Electrodynamics III: Assignment 3. Due May 3 at 11:00am in class or 10:45am in the instructor's mailbox.

1. Simultaneity. This is a classic special-relativity problem recast some time ago into the Star Trek universe by Prof. Miguel Morales. A Federation spaceship is in Federation territory at rest with respect to the border between Federation and Klingon space. According to instruments on the Federation spaceship, the border is 6 light minutes distant. A Klingon spaceship flies close by the Federation ship directly towards the border at speed $\beta=0.6$. Exactly 5 minutes later, according to a clock on the Federation spaceship, the Klingon ship emits a photon torpedo that eventually hits the Federation ship. Then, a short time later, according to instruments on the Federation ship, the Klingons cross back into their own space. Prof. Morales informs us a photon torpedo travels at the speed of light.

a. Make a Federation-spaceship-based spacetime drawing of these events, including the Federation spaceship, the Klingon spaceship, the photon torpedo, and the boundary.

b. According to instruments on the Federation ship, how long after the Klingon flies by the Federation ship does the photon torpedo hit the Federation ship? Hint: you can infer this from the spacetime diagram.

c. Also, according to instruments on the Federation ship, how long after the Klingon flies by the Federation ship does the Klingon ship cross the border? Hint: This can also be inferred from the spacetime diagram.

d. An interstellar war is at stake in your answer to this question: The Klingon commander claims that the Federation is wrong: according to instruments on the Klingon ship, the Federation ship was hit when the Klingon ship was in Klingon territory. According to instruments on the Klingon ship, when the Federation ship was hit, was or was not the Klingon ship on its own side of the border? Hint: This can also be inferred from the spacetime diagram.

2. Prove the assertion in class: If a divergenceless 4-vector A^{μ} has components non-zero only in some bounded spatial region, then the regular-volume integral over 3D space $\iiint A^4 dv$ is a Lorentz invariant. Hint: you may want to apply the 4D analog of Gauss's law we introduced in class. Also note for the surface integration: since da_{μ} is a component of the 3D "surface" normal to the vector A^{μ} in 4-space, $da_4 = dv$.

3. In some frame there are *N* static charges within some small volume δV_0 where the small volume is a cube of side *d* with the sides of the cube parallel with the inertial frame Σ_0 axes. In some other (non-rest) inertial frame Σ the element has volume δV and is moving (with its enclosed charges) along the x-axis at velocity **U**.
a. Show that, in general, if you have two frames moving at relative velocity **U**, the volumes are related by
$\delta v' = \delta V \frac{\sqrt{1-v^2/_{c^2}}}{1-\mathbf{U}\cdot\mathbf{v}/_{c^2}}$ where **U** is the velocity of the volume in the unprimed frame and the relative velocity of the two frames is **V**.
To understand the denominator, you may want to consider the simpler case of the length of a corresponding meter stick instead of a small volume.
b. Hence, show that the charge density and current density transform as
$J' = \gamma \boldsymbol{J}_{\parallel} + \boldsymbol{J}_{\perp} - \gamma \mathbf{V}\rho$, $\rho' = \gamma(\rho - \mathbf{v}\cdot\mathbf{J}/c^2)$ where the current in the first case is decomposed unto currents along- and parallel-components to **U**. You may want to consider the simpler case of the velocity transformation in along- and parallel-components of velocity. For now, assume the amount of charge in the volume is a Lorentz invariant. We'll come back to this transformation when we bundle the current density and charge density into a 4-vector.

This is a homework problem in development, it will not likely be graded. But please attempt it.

4. Relativistic mechanics. Show, e.g., by direct differentiation, that the time derivative of the "time" component of the momentum can be written in the "sensible" form:

$$\frac{dP^4}{dt} = \vec{v} \cdot \frac{d}{dt}\left(\frac{m_0 \vec{v}}{\sqrt{1 - v^2/c^2}}\right)$$