Physics 515, Electrodynamics III
Department of Physics, University of Washington
Spring quarter 2020
April 3, 2020, 11am
On-line lecture

Administrative
Homework 1 posted today. (Still working out submission/grading/return details.)

Lecture
Jackson Chapter 9. Radiating systems.
J.C.9.1-3 Usual treatment of the infinitesimal electric dipole.

RADIATION II J. C. 9.

RADIATING SYSTEMS.

WORK IN LORENTZ GAUGE.

FOR SOURCES $\rho(\vec{r},t)$ AND $\vec{J}(\vec{r},t)$ FIND EXPLICIT SOLUTIONS TO RETARDED POTENTIALS

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\iiint \frac{[\rho(\vec{r}',t)]_{RET}}{|\vec{r}-\vec{r}'|}\,d\tau'$$

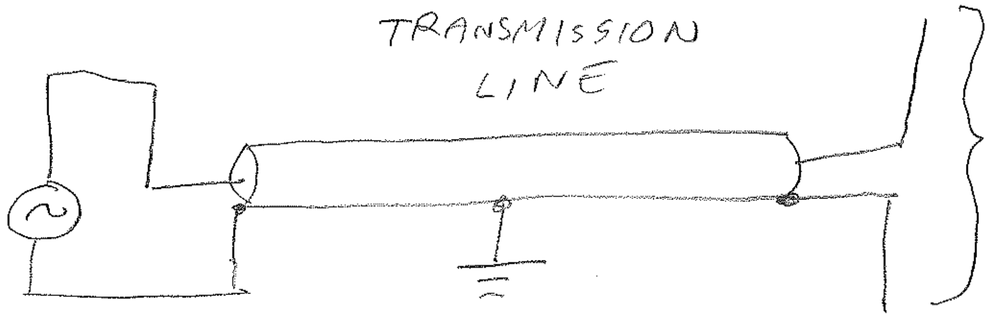
$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\iiint \frac{[\vec{J}(\vec{r}',t)]_{RET}}{|\vec{r}-\vec{r}'|}\,d\tau'$$

AND THEREBY FIND $\vec{E}$ AND $\vec{B}$, ETC.

ρ AND J ARE BULK CHARGES.

RADIATION FROM MOVING SINGLE CHARGES IS TREATED IN J.C. 14 (LIÉNARD-WIECHERT POTENTIALS).

THE INFINITESIMAL ELECTRIC DIPOLE
J. C. 9.4

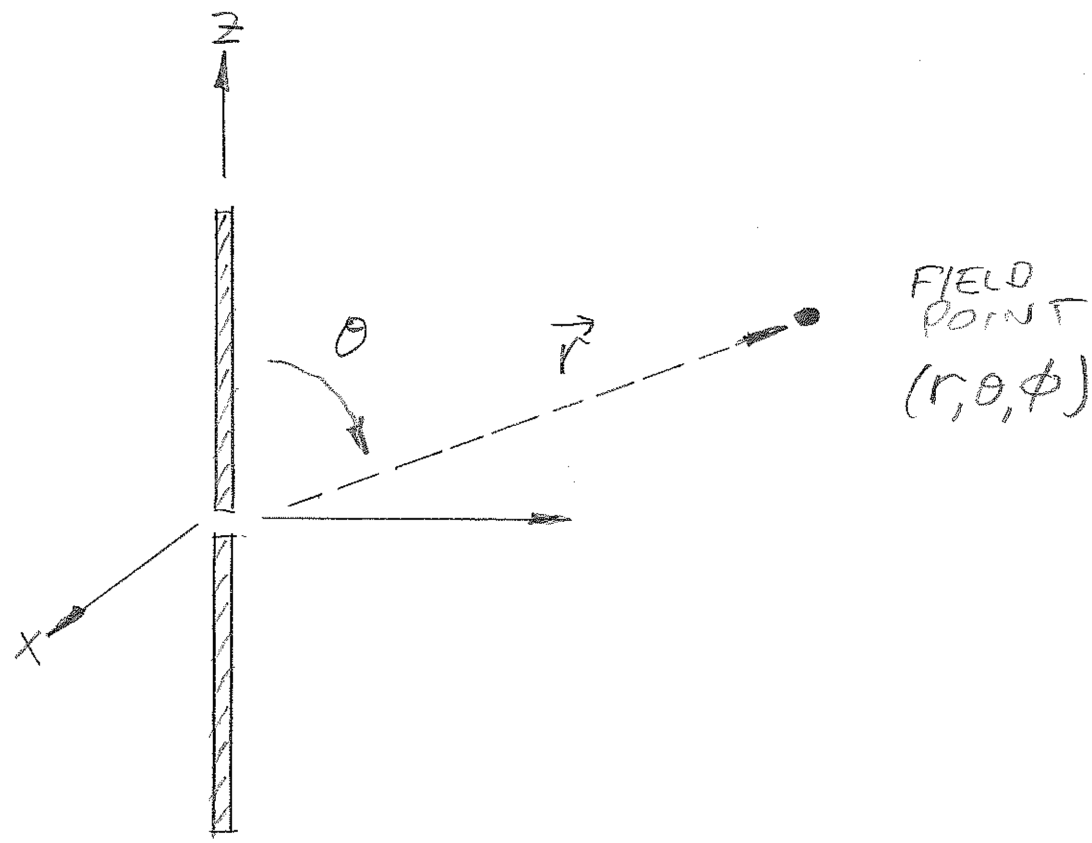


$\ell \ll \lambda_0$ (ALSO $\ell \ll r$)
λ_0 THE FREE-SPACE WAVELENGTH

- ASSUME THE CURRENT I IS UNIFORM ALONG THE DIPOLE.
- ASSUME RADIATION DOMINATED BY CURRENT I.
- IGNORE RADIATION CONTRIBUTION OF CHARGE BUILD-UP ON ANTENNA ENDS.
- THIS IS NOT A TERRIBLE ASSUMPTION
- THE DRIVING SOURCE IS A CURRENT SOURCE

$$I(t) = I_\omega e^{i\omega t}$$

ESTABLISH COORDINATE SYSTEM



FROM ASSUMPTION CURRENT ALONG ANTENNA CONSTANT:

$$\vec{I}(z,t) = I_\omega \hat{z} e^{i\omega t}$$

FIND $\vec{A}_\omega$ OR Φ_ω?

LET'S FIND $\vec{A}$. THIS GIVES

$$\vec{B}_\omega = \vec{\nabla} \times \vec{A}_\omega,$$

THEN APPLY AMPÈRE'S LAW

$$\frac{1}{c^2} i\omega \vec{E}_\omega = \vec{\nabla} \times \vec{B}_\omega$$

EVALUATE THE RETARDED POTENTIAL

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\iiint \frac{\left[\vec{J}(\vec{r}',t)\right]_{RET}}{|\vec{r}-\vec{r}'|}\,dr'$$

WE'RE INTERESTED IN ONE PARTICULAR FREQUENCY ω IN

$$\vec{A}(\vec{r},t) = \int_{\omega'} \vec{A}_{\omega'}(\vec{r})\,e^{i\omega' t}\,d\omega'$$

$$= \frac{\mu_0}{4\pi}\iiint_{r'} \frac{1}{|\vec{r}-\vec{r}'|}$$

$$\times \int_{\omega'} \vec{J}_{\omega'}(\vec{r}')\,e^{i\omega'\left(t-\frac{|\vec{r}-\vec{r}'|}{c}\right)}\,d\omega'\,dr'$$

THIS EXPRESSION APPLIES TO BULK CURRENTS $\vec{J}$. FOR LINE CURRENTS I:

$$\vec{A}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\text{ANTENNA}} \frac{\vec{I}_\omega(\vec{r}')}{|\vec{r}-\vec{r}'|}\,e^{-i\omega\frac{\vec{r}-r'}{c}}\,d\ell'$$

THIS EXPRESSION IS APPLIED TO OUR ANTENNA.

CURRENT: $\vec{I}_\omega(\vec{r}\,') \rightarrow I_\omega \hat{z}$

$$d\ell \rightarrow dz$$

INFINITESIMAL DIPOLE:

$$\ell \ll \lambda_0, \quad \ell \ll r, \quad r' \ll r$$

$$|\vec{r} - \vec{r}\,'| \rightarrow r$$

SLIGHTLY MORE SUBTLE, THE PHASE $\omega \frac{|\vec{r} - \vec{r}\,'|}{c} \rightarrow \omega \frac{r}{c}$ DOESN'T MUCH CHANGE OVER THE INTEGRATION.

$$\vec{A}_\omega(\vec{r}) = \hat{z} \underbrace{\frac{\mu_0}{4\pi} \frac{I_\omega}{r} e^{-i\omega \frac{r}{c}} \int_{-\ell/2}^{+\ell/2} dz'}_{A_z}$$

THIS IS A PRETTY SIMPLE EXPRESSION.

WE'LL NEED $\vec{\nabla} \times \vec{A}_\omega$ IN SPHERICAL COORDINATES. RECALL

$$A_{\omega\phi} = 0$$

$$A_{\omega\theta} = -A_z \sin\theta$$

$$A_{\omega r} = +A_z \cos\theta$$

NOW EVALUATE $\vec{B}_\omega = \vec{\nabla} \times \vec{A}_\omega$.

WITH $A_{\omega\phi} = 0$ AND TERMS WITH $\frac{\partial}{\partial\phi} \to 0$, ONLY THE $\hat{\phi}$ TERM REMAINS.

$$\vec{B}_\omega = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_{\omega\theta}\right) - \left(\frac{\partial}{\partial\theta} A_{\omega r}\right)\right]\hat{\phi}$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(-rA_{\omega z} \sin\theta\right) - \frac{\partial}{\partial\theta}\left(+A_{\omega z} \cos\theta\right)\right]\hat{\phi}$$

$$= \frac{1}{r}\frac{\mu_0}{4\pi} I_\omega l \cdot \left[\frac{\partial}{\partial r}\left(-r\frac{e^{-i\omega\frac{r}{c}}}{r} \sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{e^{-i\omega\frac{r}{c}}}{r}\cos\theta\right)\right]\hat{\phi}$$

$$= \frac{1}{r}\frac{\mu_0}{4\pi} I_\omega l \cdot \left[i\frac{\omega}{c} e^{-i\omega\frac{r}{c}} \sin\theta + \frac{e^{-i\omega\frac{r}{c}}}{r}\sin\theta\right]\hat{\phi}$$

THIS IS USUALLY WRITTEN AS

$$\vec{B_\omega} = i \frac{\mu_0}{4\pi} \frac{\omega}{c} I_\omega \ell \sin\theta$$

$$\cdot \frac{1}{r} \left[1 - \frac{i}{\omega/c \; r} \right] \hat{\phi}$$

FIND THE ELECTRIC FIELD

$$i\omega \frac{1}{c^2} \vec{E_\omega} = \vec{\nabla} \times \vec{B_\omega} \quad \text{(AMPÉRE'S LAW)}$$

THE CURL HAS NON-ZERO DERIVATIVES $\frac{\partial}{\partial\theta} B_{\omega\phi}$ AND $\frac{\partial}{\partial r} B_{\omega\phi}$, SO THE CURL HAS r AND θ COMPONENTS.

$$E_{\omega r} = \frac{1}{c\varepsilon_0} \frac{1}{2\pi} I_\omega \ell \cos\theta$$

$$\cdot \frac{1}{r^2} \left[1 - \frac{i}{\omega/c \; r} \right] e^{-i\omega r/c}$$

$$E_{\omega\theta} = i \frac{1}{c\varepsilon_0} \frac{1}{4\pi} I_\omega \ell \frac{\omega}{c} \sin\theta$$

$$\cdot \frac{1}{r} \left[1 - \frac{i}{\omega/c \; r} - \frac{1}{(\omega/c \; r)^2} \right] e^{-i\omega r/c}$$

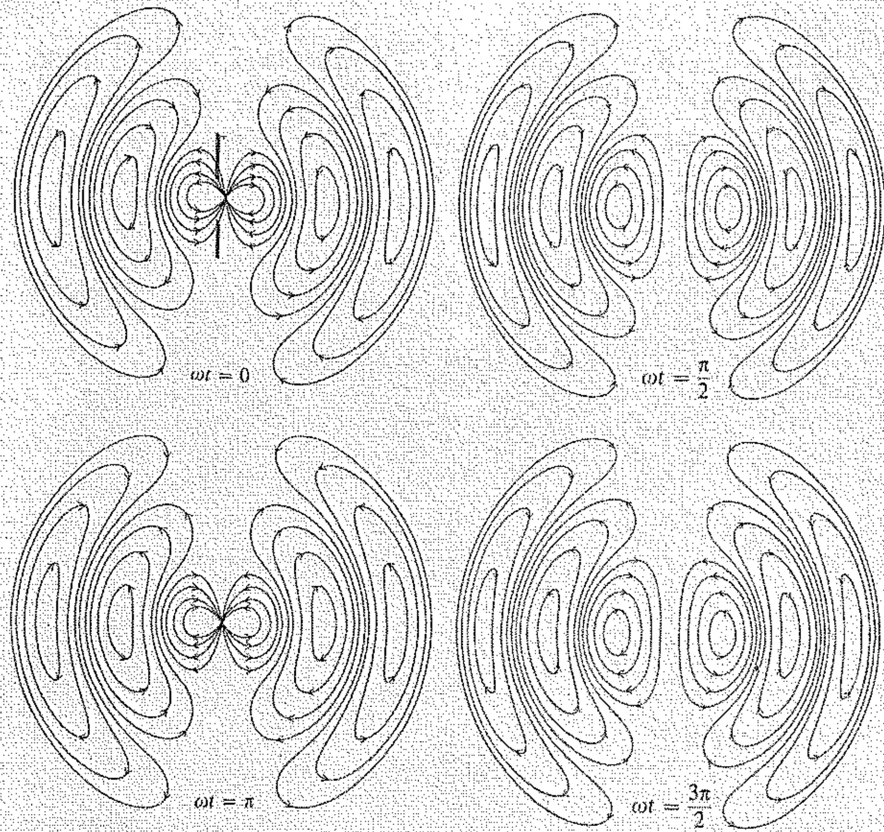
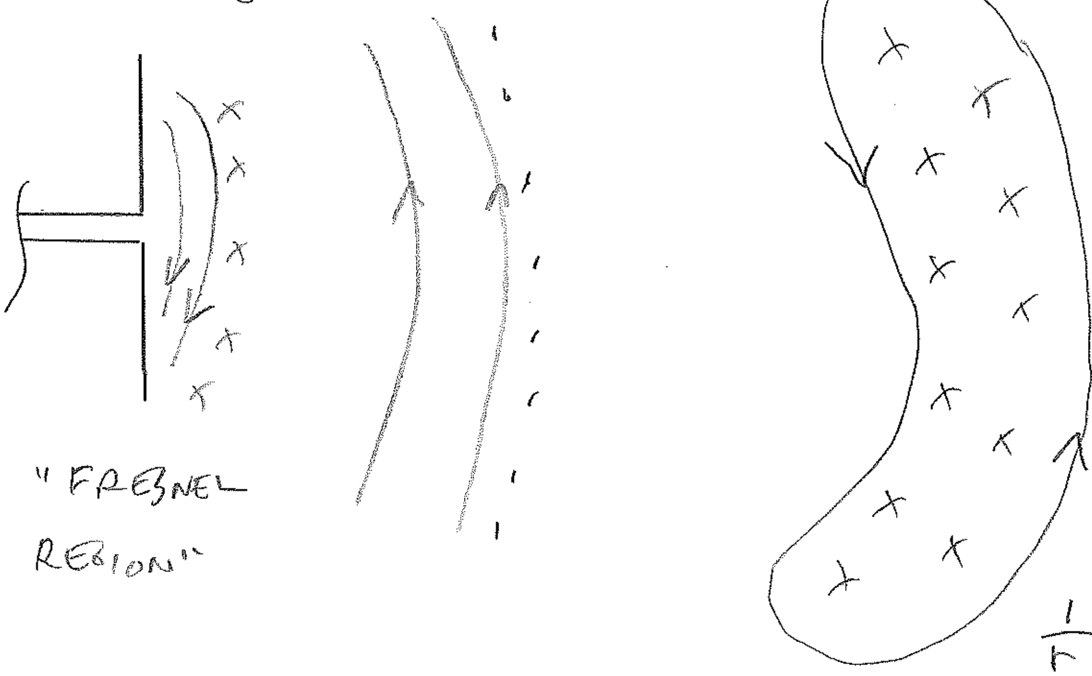


Figure 14-9. The electric lines of force of an oscillating dipole for $\omega t = 0, \pi/2, \pi$, and $3\pi/2$. The dipole is situated in the center and is oriented in the vertical direction. The decrease in wavelength with distance can be observed on these figures. The magnetic lines of force are circles perpendicular to the paper and centered on the axis of the dipole.

"NEAR ZONE" "INTERMEDIATE ZONE" "FAR ZONE"



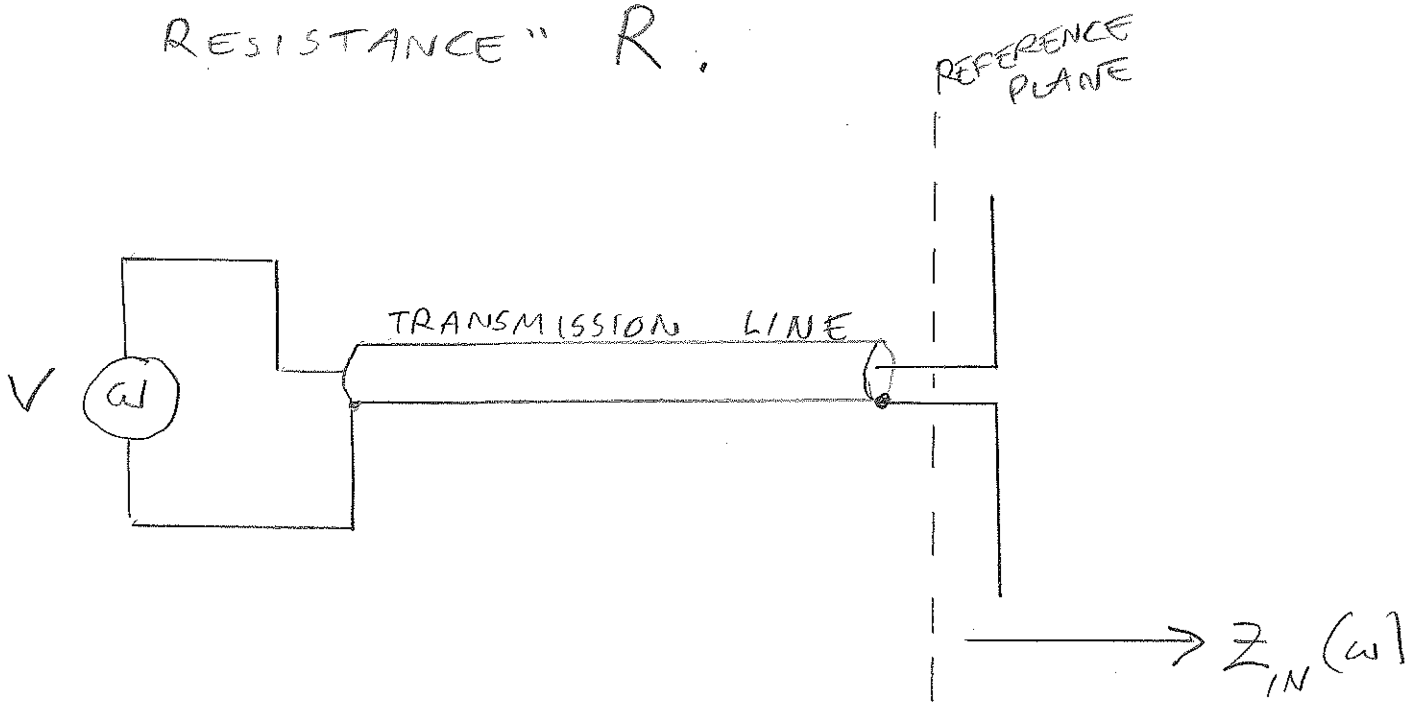
"FRESNEL REGION"

"RADIATION ZONE"

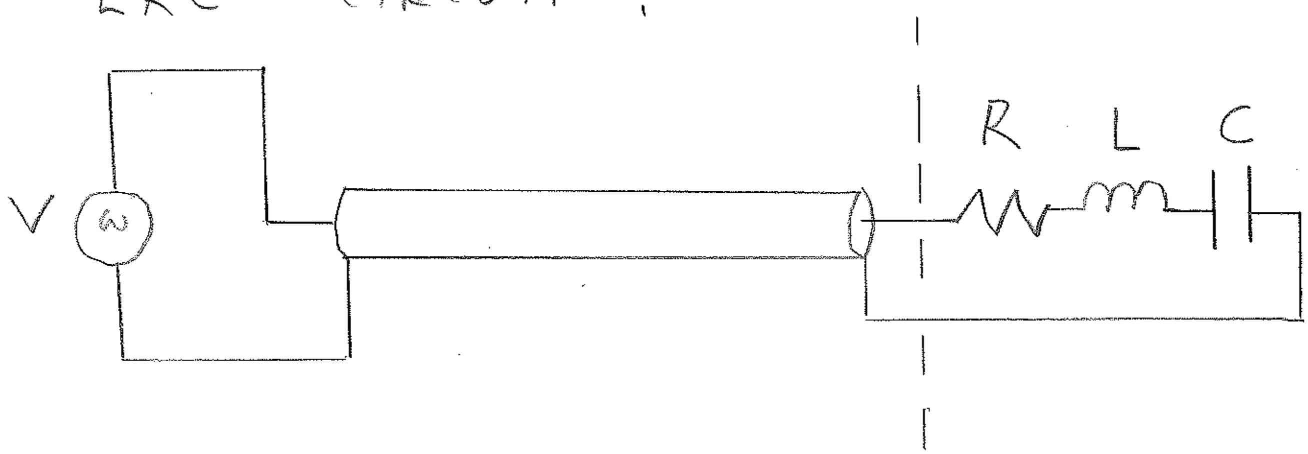
"FRAUNHOFER REGION"

$\frac{1}{r}$ $\vec{E}$ & $\vec{B}$ FIELDS

THE ANTENNA "RADIATION RESISTANCE" R.



AS FAR AS THE SOURCE $V(\omega)$ AND TRANSMISSION LINE ARE CONCERNED, THE ANTENNA IS A LOAD $Z_{IN}(\omega)$, WHICH IS EQUIVALENT TO A LRC CIRCUIT:



THIS R IS THE "LOSS" TERM AND REPRESENTS RADIATION SENT TO ∞.

THE STRATEGY IS TO EQUATE THE POYNTING FLUX ESCAPING THE SPHERE AT ∞ TO THE POWER PROVIDED BY THE SOURCE $\langle P_{IN} \rangle = \frac{1}{2} Re\{ I R I^* \}$

BEFORE WE START, ESTIMATE R. THE ONLY THING IN THIS PROBLEM WITH UNITS OF RESISTANCE IS THE IMPEDANCE OF FREE SPACE $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377.\ \Omega$.

THERE'S ALSO l AND λ_0. WE COULD HAVE TERMS LIKE, e.g., l/λ_0 OR λ_0/l.

SO, YOU SEE THIS IS A TERRIBLE ANTENNA. THE TRANSMISSION LINE HAS AN IMPEDANCE $\Theta(100\Omega)$, SO ALMOST ALL THE POWER INCIDENT ON THE ANTENNA REFLECTS BACK. SIGH..

(THAT SAID, WITH A MUCH MORE COMPLICATED CALCULATION, WITH $l = \lambda_0/2$, $R \sim 75$, A GOOD MATCH TO A 75Ω TRANSMISSION LINE. BUT, ALAS, IT'S NOT SO SIMPLE...)

BACK TO FINDING R FOR THE INFINITESIMAL ELECTRIC DIPOLE:

STEP 1: FIND POYNTING FLUX AT ∞.

$$\langle \vec{S} \rangle = \frac{1}{2} \mathrm{Re}\left\{ \frac{1}{\mu_0} \vec{E} \times \vec{B}^* \right\}$$

$$= \frac{1}{2\mu_0} \mathrm{Re}\left\{ (E_r \hat{r} + E_\theta \hat{\theta}) \times (B_\phi^* \hat{\phi}) \right\}$$

WE CAN IGNORE THE $\hat{r} \times \hat{\phi}$ TERM AS THIS DOESN'T ESCAPE THE SPHERE AT ∞.

$$[\vec{S}]_r = \frac{1}{2\mu_0} \frac{1}{c} \frac{\mu_0}{\varepsilon_0} \left[\frac{\omega/c\, I_\omega l}{4\pi r} \sin\theta \right]^2 \hat{r}$$

+ TERMS $\sim 1/r^3, \frac{1}{r^4}, \ldots$ WHICH ARE IGNORED.

$$\langle P \rangle = \oiint_{AT\,\infty} [\vec{S}]_r \, r^2 d\Omega$$

$$= \sqrt{\frac{\mu_0}{\varepsilon_0}} \left[I_\omega \frac{l}{\lambda_0} \right]^2 \pi/3$$

WHERE I USED $\lambda_0 = 2\pi \frac{c}{\omega}$.

STEP 2. POWER DELIVERED BY SOURCE.

FROM CIRCUIT THEORY

$$\langle P \rangle = \frac{1}{2} I_\omega^2 R$$

STEP 3. EQUATE THE TWO POWERS.

$$R = \sqrt{\frac{\mu_0}{\varepsilon_0}} \left[\ell / \lambda_0 \right]^2 \frac{2}{3} \pi$$

$$\ll Z_0$$ ⟸ IMPEDANCE OF TRANSMISSION LINE.

TWO EXAMPLES.

- WIFI LOW BAND

  $f \approx 1\ \text{GHz}, \quad \lambda \approx 0.3\ \text{m}$

ANTENNA IN YOUR LAPTOP, DIPOLE

$\ell \approx 0.1\ \text{m}$

$$R \approx 740\,\Omega \left(\frac{0.1\ \text{m}}{0.3\ \text{m}}\right)^2 \approx 75\,\Omega .$$

NOT BAD AT ALL.

- FM RADIO.

  $f \approx 1\ \text{MHz}, \quad \lambda \approx 300\ \text{m}.$

$$R \approx 740\,\Omega \left(\frac{0.1\ \text{m}}{300\ \text{m}}\right)^2 \approx 10^{-4}\,\Omega .$$

OUCH! THE PHYSICAL RESISTANCE OF THE ANTENNA IS PERHAPS $1/10\ \Omega$, SO ALMOST ALL THE POWER RECEIVED (OR TRANSMITTED) GOES INTO HEATING THE ANTENNA.

FORTUNATELY, THERE ARE TRICKS YOU CAN PLAY TO IMPROVE THINGS, BUT THAT'S FOR NEXT YEAR.

LOOSE ENDS, APPLICABLE TO ANY ANTENNA.

- DIRECTIVITY.

RECALL $\langle S \rangle \sim \sin^2\theta$.

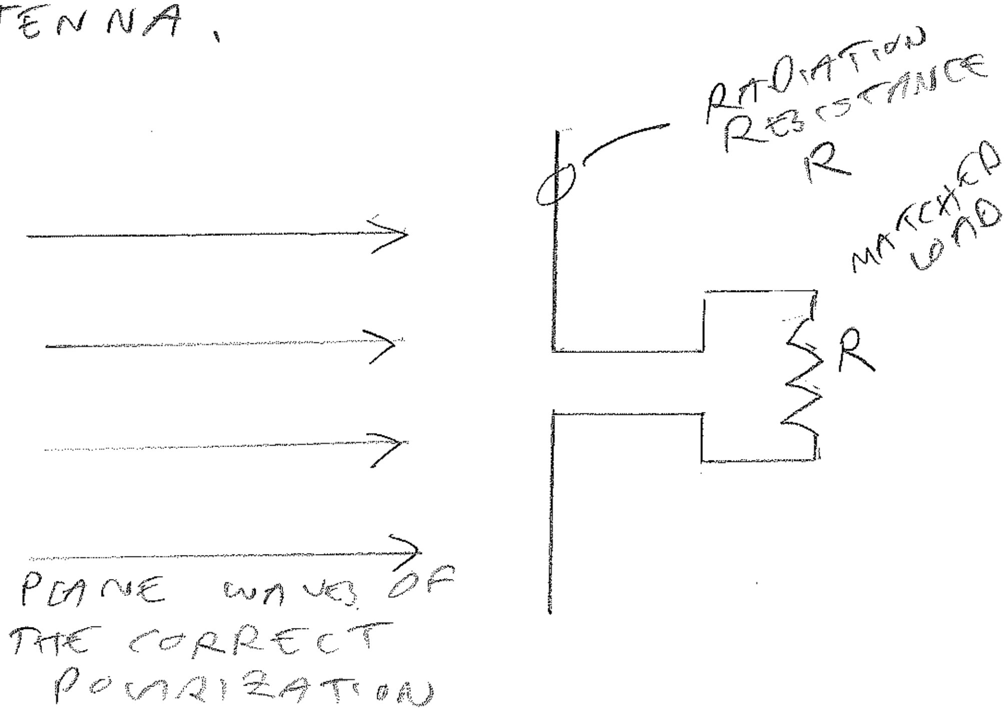
THERE'S NO POWER RADIATED ALONG THE ANTENNA AXIS $\theta = 0$, THE MAXIMUM POWER IS IN THE PLANE $\theta = \pi/2$.

DIRECTIVITY. THE RATIO OF THE POWER PER AREA TOWARDS THE MAXIMUM-POWER DIRECTION IN UNITS OF THE POWER PER AREA IF THE EMITTED POWER WERE ISOTROPIC IS THE DIRECTIVITY D.

EXERCISE: FOR THE INFINITESIMAL ELECTRIC DIPOLE $D = 3/2$, PRETTY BAD.

- EFFECTIVE APERTURE.

WHAT'S THE ELECTROMAGNETIC CROSS-SECTION σ_{EM} OF THE ANTENNA.

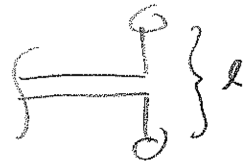


DIVIDE THE POWER DELIVERED TO THE LOAD BY THE POYNTING VECTOR!

$$\sigma_{EM} = \frac{3}{8\pi} \lambda^2 .$$

YOU SHOULD ALSO LOOK AT THE THOMPSON CROSS-SECTION FROM 514. (WE'VE SEEN THIS BEFORE.)

VARIANTS ON INFINITESIMAL ELECTRIC DIPOLE

1. 

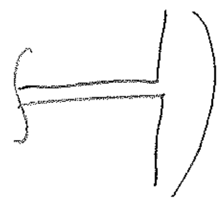
"LOADED" DIPOLE.

FIND Q_0 VIA $\dot{Q} = i\omega Q_0 = I$,

FIND p_0 VIA $Q_0 \ell$.

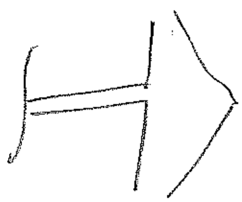
AND USE p_0 TO REPLACE I.

OR USE POLARIZATION POTENTIALS.

2. 

NON-CONSTANT CURRENT $\sim \frac{1}{2}$ SIN...

JACKSON PROBLEM 9.4.

3. 

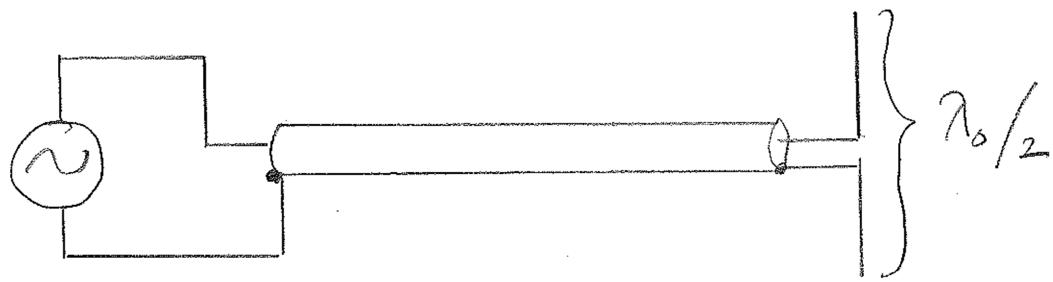
TRIANGLE CURRENT

JACKSON EQN 9.25 VIA POLARIZATION.

⋮

THINGS GET VERY RAPIDLY MORE DIFFICULT.

4. YOU DON'T NEED TO KNOW THIS! THE "$\frac{1}{2}\lambda$ DIPOLE."



$D \sim 1.64$ (PRETTY BAD).

$\sigma_{EM} \approx 0.13 \ \lambda^2$ (PRETTY BAD)

$$R = Z_0 \frac{1}{2\pi} C_{IN}(2\pi)$$
$$\approx 30\,\Omega\,(2.4) \approx 73\,\Omega\,!$$

WITH THE "RADIATION INTEGRAL"

$$C_{IN}(x) \equiv \int_0^x \frac{1-\cos y}{y}\,dy$$

(THIS INTEGRAL FREQUENTLY APPEARS IN ANTENNA PROBLEMS. YOU WON'T NEED TO USE IT IN THIS COURSE.)