Physics 515, Electrodynamics III
Department of Physics, University of Washington
Spring quarter 2020
May 1, 2020, 11am
On-line lecture

Administrative:

1. HW#4 due now (with some exceptions).
2. You should be getting your homework back; if not let me know.
3. Office hours Wednesdays after class at URL https://washington.zoom/us/j/712804010
4. The midterm exam will be posted today, May 1, at 3 pm PDT. The exam is due via email Monday, May 4, at 11am. The exam is open book, you may use Jackson. See course web site for exam information. It should take around an hour and 20 minutes to complete. See the exam for email submission instructions.

Lecture: **J. Chapter 11: Covariant and relativistic electrodynamics.**

1. Finish up miscellaneous tensor relations.
2. J. Chapter 11.5: Relativistic kinematics.
3. J. Chapter 11.9: Covariant form of the wave equation.
4. Liénard–Wiechert potentials I.

MISCELLANEOUS TENSOR RELATIONS (CONTINUED).

- SYMMETRIES,

  IF UNDER EXCHANGE OF TWO CONTRAVARIANT (OR COVARIANT) INDICIES:

  TENSOR UNCHANGED:
  "SYMMETRIC" OR "EVEN"

$$T^{\mu\nu} = +T^{\nu\mu}$$

  TENSOR CHANGES SIGN
  "ANTI-SYMMETRIC" OR "ODD"

$$\varepsilon_{\alpha\beta\nu\mu} = -\varepsilon_{\beta\alpha\nu\mu}$$

Q: CAN THERE BE TENSORS WHICH ARE NEITHER EVEN OR ODD?

PLAN FOR THE COMING TWO LECTURES:

EXTEND RELATIVITY TO MECHANICS (TRICKY) AND ELECTRODYNAMICS (EASY).

MY WAY OF PROCEEDING IS TO START WITH A VECTOR EQUATION IN PHYSICS, THEN FIND ITS COVARIANT FORM.

THAT IS, WE KNOW THE VECTOR EQUATION IN A SPECIAL FORM (SAY, WHERE THE SYSTEM IS AT REST). IF YOU CAN SOMEHOW DEVINE A TENSOR EQUATION WHICH REDUCES TO THE STARTING VECTOR EQUATION IN THE SPECIAL FRAME, THEN PERHAPS THE TENSOR EQUATION IS CORRECT.

A SIDE NOTE: THERE ARE ALTERNATIVE WAYS TO PROCEED,

YOU COULD START WITH THE VECTOR EQUATION IN THE SPECIAL FRAME, THEN TRANSFORM THOSE QUANTITIES IN THE EQUATION WHOSE TRANSFORMATION PROPERTIES ARE KNOWN, IF NOT KNOWN, PERHAPS DERIVE THE TRANSFORMATION PROPERTIES; THEN EXPRESS EVERYTHING IN TENSOR FORM. THIS IS THE "DIRECT TRANSFORMATION" AND I FIND IT A TEDIOUS PATH.

J. C. 11. 5 RELATIVISTIC KINEMATICS

APPLY CONSERVATION-OF-MOMENTUM TO AN ASSEMBLAGE OF MASS POINTS!

PRE-RELATIVITY: $\vec{p} = m\vec{v}$.

p IS NOT THE SPATIAL PART OF A 4-VECTOR SINCE

$$\vec{v} = \frac{d\vec{r}}{dt}$$

AND dt IS NOT AN INVARIANT

HOWEVER: THIS IS A 4-VECTOR

$$v^\alpha = \frac{\partial x^\alpha}{\partial s}$$

WITH ds THE LINE ELEMENT

$$ds = \frac{1}{\gamma}\, c\,dt.$$

WITH THIS DEFINITION OF v^α!

$$v^\alpha = \left(\gamma \frac{\vec{v}}{c}, \gamma\right).$$

NOTE THIS VELOCITY IS DIMENSIONLESS (A CONVENTION).

v^α IS THE "COVARIANT VELOCITY".

LET'S GENERATE ANOTHER COVARIANT EXPRESSION IN MULTIPLYING BY $m_0 c^2$

Q: WHY IS $m_0 c^2$ A SCALAR?

$$p^\alpha = m_0 \frac{dx^\alpha}{ds} c^2;$$

THIS HAS DIMENSIONS OF ENERGY (A CONVENTION).

CONSERVATION-OF-MOMENTUM/ENERGY FOR TWO PARTICLES INTERACTING AT A POINT TAKES THE FORM

$$p_1^\alpha + p_2^\alpha = \text{CONSTANT}$$

(BEFORE AND AFTER THE POINT-INTERACTION).

THIS REPLACES THE NEWTONIAN

$$m_1 \vec{v_1} + m_2 \vec{v_2} = \text{CONSTANT}$$

NOTICE A SUBTLETY: THIS DEFINITION IS RESTRICED TO DIRECT INTERACTION BETWEEN TWO PARTICLES, NOT "ACTION AT A DISTANCE". THE POINT-INTERACTION AVOIDS HAVING TO DEFINE COMBINED MOMENTUM OF TWO SEPARATED PARTICLES AT THE "SAME" TIME. FURTHER, EXCHANGE OF MOMENTUM BETWEEN SEPARATED PARTICLES MAKES SENSE ONLY IF EACH PARTICLE CONSERVES MOMENTUM WHILE A FIELD ACTS ON IT. SO, STRICTLY SPEAKING, THE MASS-POINTS WE CONSIDER IN DEVELOPING RELATIVISTIC KINEMATICS ARE TRULY POINT-LIKE.

THE COMPONENTS OF 4-MOMENTUM ARE

$$p^{\alpha} = (\gamma m_0 \vec{v} c,\ \gamma m_0 c^2).$$

NOTICE THE SPATIAL PART HAS FORM

$$c\vec{p} = c m \vec{v}; \quad m \equiv \gamma m_0 .$$

HENCE, IN DEMANDING NEWTONIAN CONSERVATION-OF-MOMENTUM, AND DEMANDING IT BE LORENTZ COVARIANT, IMPLIES THE MASS m IS NOT AN INVARIANT (IT DEPENDS ON THE FRAME): TO REITERATE, THIS IS A CONSEQUENCE OF FORMULATING CONSERVATION-OF-MOMENTUM IN COVARIANT FORM.

COVARIANCE ALSO APPLIES TO THE TO THE "TIME" COMPONENT OF p^α.

SO WE NEED TO EXAMINE p^4.

(RECALL $p^4 = mc^2$).

THE USUAL APPROACH IS TO SHOW, AFTER SOME ALGEBRA,

$$\frac{dp^4}{dt} = \vec{v} \cdot \frac{d\vec{p}}{dt}$$

(I'M WORKING THIS INTO A HOMEWORK PROBLEM.)

WE'D LIKE TO PRESERVE OUR CONCEPT OF HAVING $\vec{F}$ RELATED TO $\frac{d\vec{p}}{dt}$. HENCE

$$\vec{F} \cdot \vec{v} = \vec{v} \cdot \frac{d\vec{p}}{dt}$$

IS THE RATE OF DOING WORK ON THE SYSTEM.

IF WE HAVE CONSERVATION-OF-MOMENTUM IN THIS FRAME,

$$\frac{dP^4}{dt} = \frac{dE}{dt} \Rightarrow P^4 = E + \text{CONSTANT}.$$

SINCE ENERGY MATTERS WHEN IT CHANGES!

$$E = P^4 = mc^2 = \gamma m_0 c^2.$$

EXERCISE! SHOW FOR $\beta \ll 1$

$$E \approx m_0 c^2 + \frac{1}{2} m_0 v^2$$

AS IT MUST.

SUMMARIZING

$$P^\alpha = (c\vec{P}, E).$$

$$P^\alpha P_\alpha = E^2 - c^2 P^2 = (m_0 c^2)^2$$

IS AN INVARIANT.

P^α TRANSFORMS LIKE ANY OTHER CONTRAVARIANT VECTOR!

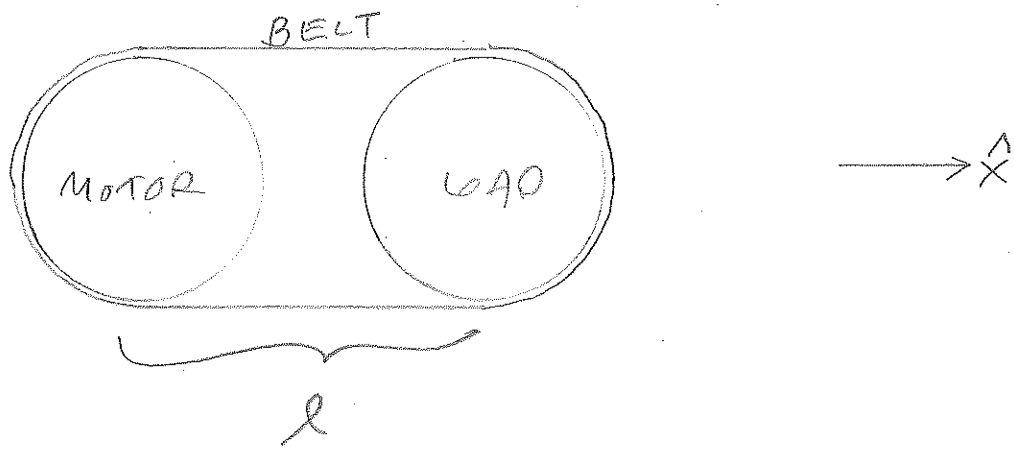
e.g., $cP_x' = \gamma(cP_x - \beta E)$

$P_y' = P_y$; $P_z' = P_z$

$E' = \gamma(E - \beta c P_x)$.

IN THIS FORMALISM, MASS, MOMENTUM AND ENERGY ARE INTERTWINED.

EXAMPLE: MOTOR, BELT, LOAD.



THE POINT OF THIS EXAMPLE IS TO SHOW THAT MOMENTUM IS ASSOCIATED WITH ANY AGENT THAT "TRANSFORMS" ENERGY.

SUPPOSE THE MOTOR TRANSFERS ENERGY AT RATE dE/dt OVER A DISTANCE ℓ.

WITH ENERGY TRANSFERRED AT $dE/dt = \frac{d}{dt}(mc^2)$, MASS IS BEING TRANSFERRED AT RATE $(1/c^2)\, dE/dt$.

WITH MASS TRANSFERRED AT RATE $\frac{1}{c^2}\,dE/dt$ OVER DISTANCE ℓ, THE SYSTEM HAS MOMENTUM

$$p_x = \ell \, \frac{1}{c^2} \frac{dE}{dt}$$

(RECALL THE "ROCKET EQUATION")

ELECTRODYNAMICS GENERALIZES THIS. IF A BODY ABSORBS ENERGY AT A CERTAIN RATE, THE MOMENTUM OF THE BODY INCREASES. IN ORDER TO CONSERVE MOMENTUM, WE ASSOCIATE THE MOMENTUM-DENSITY $\vec{g}$ WITH AN "AGENT" SUPPLYING ENERGY PER TIME PER AREA $\vec{S}$

$$\vec{g} = \frac{1}{c^2}\vec{S}.$$

YOU MAY RECOGNIZE THIS AS THE EXPRESSION FOR RADIATION PRESSURE. THIS IS ALSO THE EXPRESSION FOR MOMENTUM IN FIELDS.

COMMENT: "ETHER" THEORIES DO NOT REQUIRE THIS.

MINKOWSKI FORCE.

YOU HAVE $\vec{F}$ AND $\vec{p}$ IN SOME FRAME. $\vec{F} = \frac{d\vec{p}}{dt}$ IS NOT THE "SPACE" PART OF A 4-VECTOR.

HOWEVER, THIS IS A 4-VECTOR

$$F^\alpha = \frac{dp^\alpha}{ds} = \frac{d}{ds}(c\vec{p},\ mc^2)$$

$$= \left(\gamma \frac{d\vec{p}}{dt},\ \gamma \frac{d}{dt}\{mc\}\right)$$

$$= \left(\gamma \vec{F},\ \gamma \frac{1}{c} \vec{F}\cdot\vec{v}\right)$$

WHERE WE USED

$$\frac{dp^4}{dt} = \frac{d}{dt}(mc^2) = \vec{F}\cdot\vec{v}$$

FROM EARLIER.

THIS TRANSFORMS LIKE ANY OTHER CONTRAVARIANT 4-VECTOR.

TRANSFORMATION OF

$$F^{\alpha} = (\gamma \vec{F}, \gamma \tfrac{1}{c} \vec{F} \cdot \vec{v}).$$

IN THE REST FRAME

$$F^{\alpha} = (\vec{F}, 0).$$

$$F^1 = F_x; \quad F^2 = F_y; \quad F^3 = F_z.$$

- $F'^1 = \gamma F^1; \quad F'^2 = F^2; \quad F'^3 = F^3.$
- $F'^{\alpha} = (\gamma \vec{F}', \gamma \tfrac{1}{c} \vec{F}' \cdot \vec{v})$

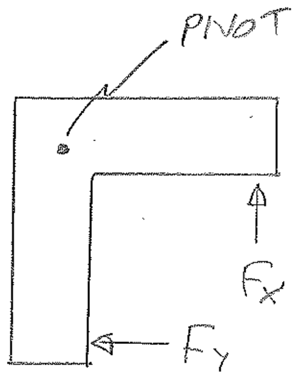
SO $F_x' = F'^1/\gamma = F^1 = F_x$

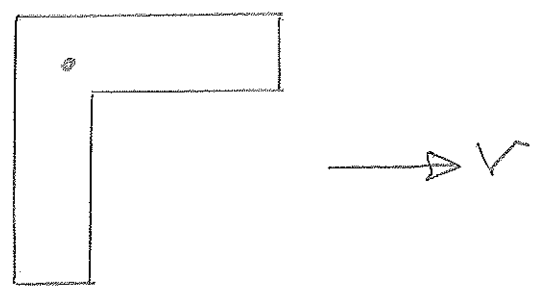
$$F_y' = F'^2/\gamma = F^2/\gamma = F_y/\gamma$$

$$F_z' = F'^3/\gamma = F^3/\gamma = F_z/\gamma.$$

THE TRANSVERSE VECTOR-FORCES TRANSFORM.

INTERESTING APPLICATION!
"MINKOWSKI LEVER"





Σ

LEVER AT REST IN EQUILIBRIUM

Σ'

LEVER MOVING. IS THERE A TORQUE? DOES THE LEVER ROTATE?

FINAL COMMENT: WE'VE MODIFIED NEWTONS 2^{ND} LAW. NOW LOOK AT A PARTICLE SUBJECT TO AN EXTERNAL FORCE

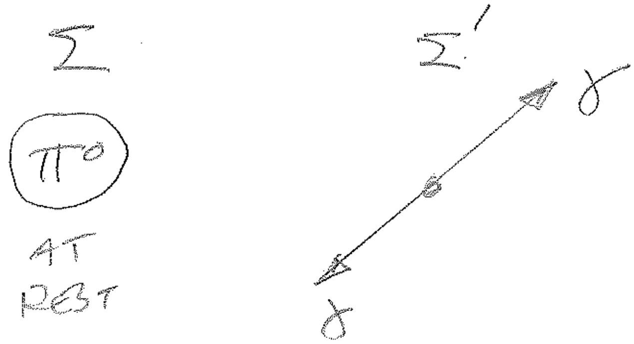
$$\vec{F} = \frac{d}{dt}\vec{p} = \frac{d}{dt}(\gamma m_0 \vec{v})$$

$$= \gamma m_0 \frac{d\vec{v}}{dt} + \underbrace{\frac{\gamma^3}{c^2} m_0 v \vec{v} \frac{dv}{dt}}$$

THE SECOND TERM IS OF A COMPLETELY NEW CHARACTER.

IT'S USUAL IN A COURSE ON SPECIAL RELATIVITY TO INTRODUCE COLLISIONS IN THE CONTEXT OF CONSERVATION-OF-MOMENTUM.

e.g., 2γ DECAY OF THE π^0



Q: WHAT'S E' AND $\vec{p}'$ OF THE PHOTONS?

A: Σ: $p^\alpha = (0, m_0c^2)$

$$p^\alpha p_\alpha = (m_0c^2)^2 \quad \text{INVARIANT.}$$

Σ': $p'^\alpha = (-c\vec{p}' + c\vec{p}', \; 2E')$

$$= (0, 2E').$$

$$p'^\alpha p'_\alpha = (2E')^2$$

$$= (m_0c^2)^2$$

$$E' = m_0c^2/2$$

FOR EACH MASSLESS PHOTON,

$E'^2 = c^2 p'^2$ so $p' = \frac{1}{c}\frac{m_0c^2}{2}$.

J, C, 11, 9

COVARIANT FORMULATION OF ELECTRODYNAMICS.

THE HISTORY AND DEVELOPMENT OF SPECIAL RELATIVITY HEAVILY RELIED ON FEATURES OF THE WAVE EQUATION

$$\left\{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}\left\{\begin{matrix}\Phi \\ \vec{A}\end{matrix}\right\} = \left\{\begin{matrix}-4\pi\rho \\ -\frac{4\pi}{c}\vec{J}\end{matrix}\right\}$$

(CGS).

THIS IS EASILY PLACED INTO COVARIANT FORM.

WE'VE ALREADY SEEN THE D'ALEMBERTIAN

$$\Box = -\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x_\alpha} = -\partial_\alpha \partial^\alpha.$$

WE NEED COVARIANT SOURCES.

LET ρ_0 BE THE "PROPER" CHARGE DENSITY.

DEFINE $J^\alpha = \rho_0 \frac{dX^\alpha}{dS}$ IS A 4-VECTOR

WITH $\rho = \gamma \rho_0$ (WHY?)

$$J^\alpha = \left(\rho \frac{\vec{v}}{c}, \rho\right)$$

THIS REDUCES TO ρ_0 IN THE "PROPER" FRAME.

CURRENT CONSERVATION.

$\vec{\nabla} \cdot \vec{J} + \frac{\partial}{\partial t} \rho = 0$ BECOMES

$$\frac{\partial J^\alpha}{\partial X^\alpha} = 0, \quad \partial_\alpha J^\alpha = 0.$$

THE TOTAL PROPER CHARGE IS AN INVARIANT (SEE LAST LECTURE "CONSERVATION LAWS").

WE COULD ALSO INFER THIS BY BRUTE FORCE.

- WE HAVE $\rho = \gamma \rho_0$.
- HOW DOES A VOLUME ELEMENT dV TRANSFORM?

$$dV = \frac{1}{\gamma} dV_0 \quad \text{(LORENTZ CONTRACTION)}$$

SO $\int \rho \, dV = \rho_0 V_0$

AND CHARGE WITHIN A GIVEN VOLUME IS AN INVARIANT.

SINCE CHARGES ARE QUANTIZED, YOU CAN FIND THE TOTAL CHARGE FROM A SIMPLE COUNTING OPERATION (A NUMBER FROM A COUNTING OPERATION IS AN INVARIANT), SO, AGAIN, TOTAL CHARGE IS AN INVARIANT.

FINAL, THE 4-POTENTIAL.

$$A^\alpha = (c\vec{A}, \Phi), \quad \text{SO}$$

$$\Box A^\alpha = -4\pi J^\alpha \quad \text{(CGS).}$$

THE LORENTZ CONDITION.

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial}{\partial t}\Phi = 0 \quad \text{BECOMES}$$

$$\partial_\alpha A^\alpha = 0.$$

GAUGE TRANSFORMATIONS.

$$\vec{A}' = \vec{A} - \vec{\nabla}\lambda \quad \text{AND}$$

$$\Phi' = \Phi + \frac{\partial}{\partial t}\lambda \quad \text{BECOME}$$

$$A'^\alpha = A^\alpha + \partial^\alpha c\lambda.$$

OK. NOW THE COVARIANT FORMALISM FOR A SINGLE MOVING CHARGE: LIÉNARD - WIECHERT POTENTIALS.