Phys 515
Prob. Set # 6
Grader Notes V1.0

1. We use the symmetrized stress tensor Jackson Eqn 12.133.
   (Jackson calls it $\Theta_{\mu\nu}$; I call it $T_{\mu\nu}$ here).

   This has structure:

   $$T_{\mu\nu} = \begin{cases} 
   3 \times 3 & 1 \times 3 \\
   T_{\mu \nu} & 5 \\
   -5 & H
   \end{cases}$$

   Where $T_{\mu \nu}$ is the (symmetric) Maxwell stress tensor;
   $\mathbf{\mathbf{S}}$ is the Poynting vector (here representing momentum density);
   and $H$ is the field energy density, see Jackson Eqn 12.119.

   $T_{\mu\nu}$ is symmetric by construction.

   $$\text{Trace}(T) = T_{\mu\mu} = T_{\mathbf{\mathbf{KK}}} + T_{\mathbf{\mathbf{44}}}$$

   $$= \frac{1}{4\pi} \left[ E^2 + B^2 - 3 \cdot \frac{1}{2} (E^2 + B^2) \right] + \frac{1}{8\pi} (E^2 + B^2)$$

   $$= 0.$$
2. This is from Landau & Lifschitz, "Classical Fields".

Let note there are an infinite number of these frames. If you've found one such, and boost along perpendicular direction to $\vec{E}$ (or $\vec{B}$) also preserves $\vec{E}$ parallel to $\vec{B}$, this follows from duality; $\vec{E}$ & $\vec{B}$ transform the same way.

So, find one such frame, choose this boost along $\hat{x}$, and since $\hat{x}$ is perpendicular to both $\vec{E}'$ & $\vec{B}'$, with primes denoting the fields in the boosted frame.

We have $E'_2 = B'_2 = 0$ and

$$E'_1 B'_2 - E'_2 B'_1 = 0,$$

then

From Jackson eqns. 11.148

$$\chi \frac{\vec{E}}{\sqrt{E^2 + B^2}} = \frac{\vec{E} \times \vec{B}}{E^2 + B^2} \quad \text{(C65)}.$$

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3. a. The source-free Proca Lagrangian

\[ L = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{m^2}{8\pi} A^\mu A_\mu \]

Recall \( F^{\mu\nu} = J^{\mu\nu} - J\cdot A' \)

_Jackson Eqn. 11, 136_

The Euler-Lagrange equations read (Jackson Eqn. 12.33)

\[ \frac{\partial L}{\partial A^\mu} - \frac{\partial}{\partial x^\nu} \frac{\partial L}{\partial (J^{\nu} A^\mu)} = 0. \]

Following the steps leading to

_Jackson Eqn. 12.89, with the extra mass term, leads to_

\[ \nabla^\mu F^{\mu\nu} + m^2 A^\nu = 0. \]

The source-free Proca equation

b. The divergence of the first term vanishes: \( \nabla^\mu J^{\mu\nu} = 0. \)

Hence \( m^2 A^\nu = 0. \)

**Hence** \( \nabla^\mu F^{\mu\nu} = \nabla^\mu \left( J^{\mu\nu} - J\cdot A' \right) \)

\[ = J^{\mu\nu} \left( \nabla^\mu - J\cdot \nabla A^\mu \right) \]

\[ = \nabla^\nu J^\mu - J\cdot \nabla A^\mu. \]
This leads to the Klein-Gordon equation

\[(d^2 + m^2)A^m = 0.\]

6. Treating \(d^m\) as a momentum operator (per quantum mechanics with \(\psi\) in Schrödinger's equation), giving \(d^2 + m^2\) the source-free energy (E) operator, the dispersion relation reads

\[E^2 = p^2 + m^2.\]

Per Einstein, \(E = \frac{\hbar}{\tau} w\) and

\[p = \frac{\hbar}{\tau} k,\]

or

\[w^2 = k^2 + m^2.\]

The familiar dispersion relation for a massive particle.
4. This is from Landau & Lifshitz, "Classical Fields," with Part C from Marion, "Advanced Electrodynamics."

a. The Lagrangian has a mass term, kinetic energy, and potential energy per Jackson Eqn. 12.7:

\[ L = - \frac{mc^2}{2} - \frac{e^2}{r} \]

b. This is the \( 1/r^2 \) force problem in mechanics. From, e.g., Goldstein, express the Lagrangian in polar coordinates \((r, \theta, \phi)\) with \( \theta \) the polar angle and \( \phi \) the azimuthal angle; this applies to

\[ L = -mc^2 \left\{ 1 - \frac{r^2}{2} \right\}^{1/2} + \frac{e^2 \phi}{r} \]

This is the usual \( 1/r^2 \) force problem with the angular momentum and total energy constants of the motion. (What's the third constant of the motion?)
c. with the total energy \( E \)

a constant of the motion,

\[
E = \sqrt{p^2 + m^2} + \frac{eZ}{r}
\]

is a constant of the motion.

up to a constant,

\[
(E - \frac{eZ}{r})^2 = p^2 + m^2
\]