Physics 515, Electrodynamics III
Department of Physics, University of Washington
Spring quarter 2020
June 3, 2020, 11am
On-line lecture

Administrative

1. You should be getting your graded homework back; if not let me know asap.
2. Final exam posted this Friday, due this Monday
3. Office hours Wednesday after class 12:30 at URL https://washington.zoom/us/j/712804010

Lecture

Chapter 16: Radiation damping, radiation reaction.
1. J. C. 16.1-2. Radiation reaction from conservation of energy. What is mass?
2. J. C. 16.3. Abraham-Lorentz equation.
3. “Runaway” solutions and energy non-conservation.
4. Integral-differential equation with an ad-hoc assumption for the solution to the Abraham-Lorentz equation and non-causality.

THE CONCEPT OF "REACTION FORCE".

IF, e.g., AN ELECTRON RADIATES BECAUSE OF AN ACCELERATION DUE TO AN EXTERNAL FORCE APPLIED BY AN EXTERNAL AGENT, THE EXTERNAL AGENT MUST SUPPLY BOTH THE ENERGY AND MOMENTUM REQUIRED REQUIRED BY THE CHANGE IN FIELDS. SINCE THE FIELDS CARRY MOMENTUM, THIS IMPLIES THERE'S A "REACTION FORCE", A RESULT OF THE RADIATION FIELD ACTING ON THE ELECTRON ITSELF.

THIS IS A VERY TRICKY ASPECT OF CLASSICAL ELECTRODYNAMICS: DO ACCELERATED CHARGES ACT ON THEMSELVES?

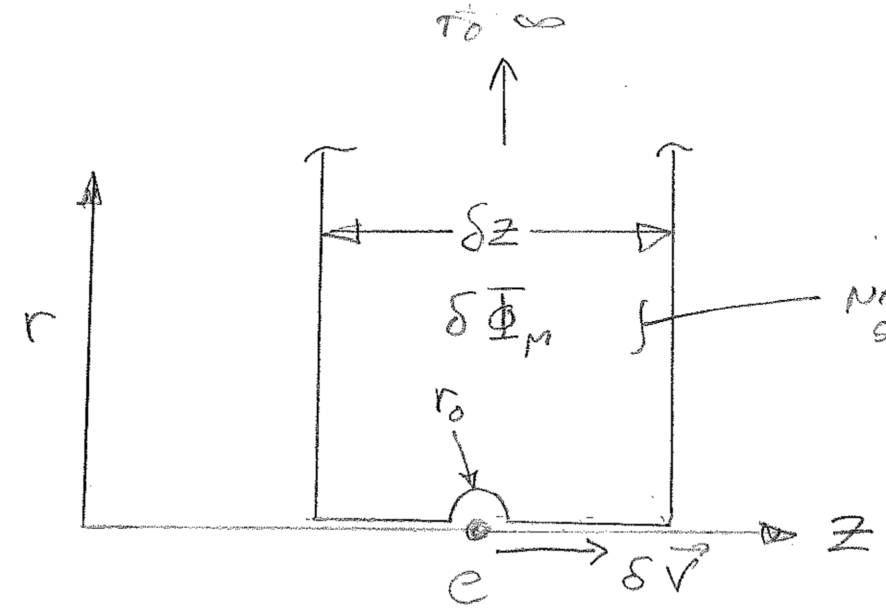
EXAMPLE: A CHARGE e MOVING WITH UNIFORM NON-RELATIVISTIC VELOCITY $\vec{v}$ (DUE TO PAULI). ... BUT THEN...

AT SOME POINT IN THE PART IN THE PARTICLE'S TRAJECTORY, ITS VELOCITY CHANGES BY $\delta\vec{v}$ (PRESUMABLY DUE TO SOME EXTERNAL AGENT).

THE VECTOR POTENTIAL, TOO, CHANGES BY $\delta\vec{A} = \frac{1}{c}\frac{e\,\delta\vec{v}}{r}$ (CGS).

NOW EVALUATE THE MAGNETIC FLUX PASSING THROUGH A NARROW STRIP FROM THE CHARGE TO ∞;

THIS HAS GEOMETRY



THE CHANGE IN THE MAGNETIC FLUX $\delta\Phi_M$ IS

$$\delta\Phi_M = \oint \delta\vec{A}\cdot d\vec{l} \quad \text{AROUND THE STRIP.}$$

(WE'LL COME TO r_0 SHORTLY... IT SPEAKS TO A "STRUCTURE" OF THE CHARGE e.)

FROM LENZ' LAW, $\delta\Phi_M$ INDUCES AN ELECTRIC FIELD AT THE POSITION OF THE CHARGE SO AS TO OPPOSE $\delta\vec{v}$.

WE CAN THINK OF THIS AS A REACTION FORCE:

$$\vec{F} = e\vec{E} = -e\frac{1}{c}\frac{d\vec{A}}{dt}$$

$$= -e\frac{1}{c}\frac{d}{dt}\frac{1}{c}\frac{e}{r_0}\delta\vec{v}$$

r_0 IS SOME MINIMUM VALUE OF r AT WHICH $\delta\vec{A} = \frac{1}{c} e \frac{\delta\vec{v}}{r}$ BREAKS DOWN OWING TO THE STRUCTURE OF e, EVEN IF e IS AN ELECTRON. THE SMALLER r_0, THE LARGER THE REACTION FORCE.

- THERE SEEMS TO BE AN "EFFECTIVE" ELECTROMAGNETIC MASS.

WRITING $\vec{F} = -M_{EFF} \frac{d\vec{v}}{dt}$,

$$M_{EFF} = \frac{e^2}{r_0 c^2} \quad (cgs)$$

- THIS "EFFECTIVE" MASS WAS OBTAINED BY LOOKING AT THE MAGNETIC "INDUCTION" FIELD PRODUCING AN ELECTRIC FIELD THAT ACTS ON A CHARGE. WE COULD ARRIVE AT THIS ANOTHER WAY.

THE ELECTROMAGNETIC ENERGY

$$U = \frac{1}{8\pi} \iiint E^2 \, dV$$

... APPLIED TO A SPHERICAL CHARGE DISTRIBUTION WITH TOTAL CHARGE e AND RADIUS r_0 IS

$$U \approx \frac{e^2}{r_0} \qquad (CGS).$$

HENCE $U = mc^2$ YIELDS

$$m_{EFF} = \frac{e^2}{r_0 c^2} \qquad (CGS)$$

- IT SEEMS ACCELERATED CHARGES ACT ON THEMSELVES.
- UNFORTUNATELY, THERE'S r_0 - DEPENDENCE. WE'LL COME BACK TO THIS ... POINCARÉ'S ARGUMENT.

BACK TO THE REACTION FORCE.

WE RECALL NON-RELATIVISTIC ACCELERATION ON A CHARGE LEADS TO LARMOR'S FORMULA

$$-\frac{dU}{dt} = \frac{2}{3}\frac{e^2}{c^3}\dot{\vec{v}}^2.$$

ENERGY CONSERVATION REQUIRES THE RADIATED POWER IS BALANCED BY THE REACTION FORCE

$$\vec{F}\cdot\vec{v} + \frac{2}{3}\frac{e^2}{c^3}\dot{\vec{v}}^2 = 0.$$

WE CAN'T SOLVE THIS FOR $\vec{F}$ WHICH IS INSTANTANEOUSLY CORRECT FOR ALL TIMES SINCE IT'S POSSIBLE TO ADJUST $\vec{v}$ AND $\dot{\vec{v}}$ INDEPENDENTLY.

WE'LL MAKE DO WITH A SOLUTION REPRESENTING AN AVERAGE OVER A TIME LONG COMPARED TO CHANGES IN $\vec{v}$ AND $\dot{\vec{v}}$. THIS INTRODUCES A SUBTLETY: THERE'S ONLY A LONG-TERM BALANCE BETWEEN FORCE AND RADIATION, SINCE ENERGY CAN BE TEMPORARILY STORED IN "REACTIVE" (OR "INDUCTION") FIELDS.

THIS LONG-TERM AVERAGE IS

$$\int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt + \int_{t_1}^{t_2} \frac{2}{3}\frac{e^2}{c^3}\dot{\vec{v}}^{\,2}\,dt = 0.$$

INTEGRATE BY PARTS

$$\int_{t_1}^{t_2} \dot{\vec{v}}^{\,2}\,dt = \dot{\vec{v}}\cdot\vec{v}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \vec{v}\cdot\ddot{\vec{v}}\,dt$$

$$0 = \int_{t_1}^{t_2}\left(\vec{F} - \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}}\right)\cdot\vec{v} + \underbrace{\frac{2}{3}\frac{e^2}{c^3}\vec{r}\cdot\dot{\vec{v}}\Big|_{t_1}^{t_2}}_{\text{"REACTIVE"}}$$

THE LAST TERM IS THE "REACTIVE" ENERGY STORED IN FIELDS.

ON LONG-TERM AVERAGE, ENERGY IS CONSERVED WHEN

$$\vec{F}_{REACTION} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}}$$

THIS FORCE IS IN ADDITION TO THE FORCE REQUIRED BY CONSERVATION OF MOMENTUM.

THE TOTAL ELECTROMAGNETIC REACTION FORCE IS THEN

$$\vec{F}_{\text{REACTION}} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}} - m_{EM}\dot{\vec{v}}$$

WITH m_{EM} THE "ELECTROMAGNETIC MASS" EVALUATED PREVIOUSLY

... MORE ON MASSES LATER.

THIS IS ONE FORM OF THE ABRAHAM-LORENTZ EQUATION (J. EQN. 16.8).

WE'LL DISCUSS LATER ANOTHER FORM OF THE ABRAHAM-LORENTZ EQUATION J. EQN. 16.9

$$m\dot{\vec{v}} = \vec{F}_{EXT} - \vec{F}_{REACTION}$$

WITH $\vec{F}_{EXT}$ THE FORCE APPLIED BY AN EXTERNAL AGENT AND $\vec{F}_{\text{REACTION}} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}}$

EXAMPLE: $\vec{F}_{EXT} = 0$.

$$m\dot{\vec{v}} = -\frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}}$$

THIS HAS SOLUTIONS

$$\dot{\vec{v}}(t) = \dot{\vec{v}}(0)e^{t/\tau}$$

$$\tau = \frac{2}{3}\frac{e^2}{c^3}\frac{1}{m} \approx 10^{-23}\text{ sec}$$

FOR $m = m_e$.

THIS IS A "RUNAWAY" SOLUTION.
IN A VERY SHORT TIME THE CHARGE ACQUIRES INCREDIBLE ENERGY.

N.B. WE'LL SEE THAT BY MAKING AN AD HOC ASSUMPTION WE CAN TURN THIS INTO AN INTEGRAL-DIFFERENTIAL SOLUTION THAT CURES THE ENERGY NON-CONSERVATION, BUT INTRODUCES ISSUES OF NON-CAUSALITY.

THE ORIGINAL POINCARÉ ARGUMENT FOR THE NON-ELECTROMAGNETIC PART OF THE ELECTRON'S BINDING ENERGY.

RECALL THE MOMENTUM DENSITY $\vec{g}$

$$\vec{g} = \frac{1}{4\pi c} \vec{E} \times \vec{H}$$

AND TOTAL MOMENTUM $\vec{G}$

$$\vec{G} = \frac{1}{4\pi c} \iiint \vec{E} \times \vec{H} \, d\tau ;$$

FOR A TRUE POINT CHARGE THE INTEGRAL DIVERGES.

INTRODUCE AN "ELECTRON RADIUS" r_0; THIS ALLOWS THE INTEGRAL TO CONVERGE.

RECALL FOR NON-RELATIVISTIC ELECTRON MOTION $\vec{H} = \frac{1}{c} \vec{v} \times \vec{E}$

HENCE...

$$\vec{G} = \frac{1}{4\pi c^2} \iiint \vec{E} \times (\vec{v} \times \vec{E})\, dv$$

$$= \frac{1}{4\pi c^2} \iiint \left\{ \vec{v} E^2 - (\vec{v} \cdot \vec{E}) \vec{E} \right\} dv$$

LET'S CHOOSE, WITHOUT SACRIFICING GENERALITY, $\vec{v}$ ALONG $\hat{x}$:

$$\vec{G} = \frac{1}{4\pi c^2} \iiint \left\{ \vec{v}(E_x^2 + E_y^2 + E_z^2) - \vec{v} E_x^2 \right\} dv$$

(IN THE LAST TERM I ASSUMED THE MOMENTUM IS ALONG $\vec{v}$ FOR A SPHERICALLY-SYMMETRIC CHARGE DISTRIBUTION.)

FOR $v \ll c$, $\vec{E}$ IS SPHERICALLY-SYMMETRIC ABOUT THE CHARGE $(E_x^2 = E_y^2 = E_z^2)$

$$\vec{G} = \frac{1}{4\pi c^2} \iiint_{r_0}^{\infty} \vec{v}(E_y^2 + E_z^2)\, dv$$

$$= \frac{1}{4\pi c^2} \iiint_{r_0}^{\infty} \frac{2}{3} \vec{v} E^2\, dv$$

RECALL $U_{em} = \frac{1}{8\pi}\iiint E^2 d\tau$. HENCE

$$\vec{G} = \frac{1}{c^2}\,\frac{4}{3}\,\vec{v}\,U_{em}$$

MOMENTA HAVE FORM $\vec{G} = m\vec{v}$,

SO WE IDENTIFY

$$M_{em} = \frac{4}{3}\,\frac{U_{em}}{c^2}$$

WE DID A "SWITCHEROO" HERE; r_0 IS HIDDEN IN U_{em}.

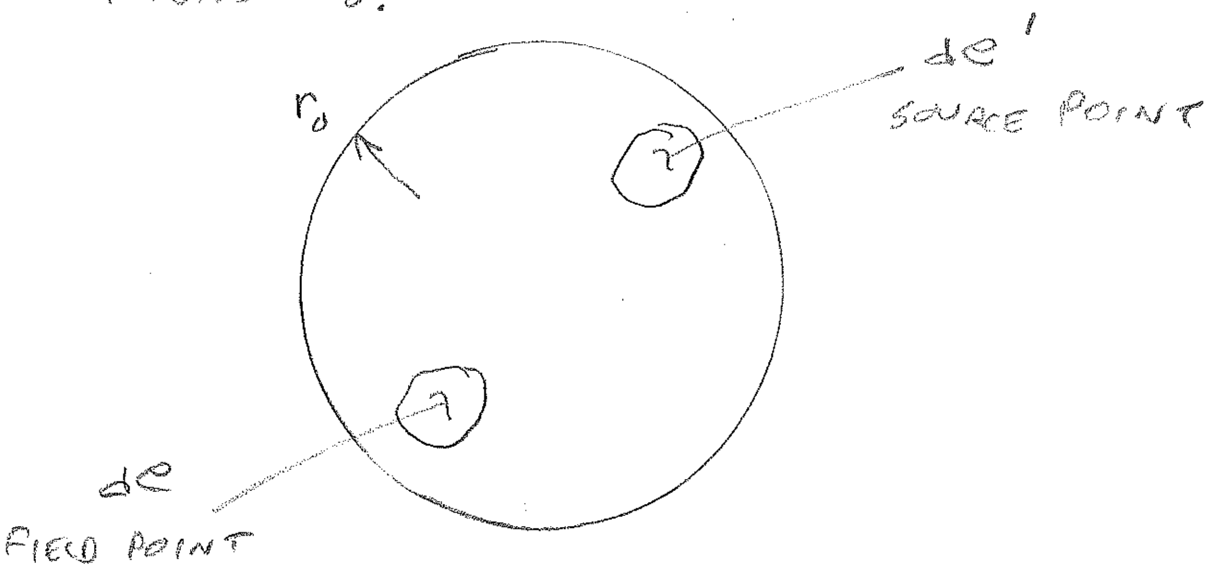
NOW WE HAVE AN INTERESTING SITUATION! THE MASS/ENERGY RELATION SHOULD BE $1 \cdot \frac{U_{em}}{c^2}$;

THERE'S AN ADDITIONAL, NON-ELECTROMAGNETIC MASS $-\frac{1}{3}\frac{U_{EM}}{c^2}$ TO ACCOUNT FOR THE RELATIVISTICALLY-CORRECT EQUATIONS.

THIS EXTRA MASS (OR ENERGY) IS THE NON-ELECTROMAGNETIC BINDING THAT MAKES THE CHARGE SYSTEM OF THE ELECTRON STABLE.

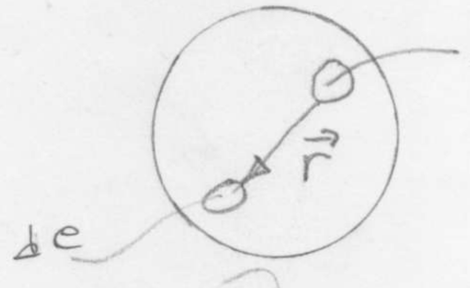
REACTION FORCE II.

LORENTZ ORIGINALLY FOUND AN ABRAHAM-LORENTZ EQUATION VIA RETARDED FIELDS (JEFEMENKO EQUATIONS) AND A SIMPLE MODEL OF THE ELECTRON: A UNIFORM DISTRIBUTION OF CHARGE, RADIUS r_0.



LORENTZ' ASSUMPTIONS

1. FRAME WITH de (FIELD POINT) AT REST.
2. THINGS LIKE $\vec{v}$, $\dot{\vec{v}}$, $\ddot{\vec{v}}$ DON'T CHANGE MUCH DURING THE LIGHT-TRAVEL TIME ACROSS THE ELECTRON
3. FIELDS DERIVED FROM LIÉNARD-WIECHERT POTENTIALS.
4. ONLY TERMS NOT EXPLICITLY CONTAINING r_0 HAVE PHYSICAL SIGNIFICANCE
5. SPHERICAL SYMMETRY



FROM $\vec{E} = \frac{q}{s^3}(\vec{r} - r\vec{\beta})(1-\beta^2)$

$$+ \frac{1}{c^2 s^2}\left\{\vec{r}\times\left[(\vec{r} - r\vec{\beta})\times\dot{\vec{\beta}}\right]\right\}$$

$$d\vec{E} = \frac{de'}{s^3}\left\{\frac{1}{c^2}\vec{r}\times\left[\left(\vec{r} - r\vec{\beta}(t')\right)\times\dot{\vec{\beta}}(t')\right] + \left[1-\beta^2(t')\right]\left(\vec{r} - r\vec{\beta}(t')\right)\right\}$$

(SUBTLETY: UNLESS YOU KNOW THE ELECTRON'S ENTIRE PAST HISTORY, YOU WON'T KNOW $t'(s)$: THIS IS TAKEN INTO ACCOUNT BY ASSUMPTION (2).)

YOU'D LIKE TO EXPRESS THE QUANTITIES WITH THE RETARDED TIME t' IN TERMS OF THE FIELD-POINT TIME t. EXPAND $\vec{r}(t')$, $\dot{\vec{v}}(t')$, $s(t')$:

(15)

$$\vec{v}(t') = \cancel{\vec{v}(t)} - \dot{\vec{v}}(t)\frac{r}{c} + \frac{\ddot{\vec{v}}(t)}{2}\left(\frac{r}{c}\right)^2 - \ldots$$

$$\dot{\vec{v}}(t') = \dot{\vec{v}}(t) - \frac{r}{c}\ddot{\vec{v}}(t) + \ldots$$

$$\frac{1}{s^3} = \frac{1}{r^2}\left[1 - \frac{3\dot{\vec{v}}(t)\cdot\vec{r}}{c^2} + \frac{3}{2}r\left(\frac{\ddot{\vec{v}}(t)\cdot\vec{r}}{c^3}\right) + \ldots\right.$$

KEEP TERMS $\Theta\left(\frac{r}{c}\right)^3$

⋮

AFTER SOME ALGEBRA

$$d\vec{E} = de'\left[-\frac{2\vec{r}(\dot{\vec{v}}\cdot\vec{r})}{r^3c^2} + \frac{1}{2}\frac{\vec{r}(\ddot{\vec{v}}\cdot\vec{r})}{r^2c^3} + \frac{\vec{r}}{r^3} + \frac{\ddot{\vec{v}}}{2c^3}\right]$$

LORENTZ USED A TRICK TO EVALUATE THIS; WRITE IN TENSOR NOTATION

$$d\vec{E}_\alpha = de'\left[-\frac{2r_\alpha(\dot{v}_\beta r_\beta)}{r^3c^2} + \frac{1}{2}\frac{r_\alpha(\ddot{v}_\beta r_\beta)}{r^2c^3} + \frac{r_\alpha}{r^3} + \frac{\ddot{v}_\alpha}{2c^3}\right]$$

HE THEN NOTED THE AVERAGE OF r_α OVER A SPHERICALLY-SYMMETRIC INTEGRATION VANISHES, AND THE AVERAGE OF $r_\alpha r_\beta$ IS $\frac{1}{3}r^2\delta_{\alpha\beta}$

HENCE,

$$\langle dE_\alpha \rangle = de' \left(-\frac{2}{3} \frac{\dot{v}_\alpha}{c^2 r} + \frac{2}{3} \frac{\ddot{v}_\alpha}{c^3} \right)$$

THEN THE TOTAL FORCE ("$e\vec{E}$") IS

$$\vec{F} = \iint de\, d\vec{E} = \frac{2}{3} \frac{e^2}{c^3} \ddot{\vec{v}} - m_{EM} \dot{\vec{v}}$$

$$\text{WITH } m_{EM} = \iint \frac{de\, de'}{2r} = \frac{4}{3c^2} U_0$$

WITH U_0 THE ELECTROSTATIC ENERGY OF THE ELECTRON DUE TO ITS OWN FIELD.

THIS IS WHAT WE FOUND BY BALANCING RADIATION LOSS WITH $\vec{F} \cdot \vec{v}$

r_0 IS "HIDDEN" IN U_0

BACK TO THE ABRAHAM–LORENTZ EQUATION.

A SUBTLETY OCCURED REGARDING MASS.

IF YOU HAVE A CHARGED PARTICLE e SUBJECT TO AN EXTERNAL FORCE $\vec{F}_{EXT}$ IN ADDITION TO THE TO THE REACTION FORCE

$$\vec{F}_{REACT} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}} - m_{em}\dot{\vec{v}},$$

THE EQUATION-OF-MOTION COMES FROM

$$\vec{F}_{Ext} - \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}} - m_{em}\dot{\vec{v}} = m_{non\text{-}em}\dot{\vec{v}}$$

WITH $m_{non\,em}$ THE NON-ELECTROMAGNETIC ("MECHANICAL") MASS.

BUT SEPARATION OF MASS INTO EM AND NON-EM ISN'T MEANINGFUL SINCE THESE HAVE MODEL-DEPENDENCE (e.g., r_0). HENCE, WE HAVE

$$\vec{F}_{EXT} = m\dot{\vec{v}} - \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}} \quad \text{WITH}$$

m THE EMPIRICAL (MEASURED) REST MASS.

BACK TO THE ABRAHAM-LORENTZ EQUATION (J. EQN. 16.9)

$$\frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}} - m\dot{\vec{v}} = \vec{F}_{REACTION}$$

$$\text{OR } m\left(\dot{\vec{v}} - \frac{2}{3}\frac{e^2}{c^3}\frac{1}{m}\ddot{\vec{v}}\right) = \vec{F}_{EXT}$$

(THE FORM OF J. EQN. 16.9)

WITH $\vec{F}_{EXT} = 0$!

$$\dot{\vec{v}}(t) = \dot{\vec{v}}(0)e^{t/\tau}$$

$$;\ \tau = \frac{2}{3}\frac{e^2}{c^3}\frac{1}{m}.$$

THIS IS OBVIOUSLY A PROBLEM.
HERE'S SOME PATHS FORWARD.

1. QUESTION THE VALIDITY OF THE ABRAHAM-LORENTZ EQUATION;
2. APPLY ARBITRARY CONSTRAINTS WHICH EXCLUDE RUNAWAY SOLUTIONS;
3. HOPE QUANTUM MECHANICS PROVIDES A SOLUTION.

1. QUESTION VALIDITY ?

- THE REACTION TERM $\frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}}$

  DOESN'T HAVE EXPLICIT MODEL DEPENDENCE;

- THE USE OF THE EMPIRICAL MASS m (MEASURED MASS) REMOVES ISSUES RELATING TO THE THEORETICAL BASIS OF THE MASS;

- NONETHELESS, THERE ARE ALTERNATIVES TO ELECTRODYNAMICS THAT SKIRT SOME ISSUES OF THE ABRAHAM-LORENTZ EQUATION.

2. AD-HOC ASSUMPTIONS?

FIND SOLUTIONS TO THE INHOMOGENEOUS ABRAHAM-LORENTZ EQUATION WITH THE INTEGRATING FACTOR $e^{-t/\tau}$:

$$e^{-t/\tau}\dot{\vec{v}}(t)$$

$$= e^{-t_0/\tau}\vec{v}(t_0)$$

$$-\frac{1}{\tau}\int_{t_0}^{t}\frac{\vec{F}_{EXT}(t')}{m}e^{-t'/\tau}\,dt'$$

OR

$$\dot{\vec{V}}(t) = e^{\frac{t-t_0}{\tau}}\dot{\vec{V}}(t_0) \quad \Big\} \text{ THIS TERM RUNS AWAY, IT SEEMS}$$

$$-\frac{1}{\tau}\int_{t_0}^{t}\frac{\vec{F}_{EXT}(t')}{m}e^{-\frac{(t'-t)}{\tau}}\,dt'$$

BUT, SINCE t_0 IS ARBITRARY, CHOOSE $t_0 \longrightarrow \infty$, HENCE

$$\dot{\vec{v}}(t) = \frac{1}{\tau}\int_{t}^{\infty}\frac{\vec{F}_{EXT}}{m}e^{-\frac{(t'-t)}{\tau}}\,dt'$$

THIS IS AN INTEGRAL-DIFFERENTIAL EQUATION. IT HAS THE NICE PROPERTY THAT

$$\dot{\vec{v}} \xrightarrow[e^2 \to 0]{} \frac{\vec{F}_{EXT}}{m} \qquad \left(\tau = \frac{2}{3}\frac{e^2}{c^3 m}\right)$$

BUT NOTICE THE INTEGRAL GOES INTO THE FUTURE: THE INTEGRAL INCLUDES FUTURE FORCES, THE INTEGRAL-DIFFERENTIAL EQUATION IS THEREFORE ACAUSAL.

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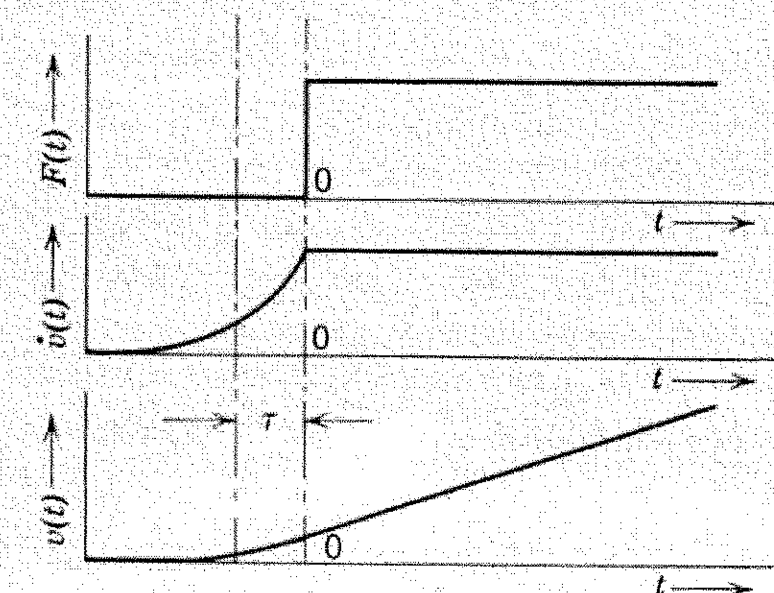

Fig. 17.1 "Preacceleration" of charged particle.

J. D. JACKSON, "CLASSICAL ELECTRODYNAMICS", 2ND ED.