PHYS 315
SPRING QUARTER 2019
MAY 10, 2019

MID-TERM EXAM
GRADER NOTES
Q1. V.110

a. Recall the (Newtonian) force transformations for a boost along $x$, $x'$:

$$ F'_x = F_x, \quad F'_y = \frac{1}{\gamma} F_y, \quad F'_z = \frac{1}{\gamma} F_z. $$

The force in the particle rest frame is

$$ F_x = 0, \quad F'_y = \frac{e^2}{\gamma}, \quad F'_z = 0 $$

(with $\gamma = \sqrt{1 - \frac{v^2}{c^2}}$ not having length contraction).

In the frame where the particles are moving

$$ F'_x = F_x = 0; $$

$$ F'_y = F_y / \gamma = \frac{e^2}{\gamma}; $$

$$ F'_z = F_z / \gamma = 0. $$

So $F'_1 = \gamma \frac{1}{\beta} \frac{e^2}{\gamma}$

b. For $\beta \to 0$, $\gamma \to 1$ and $F'_1$ is that of $F$.

c. Notice for the entire domain of $V$: $V \subseteq [c, c]$, the sign of the force is unchanged.
THIS SUGGESTS THE SIGN OF THE FORCE IS UNCHANGED ON A BOOST.
Q2. \( \gamma \), 0

a. Suppose you have fields \( \mathbf{E} \) and \( \mathbf{B} \) in some frame. In another frame boosted along \( \hat{x} \):

\[
E'^2 - B'^2 =
(E_{x}'^2 + E_{y}'^2 + E_{z}'^2) - (B_{x}'^2 - B_{y}'^2 - B_{z}'^2)
\]

\[
- E_{x}^2 + \gamma^2 (E_{y} - \beta B_{z})^2
- \gamma^2 (E_{z} + \beta B_{y})^2
- B_{x}^2 - \gamma^2 (B_{y} + \beta E_{z})^2
- \gamma^2 (B_{z} - \beta E_{y})^2
\]

\[
= E_{x}^2 + \gamma^2 \left[ E_{y}^2 - 2\beta E_{y} B_{z} + \beta^2 B_{z}^2 \right]
+ \gamma^2 \left[ E_{z}^2 + 2\beta E_{z} B_{y} + \beta^2 B_{y}^2 \right]
- B_{x}^2 - \gamma^2 \left[ B_{y}^2 + 2\beta B_{y} E_{z} + \beta^2 E_{z}^2 \right]
- \gamma^2 \left[ B_{z}^2 - 2\beta B_{z} E_{y} + \beta^2 E_{y}^2 \right]
\]

\[
= E_{x}^2 + \gamma^2 E_{y}^2 - \gamma^2 B_{z}^2 E_{y}^2 + \gamma^2 E_{z}^2 - \gamma^2 B_{y}^2 E_{z}^2
- B_{x}^2 - \gamma^2 B_{y}^2 + \gamma^2 B_{z}^2 B_{y}^2 - \gamma^2 B_{z}^2 - \gamma^2 B_{z}^2 B_{z}^2
\]

\[
= E_{x}^2 + \gamma^2 \left[ (1-\beta^2) E_{y}^2 + \gamma^2 (1-\beta^2) E_{z}^2 \right]
- B_{x}^2 - \gamma^2 \left[ (1-\beta^2) B_{y}^2 - \gamma^2 (1-\beta^2) B_{z}^2 \right]
\]

\[
= E_{x}^2 + E_{y}^2 + E_{z}^2 - B_{x}^2 - B_{y}^2 - B_{z}^2
= E^2 - B^2, \quad \text{invariant.}
6. Similarly
\[ E' \cdot B' = E'_x B'_x + E'_y B'_y = E'_z B'_z \]
\[ = E_x B_x + \kappa (E'_y - \beta B_x) \cdot (B_y + \beta B_x) \]
\[ + \kappa (E'_z + \beta B_y) \cdot (B_z - \beta B_y) \]
\[ = E_x E_y + \kappa^2 (1 - \beta^2) E_y B_y + \kappa^2 (1 - \beta^2) E_z B_z \]
\[ = E_x B_x + E_y B_y + E_z B_z \]
\[ = E \cdot B, \text{ IN Variant.} \]

C. One way to proceed is to recall the relation of \( S \) to momentum density of the fields. This relation suggests the electromagnetic field energy density \( U \) is paired with \( S' \), so there's the invariant \( S^2/c - U^2 \). You'd need to argue why the \( d^3x \) contraction doesn't affect the invariant (it wouldn't if the invariant vanishes, e.g.).

Another way to proceed is to write \( S' \) as
\[ S' = \left( \frac{c}{4 \pi} \right)^2 \left\{ E \cdot B - (E^2, B^2) \right\} \]
Notice $E' \cdot B'$ is an invariant, so we need only consider how $E'^{2} B'^{2}$ transforms.

The thing to notice is $E'^{2} B'^{2}$ is a cross-term in the square of the energy density. So, combine the "extra" term $E'^{2} B'^{2}$ from $S^{2}$ with the square of the energy density:

$$E'^{2} B'^{2} = \left\{ \frac{1}{2} E'^{2} + \frac{1}{2} B'^{2} \right\}$$

$$= \left\{ \frac{1}{2} E'^{2} - \frac{1}{2} B'^{2} \right\}.$$ 

So, $\left(\frac{S}{c}\right)^{2} - V^{2}$ is an invariant because it contains a term $E'^{2} B'^{2}$ and a term $E^{2} - B^{2},$ both invariant.
d. Recall some properties of a plane wave: $E$ and $B$ are orthogonal and $E = B$.

From (b) $E'$ and $B'$ remain orthogonal in any inertial frame, from (a) $E'^2 - B'^2$ remains zero in any inertial frame, so $E' = B'$. Hence, it seems the plane wave is a plane wave in any inertial frame.
Q 3, VI. 0

a. WE FOLLOW THE PATH FROM THE SIMILAR HOMEWORK PROBLEM; IN THE FORWARD DIRECTION

\[ r = \sqrt{x^2 + y^2 + z^2} \]

\[ \approx 2 + \frac{1}{2} \frac{x^2 + y^2}{z^2} \quad \text{Hence} \]

\[ \frac{E(r)}{E_0} \approx e^{ikr} \left( 1 + \frac{1}{2} \frac{\rho^2}{z^2} \right) e^{ikr} f(\rho) \]

with \( \rho^2 = x^2 + z^2 \)

THE RADIAL-CYLINDRICAL COORDINATE,

\[ = e^{ikr} \left( 1 + \frac{1}{2} \frac{\rho^2}{z^2} \right) \]

\[ = e^{ikr} + \frac{1}{2} e^{ikr} \]

\[ + \text{terms } \frac{1}{2^2}, \frac{1}{2^3}, \ldots \]

(Where we expanded \( \frac{1}{2^2} \) and kept \( \frac{1}{2} \)).
WE NEED TO SQUARE THIS:

\[
\left| \frac{E}{E_0} \right|^2 = \frac{E}{E_0} \left( \frac{E}{E_0} \right)^* = \left\{ e^{i \mathbf{k} \cdot \mathbf{r}} + \frac{1}{2} e^{i \mathbf{k} \cdot \mathbf{r}} e^{i \mathbf{k} \cdot \mathbf{r}} \right\} \cdot \left\{ e^{-i \mathbf{k} \cdot \mathbf{r}} + \frac{1}{2} e^{-i \mathbf{k} \cdot \mathbf{r}} e^{-i \mathbf{k} \cdot \mathbf{r}} \right\} = 1 + \frac{1}{2} e^{i \mathbf{k} \cdot \mathbf{r}} e^{i \mathbf{k} \cdot \mathbf{r}} f(0) + \frac{1}{2} e^{i \mathbf{k} \cdot \mathbf{r}} e^{-i \mathbf{k} \cdot \mathbf{r}} f^*(0)
\]

KEEPING TERMS LOWER THAN $\frac{1}{2} \mathbf{k} \cdot \mathbf{r}$,

\[
\left| \frac{E}{E_0} \right|^2 \approx 1 + \frac{1}{2} e^{i \mathbf{k} \cdot \mathbf{r}} f(0) + \frac{1}{2} e^{-i \mathbf{k} \cdot \mathbf{r}} f^*(0)
\]

RECALL FOR A COMPLEX $M$,

\[
M + M^* = \text{Re} M + i \text{Im} M + \text{Re} M - i \text{Im} M = 2 \text{Re} M
\]

\[
\left| \frac{E}{E_0} \right|^2 \approx 1 + \frac{1}{2} \text{Re} \left\{ e^{i \mathbf{k} \cdot \mathbf{r}} f(0) \right\}
\]

NOW, WE'RE IN A POSITION TO INTEGRATE THIS OVER THE DISK,
\[
\int_{\theta=0}^{2\pi} \int_{\rho=0}^{R} \frac{e^{i k \rho^2}}{2\pi} \rho \, d\rho \, d\theta = \pi R^2 + \frac{4\pi}{2} \Re \left\{ \frac{e^{i k \rho^2}}{2\pi} \rho \, d\rho \right\}
\]

We need to evaluate the integral. Let \( M = \rho^2 \), then \( R^2 \frac{e^{i k M}}{2\pi} \int_{\rho=0}^{R} \rho \, d\rho = \int_{0}^{\infty} e^{i k M} \, dM/2 \).

Because \( R^2 \gg \frac{2\pi}{k} \), the upper limit \( \to \infty \) and the integral is approximately

\[
\frac{1}{2} \int_{0}^{\infty} e^{\frac{ikM}{2}} \, dM.
\]

You can wick-rotate \( \int e^{-aq} \, dx = \frac{1}{a} \), or use Gaussian integration, etc. To find \( \int_{0}^{\infty} e^{iqx} \, dx = \frac{-1}{iq} \).

Hence,
\[ \int |E/E_0|^2 \, d\alpha = 4\pi R^2 - \frac{4\pi}{K} \text{Im} \, f(0), \]

over disk

b. \( 4\pi R^2 \) is the cross-section for scattering without the scatterer. The minus sign indicates some forward-going energy is removed from the beam; this can come from scattering or absorption. Hence, because we know the optical theorem

\[ \sigma_{\text{tot}} = (\sigma_{\text{scatt}} + \sigma_{\text{abs}}) = \frac{4\pi}{K} \text{Im} \, f(0), \]

we expect the result in (a).