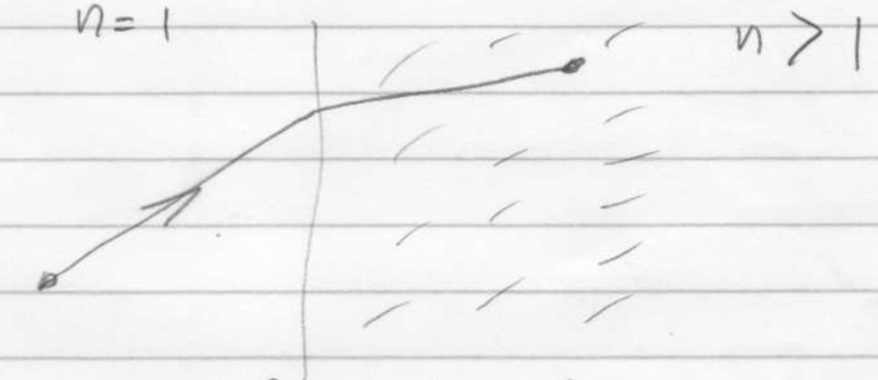
EULER-LAGRANGE DERIVATION OF
MAXWELL'S EQUATIONS.

THE LAGRANGIAN PRINCIPLE.

FERMAT "PRINCIPLE OF LEAST TIME"



DERIVED SNELL'S LAW OF
WILLEBRORD SNELLIUS.

CUTE, BUT UNHELPFUL. IT REPLACES
A SIMPLE FORMULA (SNELL'S LAW)
WITH A PRINCIPLE REQUIRING THE
WHOLE APPARATUS OF THE
CALCULUS OF VARIATIONS.

NEWTON'S LAWS

PARTICLE MASS m FEELS FORCE F, MOVES IN 1D FROM $A \to B$.

IT'S EXACT PATH IS GIVEN BY

$$F = m\ddot{x}$$

INTEGRATE THIS TO FIND $X(t)$.

THIS RUNS INTO PROBLEMS W/ QUANTUM MECHANICS: $X(t)$ IS THE POSITION AT EVERY TIME t FROM $t=0$ TO $t=\tau$.

BUT QUANTUM MECHANICS SAYS AT $t=0$ IT CAN BE AT A, AND AT $t=\tau$ IT CAN BE AT B. BUT YOU CAN'T KNOW PRECISELY WHAT IT DID IN-BETWEEN.

NEED DYNAMICS OF
JOSEPH-LOUIS LAGRANGE &
WILLIAM ROWAN HAMILTON.

LAGRANGE & HAMILTON ASKED HOW THE TOTAL ENERGY $E = T + V$ VARIES OVER THE PARTICLE TRAJECTORY

$E = T + V$ IS CONSTANT,
BUT T & V CAN VARY
OVER THE TRAJECTORY

OVER THE TRAJECTORY

$$\overline{E} = \overline{T} + \overline{V} \qquad (\overline{E} = \text{AVERAGE ENERGY})$$

$$\overline{T} = \frac{1}{\tau}\int_0^{\tau} \frac{1}{2} m \left\{ \dot{x}(t) \right\}^2 dt$$

$$\overline{V} = \frac{1}{\tau}\int_0^{\tau} V(x)\, dt$$

BACK TO LAGRANGE & HAMILTONS' QUESTION

RECALL CALCULUS OF VARIATIONS;
A FUNCTION VS. A FUNCTIONAL

A FUNCTION TURNS A NUMBER INTO ANOTHER NUMBER: e.g.,

$$f(x) = 3x^2$$

FUNCTIONAL TURNS A FUNCTION INTO A NUMBER. e.g.,

$$F(f) = \int_0^1 f(x)\,dx$$

$f(x)$ COULD BE x^2 OR $\sin x$, e.g.

FUNCTION DERIVATIVE

$$\frac{df}{dx} = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

FUNCTIONAL DERIVATIVE

$$\frac{\delta F}{\delta f(x)} = \lim_{\varepsilon \to 0} \frac{F\left[f(x') + \varepsilon\delta(x-x')\right] - F\left[f(x')\right]}{\varepsilon}$$

KEY EXAMPLES:

1. LET $\bar{V}[x] = \frac{1}{T}\int_0^T V[x(t)]\,dt$

THEN $$\frac{\delta \bar{V}[x]}{\delta x(t)} = \frac{1}{T} V'[x(t)]$$

2. LET $\bar{T}[x] = \frac{1}{T}\int_0^T \frac{1}{2} m [\dot{x}(t)]^2\,dt$

THEN $$\frac{\delta \bar{T}[x]}{\delta x(t)} = -\frac{1}{T} m \ddot{x}$$

BACK TO LAGRANGE & HAMILTON.
FOR THE CLASSICAL TRAJECTORY
WE HAVE $m\ddot{x} = -\frac{dV}{dx}$

HENCE NEAR THE CLASSICAL TRAJECTORY

$$\frac{\delta \bar{V}[x]}{\delta x(t)} = \frac{\delta \bar{T}[x]}{\delta x(t)}$$

$\bar{V}$ & $\bar{T}$ CHANGE BY THE SAME AMOUNT

AND THEREFORE

$$\frac{\delta}{\delta x(t)}\left\{\bar{T}[x] - \bar{V}[x]\right\} = 0$$

T − V IS SPECIAL.

DEFINING $L = T - V$ ("THE LAGRANGIAN").

THE ACTION S IS DEFINED AS

$$S = \int_0^{\tau} L \, dt$$ (S HAS UNITS OF PLANCK'S CONSTANT)

AND

$$\frac{\delta S[x]}{\delta x(t)} = 0$$

THIS IS HAMILTON'S PRINCIPLE OF LEAST ACTION. FROM MECHANICS

KEY EXERCISE:

$$0 = \frac{\delta S[x]}{\delta x(t)} = \underbrace{\frac{\delta L[x]}{\delta x(t)} - \frac{d}{dt} \frac{\delta L[x]}{\delta \dot{x}(t)}}_{\text{EULER-LAGRANGE EQUATION,}}$$

THIS GIVES "THE EQUATIONS OF MOTION" FROM L

FINALLY, THE "LAGRANGIAN DENSITY" $\mathcal{L}$ FOR A CONTINUOUS SYSTEM:

$$L = \int \mathcal{L} \, dx$$

SO

$$S = \int L \, dt = \int \mathcal{L} \, dt \, dx$$

SUPPOSE $\mathcal{L}$ DEPENDS ON SOME FUNCTION ϕ & ITS DERIVATIVE:

$\frac{\partial \phi}{\partial x_\mu}$ IN SPACE TIME 3+1 D

THE EULER-LAGRANGE EQUATION:

$$\frac{\delta S\left[\phi, \frac{\partial \phi}{\partial x_\mu}\right]}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x_\mu}\left\{\frac{\partial \mathcal{L}}{\partial\left\{\frac{\partial \phi}{\partial x_\mu}\right\}}\right\} = 0$$

THE ACTION

$$S = \iiiint \mathcal{L} \, d^4x$$

SINCE d^4x IS A SCALAR (WHY?)
$\mathcal{L}$ IS A SCALAR IF
S IS TO BE A SCALAR
(AN THE FORMALISM WOULD NOT BE IF S WERE NOT).

IN ELECTRODYNAMICS,
WHAT SCALARS DO WE HAVE?

1. $F^{\mu\nu}\tilde{F}_{\mu\nu}$ IS A PSEUDOSCALAR (BAD)
(ALSO $\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = F^{\mu\nu}F_{\mu\nu}$)
2. $F^{\mu\nu}F_{\mu\nu}$ IS A SCALAR
3. RECALL GRIFFITHS EQUATIONS 2.43 & 7.32

$$W = \frac{1}{2}\iiint \rho V \, d\tau + \frac{1}{2}\iiint \vec{J}\cdot\vec{A} \, d\tau$$

THESE LEAD TO

$$\mathcal{L} = -\frac{1}{4c\mu_0} F^{\mu\nu}F_{\mu\nu} - \frac{\mu_0}{4\pi} J_\mu A^\mu$$

FOR ELECTROMAGNETISM

RECALL $F^{\mu\nu} = \dfrac{\partial A^\nu}{\partial x^\mu} - \dfrac{\partial A^\mu}{\partial x^\nu}$

THE FIELD DEGREES-OF-FREEDOM ARE NATURALLY A^μ (A^μ APPEARS IN $\mathcal{L}$).

FOR EULER-LAGRANGE EQUATIONS,

FIRST EVALUATE $\dfrac{\partial \mathcal{L}}{\partial \left\{ \dfrac{\partial A^\mu}{\partial x^\nu} \right\}}$ (TRICKY)

$$F_{\alpha\beta} F^{\alpha\beta} = \left\{ \frac{\partial A_\beta}{\partial x_\alpha} - \frac{\partial A_\alpha}{\partial x_\beta} \right\} \left\{ \frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x^\beta} \right\}$$

$$= g_{\beta\sigma} g_{\alpha\lambda} \left\{ \frac{\partial A^\sigma}{\partial x^\lambda} - \frac{\partial A^\lambda}{\partial x^\sigma} \right\} \left\{ \frac{\partial A^\beta}{\partial x^\alpha} - \frac{A^\alpha}{x^\beta} \right\}$$

$$= g_{\beta\sigma} g_{\alpha\lambda} \left\{ \frac{\partial A^\sigma}{\partial x^\lambda} \frac{\partial A^\beta}{\partial x^\alpha} + \cdots \text{ (4 TERMS)} \right\}$$

$$\frac{\partial F_{\alpha\beta} F^{\alpha\beta}}{\partial \left\{ \dfrac{\partial A^\mu}{\partial x^\nu} \right\}} = g_{\beta\sigma} g_{\alpha\lambda} \times$$

$$\left\{ \frac{\partial A^\sigma}{\partial x^\lambda} \delta^\mu_\beta \delta^\nu_\alpha + \frac{\partial A^\beta}{\partial x^\alpha} \delta^\mu_\sigma \delta^\nu_\lambda + \cdots \right\}$$

$$= g_{\beta\sigma} g_{\alpha\lambda} \times$$

$$\left\{ \delta^\lambda_\nu \delta^\sigma_\mu F^{\alpha\beta} - \delta^\sigma_\nu \delta^\lambda_\mu F^{\alpha\beta} + \delta^\alpha_\nu \delta^\beta_\mu F^{\lambda\sigma} - \delta^\beta_\nu \delta^\alpha_\mu F^{\lambda\sigma} \right\}$$

THIS IS
PRETTY COMPLICATED "INDEX GYMNASTICS".
HOWEVER: $g_{\alpha\beta}$ IS SYMMETRIC,
& $F^{\alpha\beta}$ IS ANTI-SYMMETRIC.
THEREFORE ALL 4 OF THE PREVIOUS TERMS ARE THE SAME
SO

$$\frac{\partial \mathcal{L}}{\partial \left\{ \frac{\partial A^\mu}{\partial x^\nu} \right\}} = \frac{1}{4\pi} F_{\mu\nu}$$

IT'S EASIER TO EVALUATE $\frac{\partial \mathcal{L}}{\partial A^\mu}$

$$\frac{\partial \mathcal{L}}{\partial A^\mu} = -\frac{\mu_0}{4\pi} J_\mu$$

HENCE $\frac{\partial}{\partial x_\nu} F_{\mu\nu} = \mu_0 J_\mu$

THIS
CONTAINS THE TWO INHOMOGENEOUS MAXWELL'S EQUATIONS (FROM LAST LECTURE)

WHAT ABOUT THE HOMOGENEOUS MAXWELL EQUATIONS? SUBTLE.

FIRSTLY,

$\vec{\nabla}\cdot\vec{B} = 0$ HAS ROOTS IN CHOOSING $\vec{B} = \vec{\nabla}\times\vec{A}$

& $\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$ HAS ROOTS IN $\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$

WE ARE THEN GUARANTEED TO HAVE $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$ BY OUR CHOICE OF THE FORM

$$F^{\alpha\beta} = \frac{\partial A^\beta}{\partial X^\alpha} - \frac{\partial A^\alpha}{\partial X^\beta}$$

IF YOU INSIST!

$$\frac{\partial}{\partial X^\alpha}\tilde{F}^{\alpha\beta} = \frac{1}{2}\frac{\partial}{\partial X^\alpha}\varepsilon^{\alpha\beta\lambda\mu} F_{\lambda\mu}$$

$$= \frac{\partial}{\partial X^\alpha}\varepsilon^{\alpha\beta\lambda\mu}\frac{\partial A_\mu}{\partial X^\lambda}$$

NOTICE $\frac{\partial}{\partial X^\alpha}\frac{\partial}{\partial X^\lambda}$ IS SYMMETRIC

& $\varepsilon^{\alpha\beta\lambda\mu}$ IS ANTI-SYMMETRIC

SO $\frac{\partial}{\partial X^\alpha}\tilde{F}^{\alpha\beta} = 0$ THE TWO OTHER MAXWELL EQUATIONS

WHY DO THIS? LOTS OF REASONS
FOR INSTANCE, WHAT IF THE
PHOTON HAD A MASS?

$$\mathcal{L} = -\frac{1}{4\epsilon\mu_0} F^{\mu\nu} F_{\mu\nu} - \frac{\mu_0}{4\pi} J_\mu A^\mu$$

$$+ \frac{1}{2\epsilon\mu_0} m^2 A_\mu A^\mu$$

WE APPLY EULER-LAGRANGE
EQUATIONS TO FIND

$$\Box^2 A_\mu + m^2 A_\mu = \mu_0 J_\mu \quad (\Box^2 = \partial_\mu \partial^\mu)$$

IN THE STATIC LIMIT

$$\nabla^2 A_\mu - m^2 A_\mu = -\mu_0 J$$

SUPPOSE YOU HAVE A SOURCE CHARGE
Q AT THE ORIGIN. ONLY V IS
NON-VANISHING:

$$V(r) = q \frac{e^{-mr}}{r} \quad \left\{ \begin{array}{l} \text{IN HEAVISIDE-} \\ \text{LORENTZ UNITS} \end{array} \right\}$$

THIS IS A MODIFIED
GAUSS'S LAW: $E(r)$ FALLS
FASTER THAN $1/r^2$