Physics 323-Electrodynamics **Exam 1**
Spring 2016 **April 21, 2016**

Printed Name ___________________________________
last *first*

I certify that the work I shall submit is my own creation, not copied from any source, and that I shall abide by the examination procedures outlined below.

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READ THIS ENTIRE PAGE NOW, BEFORE THE STARTING BELL.
Do not open the exam before the starting bell.
You will have 80 minutes after the bell to complete the examination.
Exam papers will no longer be accepted after 85 minutes have elapsed.

NO CELL PHONES, TEXT MSG, etc. ALLOWED AT ANY TIME

Before the exam begins:

- Print and sign your name, and write your student ID number in the spaces on this page (above).

During the exam:

- **Important!** When the exam begins, print your name and student ID number at the top of each and **every** page. Do this **first** when you are told to open your exam.
- If you are confused about a question, raise your hand and ask for an explanation.
- If you cannot do one part of a problem, move on to the next part.
- This is a closed book examination. You may not use your own notes.
- You may use a calculator, but not a computer or other programmable device.

For all problems:

- If you need more space than is available to answer any part of a problem, use the **back side of the same page** to complete your answer. Clearly indicate to the grader that you used the back side. Scratch paper will be ignored.
- Show your work in enough detail so that the grader can follow your reasoning and your method of solution. Circle your answers, and state units if appropriate. For numerical answers significant figures should match the number of significant figures in the numerical values given in the problem (usually 2 or 3).
- You will lose 3 points on each problem for which you don't put your name and student number at the top of the problem page. You lose 3 points for not filling out information at the top of this page.

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$	(permeability of free space)
$c = 3.00 \times 10^8\ \mathrm{m}/s$	(speed of light)
$e = 1.60 \times 10^{-19}\ \mathrm{C}$	(charge of the electron)
$m = 9.11 \times 10^{-31}\ \mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$\mathbf{F} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qQ}{r^2}\hat{r}$ Coulomb's Law $\quad \mathbf{F} = Q\mathbf{E} + Q\mathbf{v}\times\mathbf{B}$ Lorentz force

$\mathbf{E} = \dfrac{1}{4\pi\varepsilon_0}\iiint \dfrac{\rho(\mathbf{r}')}{r^2}\hat{r}\,d\tau'$ electric field from continuous charge distribution

$$\oiint \mathbf{E} \bullet d\mathbf{a} = \frac{1}{\varepsilon_0} Q_{\text{encl}}$$ Gauss' Law (integral) $\oint \mathbf{E} \bullet d\mathbf{l} = 0$ (statics)

$$V(\mathbf{r}) = -\int_{\wp}^{\mathbf{r}} \mathbf{E} \bullet d\mathbf{l}$$ Electrostatic potential, and $\mathbf{E} = -\nabla V$ (statics)

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$ Poisson's equation, and $\nabla^2 V = 0$ Laplace's equation (regions of no charge)

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \quad \text{and} \quad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(\mathbf{r}')}{r} d\tau'$$ (setting reference point at infinity)

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}} \quad \text{and} \quad \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = \frac{1}{\varepsilon_0}\sigma$$ boundary conditions

$$D_{\text{above}}^{\perp} - D_{\text{below}}^{\perp} = \sigma_f$$

$$V(\mathbf{b}) - V(\mathbf{a}) = \frac{W}{Q} \qquad W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \quad \text{and} \quad W = \frac{1}{2}\iiint \rho V \, d\tau$$

$$Q = CV \qquad W = \frac{1}{2}CV^2 \qquad W = \frac{1}{2}Q^2 / C$$ capacitors

$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho$$ Poisson's equation $\qquad \nabla^2 V = 0$ Laplace's equation

$$V(x,y) = \sum_{n=0}^{\infty}(A_n e^{+kx} + B_n e^{-kx})(C_n \sin ky + D_n \cos ky)$$ solution to Laplace's equation in Cartesian coordinates in two dimensions

$$\int_0^a \sin(n\pi\frac{y}{a})\sin(n'\pi\frac{y}{a})dy = \begin{cases} 0 & n \neq n' \\ \frac{a}{2} & n = n' \end{cases}$$ orthogonality of sines

$$\int_0^a \sin(n\pi\frac{y}{a})dy = \begin{cases} 0 & \text{n even} \\ \frac{2a}{n\pi} & \text{n odd} \end{cases} \qquad e^{+kx} + e^{-kx} = 2\cosh kx$$

$\int_0^\pi \cos^2\theta \sin\theta \, d\theta = 2/3$

$P_0(x)$	$= 1$
$P_1(x)$	$= x$
$P_2(x)$	$= (3x^2 - 1)/2$
$P_3(x)$	$= (5x^3 - 3x)/2$
$P_4(x)$	$= (35x^4 - 30x^2 + 3)/8$
$P_5(x)$	$= (63x^5 - 70x^3 + 15x)/8$

TABLE 3.1 Legendre Polynomials.

$$V(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}})P_\ell(\cos\theta)$$

solution to Laplace's equation in spherical coordinates with azimuthal symmetry

$$\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\,d\cos\theta = \begin{cases} 0 & \ell \neq \ell' \\ \dfrac{2}{2\ell+1} & \ell = \ell' \end{cases}$$ orthogonality of Legendre polynomials

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$ potential outside a neutral conducting sphere in uniform field

Law of cosines $c^2=a^2+b^2-2ab\cos\gamma$, with γ the angle opposite to side c.

$$\frac{1}{\Re} = \frac{1}{r}\sum_0^\infty\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$ 1/r expansion in Legendre polynomials

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_0^\infty \frac{1}{r^{n+1}}\iiint (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')d\tau'$$ multipole expansion

$$\vec{p} = \iiint \vec{r}'\rho(\vec{r}')d\tau' \quad \vec{p} = \sum_1^n q_i\vec{r}_i' \quad V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\bullet\hat{r}}{r^2}$$ dipole moment

$$\vec{E}_{\text{dip}} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

$$\mathbf{F}_{\text{mag}} = \int I\,d\mathbf{l}\times\mathbf{B}$$

$\mathbf{K} = d\mathbf{I}/d\ell_\perp \quad \mathbf{J} = d\mathbf{I}/da_\perp$ surface and volume currents

$\nabla\bullet\mathbf{J} + \frac{d\rho}{dt} = 0$ conserved current, continuity equation

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}I\int\frac{d\mathbf{l}'\times\hat{r}}{|\mathbf{r}-\mathbf{r}'|^2} \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{K}\times\hat{r}}{|\mathbf{r}-\mathbf{r}'|^2}da' \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}I\int\frac{\mathbf{J}\times\hat{r}}{|\mathbf{r}-\mathbf{r}'|^2}d\tau'$$

$$\oint\mathbf{B}\bullet d\mathbf{l} = \mu_0 I_{\text{enc}}$$

$\mathbf{B} = \nabla \times \mathbf{A}$ magnetic vector potential.

$\nabla \bullet \mathbf{A} = 0$ "Coulomb gauge" convention $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ (for Coulomb gauge)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} da' \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{I}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{l}'$$

$$\oint \mathbf{A} \bullet d\mathbf{l} = \Phi_m$$

$$B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0 \qquad B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K \qquad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 \mathbf{K} \times \hat{n}$$

$$\mathbf{A}_{\text{above}} - \mathbf{A}_{\text{below}} = 0 \qquad \frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$$

$$\mathbf{m} = I \iint \hat{n}\, da = I\mathbf{a}$$

$$\mathbf{m} = \frac{1}{2} \oint \mathbf{r} \times \mathbf{I}\, d\ell \quad \mathbf{m} = \frac{1}{2} \iint \mathbf{r} \times \mathbf{K}\, da \quad \mathbf{m} = \frac{1}{2} \iiint \mathbf{r} \times \mathbf{J}\, d\tau$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{r}}{r^2}$$

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \quad \mathbf{F} = \nabla(\mathbf{m} \bullet \mathbf{B}) \quad U = -\mathbf{m} \bullet \mathbf{B}$$

$$\mathbf{J}_b = \nabla \times \mathbf{M} \quad \mathbf{K}_b = \mathbf{M} \times \hat{n}$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \qquad \nabla \times \mathbf{H} = \mathbf{J}_f \qquad \oint \mathbf{H} \bullet d\mathbf{l} = I_{\text{f,encl}}$$

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}) \qquad \mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{n}$$

$$\mathbf{M} = \chi_m \mathbf{H} \qquad \mu = \mu_0 (1 + \chi_m) \qquad \mathbf{B} = \mu \mathbf{H}$$

$$\mathbf{J} = \sigma \mathbf{E} \qquad V = IR \quad P = VI = I^2 R$$

$$\mathcal{E} = \oint \mathbf{E} \bullet d\ell \quad \mathcal{E} = -\frac{d\Phi_M}{dt}$$

$$\Phi_2 = M\, I_1 \quad \Phi = LI \quad \mathcal{E}_2 = -M \frac{dI_1}{dt} \quad \mathcal{E} = -L \frac{dI}{dt} \quad W = \frac{1}{2} LI^2$$

$$\nabla \bullet \left(\frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \right) + \frac{\partial}{\partial t} \left(\frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) = -\mathbf{E} \bullet \mathbf{J}$$ Poynting's theorem (differential)

$$\frac{dW}{dt} = -\frac{d}{dt} \iiint \left(\frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) d\tau - \oiint \left(\frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \right) \bullet \hat{n}\, da$$ Poynting's thm (integral)

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{V^2}\frac{\partial^2 f}{\partial t^2} = 0 \quad \lambda = 2\pi / k \quad T = 2\pi / kV \quad \nu = 1/T \quad \omega = 2\pi\nu = kV$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

vacuum $\nabla^2\mathbf{E} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \nabla^2\mathbf{B} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \quad c = 1/\sqrt{\mu_0\varepsilon_0}$

$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)} \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)} \quad E_0 = cB_0$ (vacuum)

$\langle u \rangle = \frac{1}{2}\varepsilon_0 E_0^2 \quad \langle \mathbf{S} \rangle = \frac{1}{2} c\varepsilon_0 E_0^2 \hat{k}$ (vacuum) $\quad I = \langle S \rangle \quad \langle \mathbf{p} \rangle = \frac{1}{2c}\varepsilon_0 E_0^2 \hat{k}$

lossless $\nabla^2\mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \nabla^2\mathbf{B} - \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \quad v = 1/\sqrt{\mu\varepsilon} = c/n \quad n = \sqrt{\mu\varepsilon / \mu_0\varepsilon_0}$

$$u = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right) \quad \mathbf{S} = \frac{1}{\mu}\mathbf{E}\times\mathbf{B} \quad I = \frac{1}{2}\varepsilon v E_0^2$$

$$\tan\theta_B \cong n_2 / n_1 \quad \sin\theta_c = n_2 / n_1$$

$\tilde{E}_{0R} = \frac{1-\beta}{1+\beta}\tilde{E}_{0I} \quad \tilde{E}_{0T} = \frac{2}{1+\beta}\tilde{E}_{0I} \quad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$ normal incidence

lossy $\nabla^2\mathbf{E} - \mu\sigma\frac{\partial \mathbf{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \nabla^2\mathbf{B} - \mu\sigma\frac{\partial \mathbf{B}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$

$$\tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega \quad d = 1/\operatorname{Im}\tilde{k}$$

$$\operatorname{Re}\tilde{k} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1\right]^{1/2} \quad \operatorname{Im}\tilde{k} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}$$

poor conductor $\operatorname{Re}\tilde{k} = \sqrt{\mu\varepsilon}\,\omega \quad \operatorname{Im}\tilde{k} = \sqrt{\frac{\mu}{\varepsilon}}\frac{\sigma}{2}$

good conductor $\operatorname{Re}\tilde{k} = \operatorname{Im}\tilde{k} = \sqrt{\frac{\omega\sigma\mu}{2}}$

$$\ddot{y}+\gamma\dot{y}+\omega_0^2 y=\frac{e}{m}E(t) \qquad E(t)=E_0 e^{i\omega t} \qquad y(t)=y_0 e^{i\omega t} \qquad y_0=\frac{e/m}{\left(\omega_0^2-\omega^2\right)-i\gamma\omega}E_0$$

$$\varepsilon=\varepsilon_0+\frac{Nfe^2}{m}\frac{1}{\left(\omega_0^2-\omega^2\right)}-i\gamma\omega$$

dilute $n=\frac{c}{\omega}\operatorname{Re}\tilde{k}=1+\frac{1}{2\varepsilon_0}\frac{Nfe^2}{m}\frac{\left(\omega_0^2-\omega^2\right)}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}$

$$\alpha=2\operatorname{Im}\tilde{k}=1+\frac{\omega}{c\varepsilon_0}\frac{Nfe^2}{m}\frac{\gamma\omega}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}$$

plasma $\sigma=\frac{Nfe^2}{m}\frac{1}{\gamma-i\omega}$ dilute plasma $\sigma=\frac{Nfe^2}{m}\frac{1}{-i\omega}$ $\omega_p=\sqrt{\frac{Nfe^2}{\varepsilon_0 m}}$

dilute plasma $\tilde{k}^2=\frac{1}{c^2}\left\{\omega^2-\omega_p^2\right\}$

waveguide transverse fields

$$E_x=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial x}-\omega\frac{\partial B_z}{\partial y}\right)$$

$$E_y=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right)$$

$$B_x=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial B_z}{\partial x}-\frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial B_z}{\partial y}-\frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

waveguide longitudinal fields

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\omega/c^2\right)-k^2\right]E_z=0$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\omega/c^2\right)-k^2\right]B_z=0$$

“waveguide equation” $\frac{1}{\lambda_g^2}+\frac{1}{\lambda_c^2}=\frac{1}{\lambda_0^2}$

TE & TM guided wavenumber $k^2 = \left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]$

TE & TM cutoff angular frequency $\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$

phase velocity $v = \omega / k$ group velocity $\frac{1}{dk / d\omega}$

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum: $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$