Physics 515, Electrodynamics III
Department of Physics, University of Washington
Spring quarter 2020
May 6, 2020, 11am
On-line lecture

Administrative

1. No homework this week.
2. You should be getting your graded homework back; if not let me know asap.
3. Office hours Wednesday after class at URL https://washington.zoom/us/j/712804010

Lecture

Special lecture: Liénard–Wiechert potentials and fields for a charge at constant velocity.
Chapter 11: Covariant form of electrodynamics.
Chapter 14: Radiation from an accelerated particle.
1. J. C. 14.1: Liénard–Wiechert potentials I.
2. J. C. 14.3: Liénard–Wiechert fields and radiation I.
3. J. C. 11.9: Electromagnetic field-strength tensor and Maxwell's equations in covariant form.

J.C. 11.9

COVARIANT FORMULATION OF ELECTRODYNAMICS.

SUMMARY:

WAVE EQUATION

$$\Box A^{\mu} = -4\pi J^{\mu} \quad \text{(CGS)}$$

$$\Box \equiv -\partial_{\mu}\partial^{\mu}$$

$$A^{\mu} = (c\vec{A}, \Phi)$$

$$J^{\mu} = (\rho\vec{\beta}, \rho)\,; \quad \rho = \gamma\rho_0$$

LORENTZ CONDITION

$$\partial_{\mu}A^{\mu} = 0$$

GAUGE TRANSFORMATIONS

$$A'^{\mu} = A^{\mu} + \partial^{\mu}c\lambda$$

CURRENT CONSERVATION

$$\partial_{\mu}J^{\mu} = 0.$$

$$\left\{ \longrightarrow \iiint J^4 d^3x \text{ AN INVARIANT} \right\}$$

LIÉNARD-WIECHERT POTENTIALS I.

POTENTIALS (& FIELDS) OF A POINT CHARGE IN ARBITRARY MOTION.

JACKSON STUDIES THIS IN Ch 14. WE'LL START EARLIER.

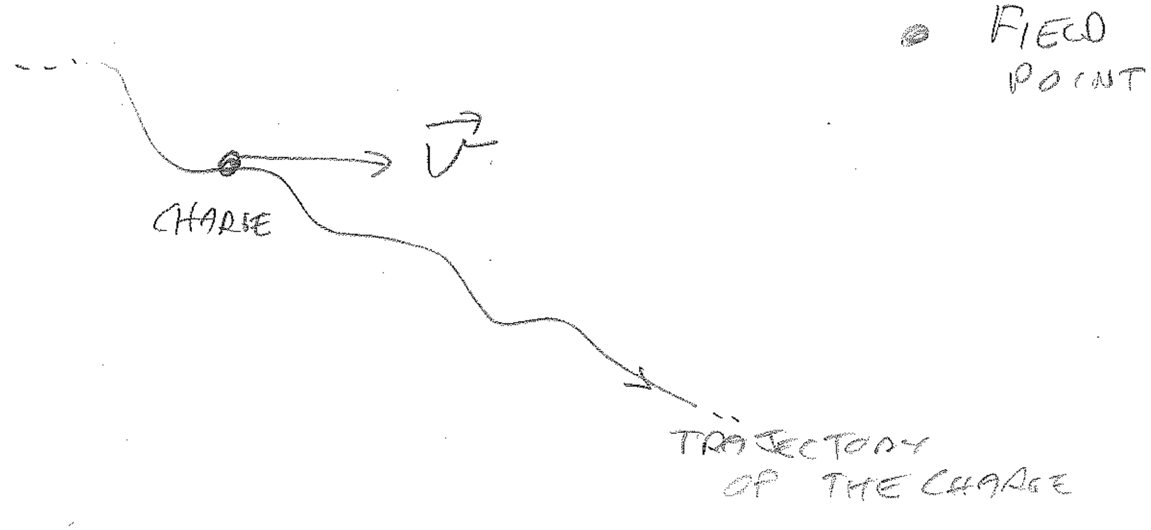
IN PRINCIPLE, WE'VE ALREADY GOT THIS THROUGH THE RETARDED POTENTIALS

$$\Phi(\vec{r}) = \iiint \frac{[\rho(\vec{r}';t)]_{RET}}{|\vec{r}-\vec{r}'|} dv$$

$$\vec{A}(\vec{r}) = \iiint \frac{[\vec{J}(\vec{r}';t)]_{RET}}{|\vec{r}-\vec{r}'|} dv .$$

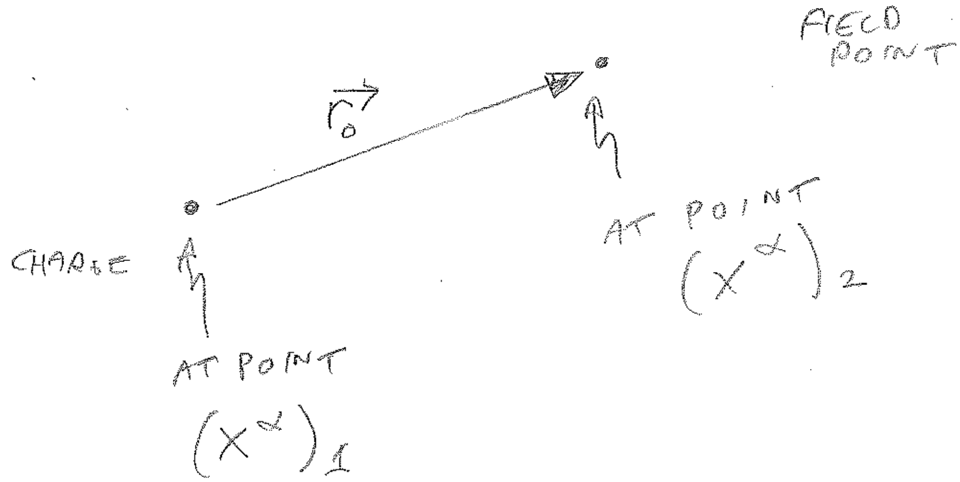
- THESE ARE NOT IN COVARIANT FORM.
- IT'S NOT OBVIOUS HOW THEY APPLY TO A SINGLE POINT CHARGE.

HERE'S A TYPICAL GEOMETRY ....



NOTICE THE CHARGE AND OBSERVER (THE FIELD POINT) ARE IN RELATIVE MOTION. WE'LL FIND SOLUTIONS TO THE RETARDED POTENTIALS IN COVARIANT FORM. WE'LL USE OUR USUAL METHOD OF FINDING A SOLUTIONS IN A "SPECIAL" FRAME, THEN GENERALIZING TO AN ARBITRARY FRAME.

FIRST: FIND SOLUTIONS IN THE "PROPER" FRAME OF THE CHARGE



IN THE FRAME WHERE THE CHARGE IS AT REST, THERE'S A SIMPLE ELECTROSTATIC COULOMB POTENTIAL A_0^μ;

$$A_0^\mu = \left(0, \frac{1}{4\pi\epsilon_0}\frac{e}{r_0}\right) \quad \text{(MKS)}$$

$$= \left(0, \frac{e}{r_0}\right) \quad \text{(CGS)}.$$

WE TAKE CARE IN EVALUATING r_0; r_0 IS THE "PROPER" EUCLIDIAN DISTANCE FROM THE CHARGE TO THE OBSERVER.

NOW, WE'LL FIND THE "4-DISTANCE" FROM SOURCE TO OBSERVER, KEEPING IN MIND THE SIGNAL PROPAGATES AT THE SPEED OF LIGHT AND IS RETARDED.

THE 4-VECTOR (DISPLACEMENT VECTOR) R^{μ} FROM CHARGE TO OBSERVER IS

$$R^{\mu} = (x^{\mu})_2 - (x^{\mu})_0$$

THE DISPLACEMENT IS "LIGHT LIKE";

$$R^{\mu} R_{\mu} = 0.$$

WITH $R^{\mu} = (\vec{r}, ct)$, THE "LIGHT LIKE" CONDITION MEANS

$$R^{\mu} = (\vec{r}, r).$$

SECONDLY, WE GENERALIZE TO AN ARBITRARY FRAME. THAT IS, WE GENERALIZE $A_0^{\mu} = (0, \frac{e}{r_0})$.

WE FOLLOW THE PATH OF GUESSING A COVARIANT FORM THAT REDUCES TO A_0^{μ} IN THE "PROPER" FRAME (THAT IS, FOR $\vec{v} = 0$).

RECALL IN THE "PROPER" FRAME THE 4-VELOCITY REDUCES TO

$$V_0^{\mu} = (\vec{0}, 1).$$

NOW, EVALUATE THE LORENTZ-INVARIANT $R^{\mu} V_{\mu}$, FIRST IN THE "PROPER" FRAME WHERE $R_0^{\mu} = (\vec{r}_0, r_0)$.

IN THE "PROPER" FRAME, $R^{\mu} V_{\mu}$ IS EASILY EVALUATED:

$$R_0^{\mu} \underset{\sim}{V}_{\mu} = r_0$$

AND IN GENERAL

$$R^{\mu} V_{\mu} = r_0.$$

THIS SUGGESTS WE GENERALIZE

$$A_0^{\mu} = \left(\vec{0}, \frac{e}{r_0}\right) \quad \text{TO}$$

$$A^{\mu} = e \frac{V^{\mu}}{R^{\alpha} V_{\alpha}}.$$

EXERCISE: SHOW THAT $A^{\mu} \longrightarrow A_0^{\mu}$ FOR $V \to 0$.

FORMALLY, WE'RE DONE.

BUT LET'S WRITE A^μ IN A MORE FAMILIAR (EUCLIDIAN) FORM.

RECALL $V^\mu = \left(\gamma \frac{\vec{v}}{c}, \gamma\right)$, SO

$$V^\mu R_\mu = -\vec{R} \cdot \gamma \frac{\vec{v}}{c} + \gamma r, \quad \text{so}$$

$$A^\mu = e \frac{V^\mu}{V^\alpha R_\alpha} \quad \text{BECOMES}$$

$$A^\mu = e\left(\frac{1}{s}\frac{\vec{v}}{c}, \frac{1}{s}\right)$$

$$\text{WITH } s = R - \vec{R} \cdot \frac{\vec{v}}{c}$$

"s" IS CALLED THE LIÉNARD-WIECHERT DENOMINATOR

$\vec{v}$ IS THE VELOCITY OF THE OBSERVER RELATIVE TO THE FRAME IN WHICH THE CHARGE WAS AT REST AT THE TIME OF EMISSION OF RADIATION.

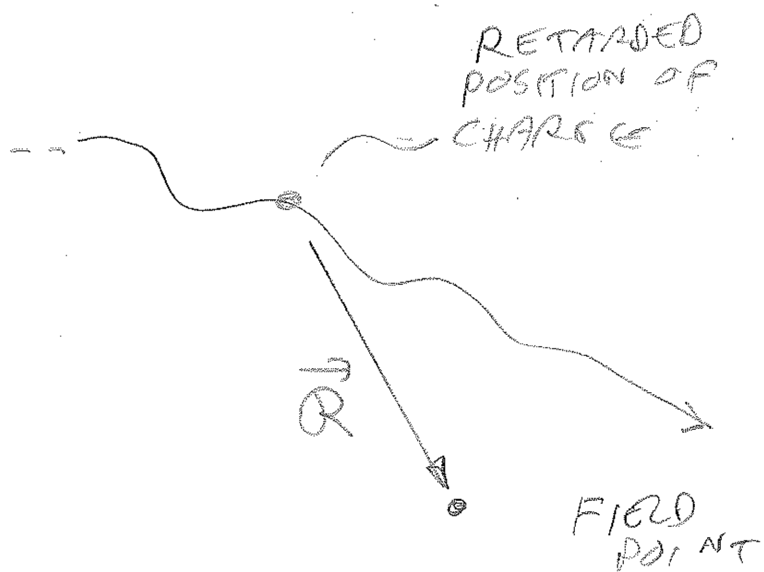
SEE, e.g., JACKSON EQNS. 14.6, .8.

$\vec{R}$ ...

WITH GEOMETRY

$$A^{\mu} = e\left(\frac{1}{s}\frac{\vec{v}}{c}, \frac{1}{s}\right)$$

$$s = R - \vec{R}\cdot\frac{\vec{v}}{c}$$



LIÉNARD - WIECHERT FIELDS.

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A};$$

$$\vec{B} = \vec{\nabla} \times \vec{A},$$

WITH $\Phi = e\frac{1}{s}$, $\vec{A} = e\frac{1}{s}\frac{\vec{v}}{c}$.

THE DIFFERENTIATION IS TRICKY;

$$s = \vec{r} - \vec{r}\cdot\frac{\vec{v}}{c}$$

NEEDS TO BE EVALUATED AT THE RETARDED VALUES OF $\vec{r}$ AND $\vec{v}$.

SEE, e.g., JACKSON EQNS 14.13-14

⋮

LET'S JUST SAY THE FORMS FOR $\vec{E}$ & $\vec{B}$ ARE COMPLICATED.

BUT NOTICE THE FIRST TERM IN EQN 14.14 CONTAINS $\vec{\beta}$, THE SECOND CONTAINS $\dot{\vec{\beta}}$.

THE TERM CONTAINING $\vec{\beta}$ IS THE "VELOCITY FIELD" AND FALLS AS $1/r^2$. THE TERM CONTAINING $\dot{\vec{\beta}}$ IS THE "ACCELERATON FIELD" OR "RADIATION FIELD" AND FALLS AS $1/r$.

WE'LL COME BACK TO THIS LATER.

WE HAVE 3 WAYS OF FINDING $\vec{E}$ AND $\vec{B}$:

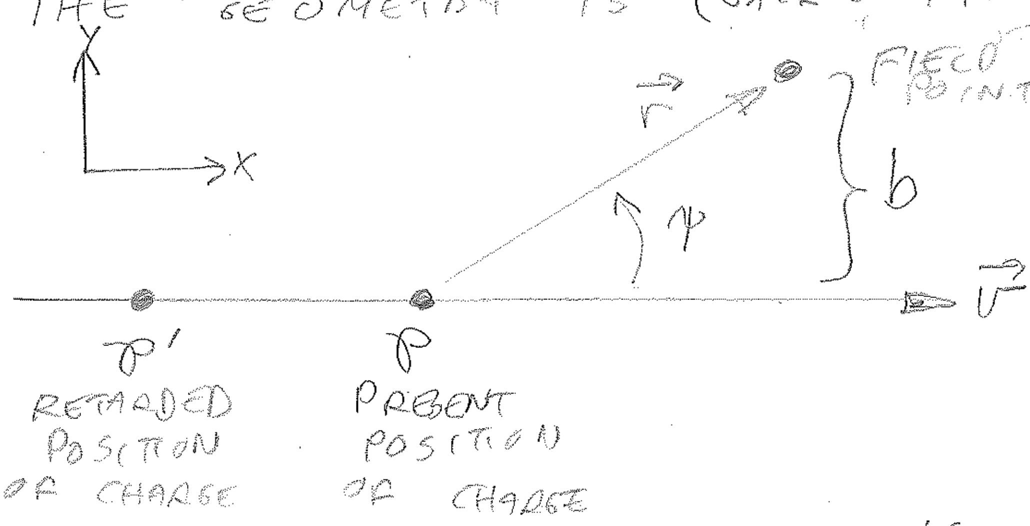
1. DIRECT DIFFERENTIATION OF THE LIENARD-WIECHERT POTENTIALS;
2. JEFIMENKO'S EQUATION;
3. VIA THE "FIELD TENSOR" (AS JACKSON § 14.).

ONE OF THE CLASSIC PROBLEMS INVOLVING THE LIÉNARD - WIECHERT POTENTIALS IS TO STUDY A CHARGE IN UNIFORM MOTION.

THIS CAN CERTAINLY BE STUDIED USING THE LIÉNARD- WIECHERT POTENTIALS (e.g., GRIFFITHS TEXT AND PROBLEM WALK YOU THROUGH THIS).

BUT IT'S EASIER TO STEAL A RESULT FROM THE NEXT LECTURE: THE LORENTZ TRANSFORMATION OF A STATIC COULOMB FIELD.

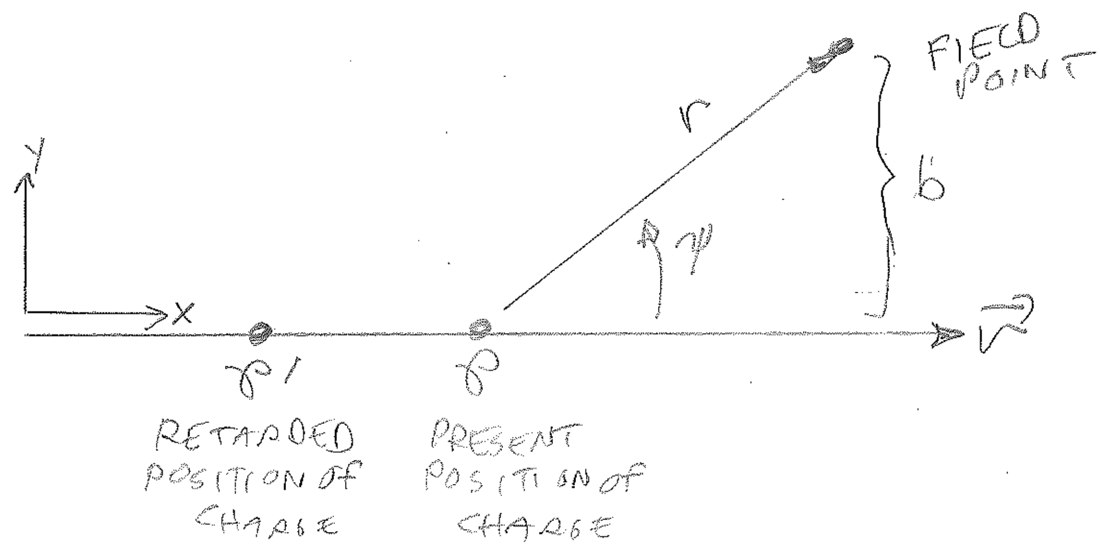
THE GEOMETRY IS (JACKSON FIG 14.2)



AND JACKSON EQN 11.152 IS

$E_y = \gamma E_y'$

THE GEOMETRY IS (JACKSON FIG. 14.2);



LET E_y BE THE FIELD IN THIS FRAME.

LET E_y' BE THE FIELD IN THE PROPER FRAME.

FROM JACKSON EQN. 11.148

(DONT WORRY, WE'LL GET TO THIS)!

$E_y = \gamma E_y'$ (WHY IS THIS NOT SURPRISING?).

IN THE STATIC FRAME

$$E_y' = \frac{e}{r^2}\frac{b}{r}$$

$$= \frac{eb}{(b^2+\gamma^2 v^2 t^2)^{3/2}},$$

So

$$E_y = \frac{\gamma e b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

(JACKSON 14.17a.)

THERE'S A JACKSON PROBLEM TO SHOW THIS CAN BE WRITTEN IN THE FORM

$$\vec{E} = \frac{e}{r^2}\,\frac{1}{\gamma^2}\,\frac{1}{(1-\beta^2\sin^2\psi)^{3/2}}\,\hat{r},$$

(NEED $\vec{E}_x$, $\vec{E}_y$, $\vec{E}_z$,

$$r^2 = v^2t^2 + b^2$$

$$b = r\sin\psi).$$

- $\vec{r}$ POINTS FROM THE PRESENT POSITION OF THE SOURCE TO THE FIELD POINT. (YOU MAY FIND THIS SURPRISING.)
- ψ IS THE ANGLE BETWEEN $\vec{r}$ AND $\vec{v}$.

IN THE FORWARD DIRECTION,

$$\vec{E} = \frac{e}{r^2}\,\frac{1}{\gamma^2}\,\hat{x}, \quad \text{REDUCED BY } \frac{1}{\gamma^2}$$

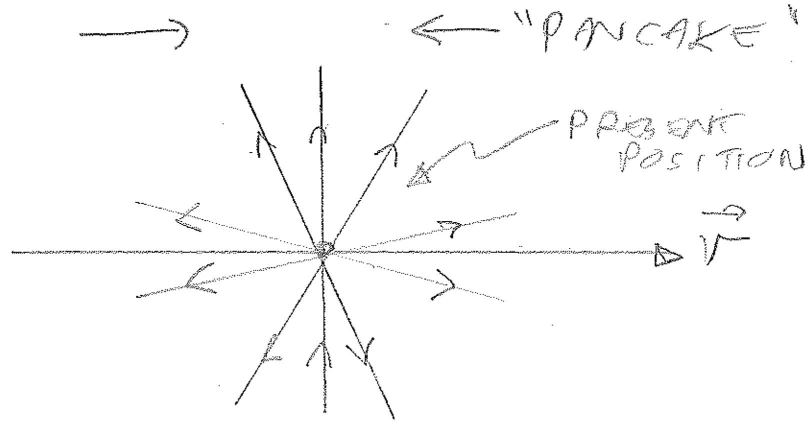
IN THE TRANSVERSE DIRECTION

$$\vec{E} = \frac{e}{r^2}\,\frac{1}{\gamma^2}\,\gamma^3\,\hat{y} = \frac{e}{r^2}\,\gamma\,\hat{x},$$

ENHANCED BY γ,

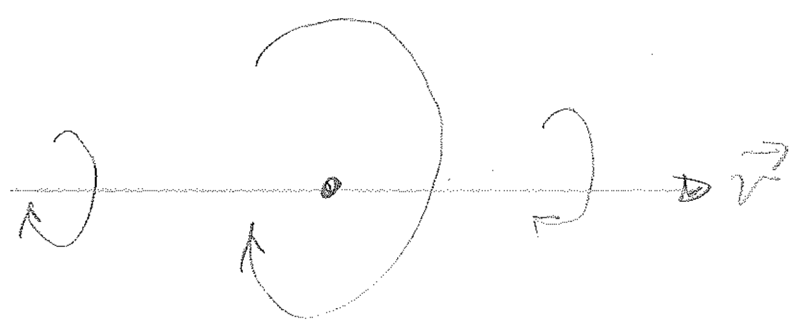
THE FIELDS HAVE CHARACTER

$\vec{E}$:



$\vec{B}$: $\vec{B} = \left[\hat{n} \times \vec{E}\right]_{RET}$

(JACKSON EQN 14.13.)



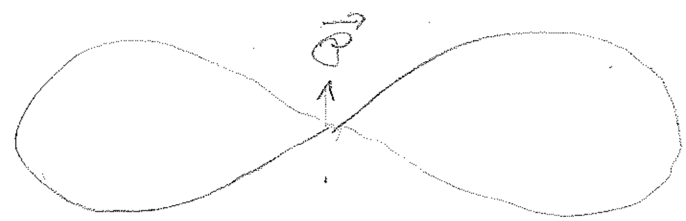
FINALLY, BY DIRECT (VERY TEDIOUS) INTEGRATION

$$\oiint \vec{E} \times \vec{B} \cdot d\vec{a} = 0;$$

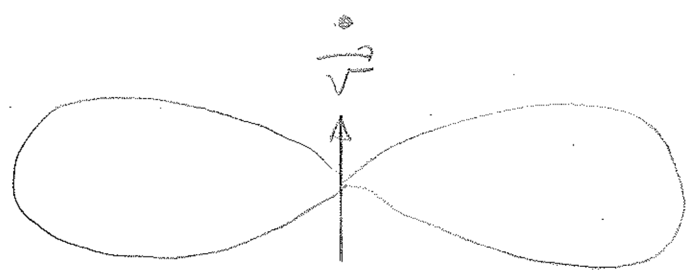
NO RADIATION.
(AS EXPECTED).

CHARGE IN ACCELERATED LINEAR MOTION, (J. Ch. 14.3). A PRELUDE.

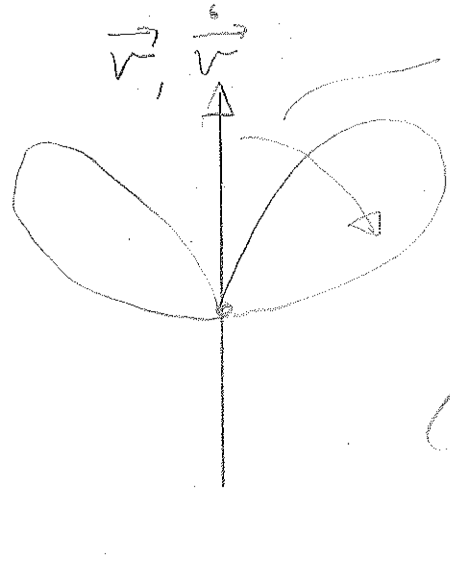
WE'VE SEEN HOW A DIPOLE RADIATES:



THIS IS SENSIBLE FOR NON-RELATIVISTIC LINEAR ACCELERATION: (WHY?)



BUT THINGS LOOK DIFFERENT FOR RELATIVISTIC VELOCITIES



SEE, e.g., (JACKSON FIG 14.4).

ELECTROMAGNETIC FIELD TENSOR $F^{\mu\nu}$

JACKSON C. 11.9.

$\vec{E}$ AND $\vec{B}$ SEPARATELY ARE NOT SPATIAL PARTS OF 4-VECTORS! IT HAPPENS THAT UNDER LORENTZ TRANSFORMATIONS A PURE COULOMB $\vec{E}$ FIELD ACQUIRES A $\vec{B}$-FIELD.

(E.G., THE CHARGE IN UNIFORM MOTION.)

SINCE $\vec{E}$ AND $\vec{B}$ AREN'T COMPONENTS OF A RANK I TENSOR (4-VECTOR), WE SEE IF THEY ARE EMBEDDED IN A RANK 2 TENSOR! $F^{\mu\nu}$, THE FIELD-STRENGTH (OR FIELD-) TENSOR.

AN ANTI-SYMMETRIC RANK 2 TENSOR HAS 6 DEGREES OF FREEDOM (WHY?).

WE NEED TO RESPECT $\vec{B}$ BEING A (EUCLEDIAN) PSEUDO-VECTOR, AND $\vec{E}$ IS A POLAR-VECTOR.

AND WE HAVE $A^{\mu} = (\vec{A}, \Phi)$.

FIELDS ARE RELATED TO DERIVATIVES OF POTENTIALS. FURTHER, $\vec{E}$ CONTAINS DERIVATIVES OF Φ AND $\vec{A}$: A SIMPLE TENSOR LIKE $\partial^\mu A^\nu$ WOULDN'T GIVE THIS TO US.

SO, WE DEFINE

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

THIS GIVES, FOR EXAMPLE,

$$F^{13} = \partial^1 A^3 - \partial^3 A^1$$
$$= \frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x = \left[\nabla \times A\right]_y = B_y.$$
$$F^{14} = \partial^1 A^4 - \partial^4 A^1 = \frac{\partial}{\partial x} \Phi - \frac{\partial}{\partial t} A_x \quad -E_x.$$
$$= \left[\vec{\nabla} \Phi - \frac{\partial}{\partial t} \vec{A}\right] = E_x$$

WITH THIS (NOW A TRIAL) FORM OF $F^{\mu\nu}$, WE'LL SHOW THIS FORMALISM LEADS TO MAXWELL'S EQUATIONS.

CAVEAT: THERE ARE CONVENTIONS GALORE; $\mu\nu$ OR $\nu\mu$? SIGNS? I RECOMMEND YOU CHOOSE A NOTATION AND STICK WITH IT.

$F^{\mu\nu}$, EXPLICITLY:

$$F_{\mu\nu} = \begin{array}{c} \xrightarrow{\nu} \\ \downarrow \mu \begin{pmatrix} 0 & -B_z & B_y & -E_x \\ B_z & 0 & -B_x & -E_y \\ -B_y & B_x & 0 & -E_z \\ E_x & E_y & E_z & 0 \end{pmatrix} \end{array}$$

OR FLIP $\mu\nu$ FOR JACKSON EQN. 11.137.

EXERCISE: SHOW THAT

$$F^{\mu\nu} = F_{\alpha\beta}\, g_\alpha^{\ \mu}\, g_\beta^{\ \nu}$$

$$= \left\{ F_{\mu\nu} \text{ WITH } \vec{E} \to -\vec{E} \right\}$$

JACKSON EQN. 11.138

{THERE'S A CLEVER WAY TO SHOW THIS USING THE PROPERTIES OF CONTRACTED POLAR AND AXIAL VECTORS.}

THE "DUAL" OF $F^{\mu\nu}$. THIS IS CLOSELY RELATED TO OUR CONCEPT OF DUALITY:

RECALL THE LEVI-CIVITA TENSOR

$\varepsilon^{\alpha\beta\mu\nu} = +1$ FOR $\alpha\beta\mu\nu = 1,2,3,4$ AND EVEN PERMUTATIONS, e.g., 2,1,4,3.

$= -1$ FOR ODD PERMUTATIONS.

$= 0$ FOR ANY TWO INDICIES ARE THE SAME.

THE DUAL OF $F^{\mu\nu}$ IS $\tilde{F}^{\mu\nu}$ OR $\mathcal{F}^{\mu\nu}$.

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

(THE $\frac{1}{2}$ IS A CONVENTION).

(JACKSON EQN. 11.140).

SYMBOLICALLY:

$\tilde{F}^{\mu\nu} = \{ F^{\mu\nu}$ WITH $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$, THE DUALITY TRANSFORMATION, COMPARE JACKSONS EQN.S 11.137 AND 11.140 $\}$.

YOU SHOULD NOTE $\varepsilon^{\alpha\beta\mu\nu}$ IS A PSEUDO-TENSOR, WE'LL GET BACK TO THIS. (JACKSON (C6.10c).)

WE HAVE COVARIANT POTENTIALS. NOW WE HAVE THE COVARIANT FIELD-STRENGTH TENSORS (AND ITS DUAL). WE CAN PROCEED TO MAXWELL'S EQUATIONS IN COVARIANT FORM.

CONSIDER $\partial_\mu F^{\mu\nu} = 4\pi J^\nu$ (C65).

$\nu = 4$: $\partial_\mu F^{\mu 4} = 4\pi J^4$

$$\frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z + 0 = 4\pi \rho.$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho,$$

$\nu = 1$: $\partial_\mu F^{\mu 1} = 4\pi J^1$

$$0 + \frac{\partial}{\partial y} B_z - \frac{\partial}{\partial z} B_y - \frac{\partial}{\partial t} E_x = \frac{4\pi}{c} J_x$$

$$[\vec{\nabla} \times \vec{B}]_x = \frac{4\pi}{c} J_x + \frac{1}{c}\frac{\partial}{\partial t} E_x$$

EXERCISE: SHOW $\partial_\mu F^{\mu\nu} = J^\nu$ CONTAINS CURRENT CONSERVATION $\partial_\mu J^\mu = 0$.

HINT: USE ANTI-SYMMETRIC PROPERTIES OF $F^{\mu\nu}$.

FOR EXAMPLE, YOU CAN SHOW FOR EACH TERM LIKE $\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^3} F^{13}$ IN $\partial_\mu J^\mu$, THERE'S ALSO A TERM $\frac{\partial}{\partial x^3}\frac{\partial}{\partial x^1} F^{31}$. THEN APPLY ANTI-SYMMETRY OF $F^{\mu\nu}$.

SO, $\partial_\mu F^{\mu\nu} = 4\pi J^\nu$ CONTAINS

$$\vec{\nabla} \cdot \vec{E} = -4\pi\rho$$

AND $$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial}{\partial t} \vec{E}.$$

NOTICE THESE ARE THE TWO MAXWELL EQUATIONS CONTAINING SOURCES.

NOTICE $\vec{\nabla} \cdot \vec{E}$ IS A POLAR SCALAR AND $\vec{\nabla} \times \vec{B}$ IS A POLAR VECTOR.

THIS IS EXPECTED FOR SEVERAL REASONS, INCLUDING NOTING A^μ IS A POLAR VECTOR AS IS ∂_μ.

BUT $\vec{\nabla} \cdot \vec{B}$ AND $\vec{\nabla} \times \vec{E}$ ARE PSEUDO-SCALARS AND PSEUDO-VECTORS! THEY CAN'T COME OUT OF THINGS LIKE $\partial_\mu F^{\mu\nu}$. WE NEED ANOTHER (COVARIANT) EQUATION.