + \cos \frac{\omega}{c} r$$

WITH BOUNDARY CONDITION

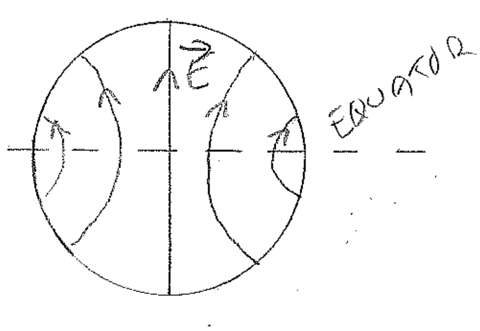
$$E_\theta \Big|_{r=R} = 0 \quad (\text{NO TANGENTIAL } \vec{E})$$

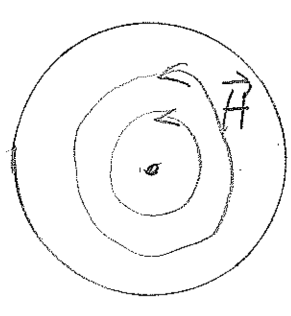
$$0 = \frac{\left(\frac{\omega}{c}\right)^2 - 1}{\frac{\omega}{c} R} \sin \frac{\omega}{c} R + \cos \frac{\omega}{c} R$$

$$\text{OR} \quad \frac{\frac{\omega}{c} R}{1 - \left(\frac{\omega}{c} R\right)^2} = \tan \frac{\omega}{c} R .$$

THIS IS A TRANSCENDENTIAL EQUATION WITH $\frac{\omega}{c} R \simeq 2.3$.

SO, WE' FOUND THE RESONANT FREQUENCY WITH THE ADVERTISED FIELDS





TOP VIEW

FINDING Q, IN PRINCIPLE:

$$Q = \omega_0 \frac{\text{STORED ENERGY}}{\text{POWER LOST PER CYCLE}}$$

STORED ENERGY W AT RESONANCE

$$W = 2 \times \iiint \frac{\mu_0}{2} |\vec{H}_\phi|^2 \, dv$$

POWER LOST P IN WALLS

$$P = \oiint \frac{1}{2} R_s |\vec{H}_\phi|^2 \; R^2 d\Omega$$

(SEE JACKSON EQN. 8.15 WITH COMMENT JUST BELOW, "... $1/\sigma\delta$ PLAYS THE ROLE OF A SURFACE RESISTANCE ...")

THESE GIVE THE NOTABLE RESULT

$Q = Z_0/R_s$. THE GEOMETRY FACTOR IN THIS CASE IS $\underline{1}$ (OPTIMAL; SEE J. EQN. 8.96.)

(SEE JACKSON PROBLEMS 7.22-3.)