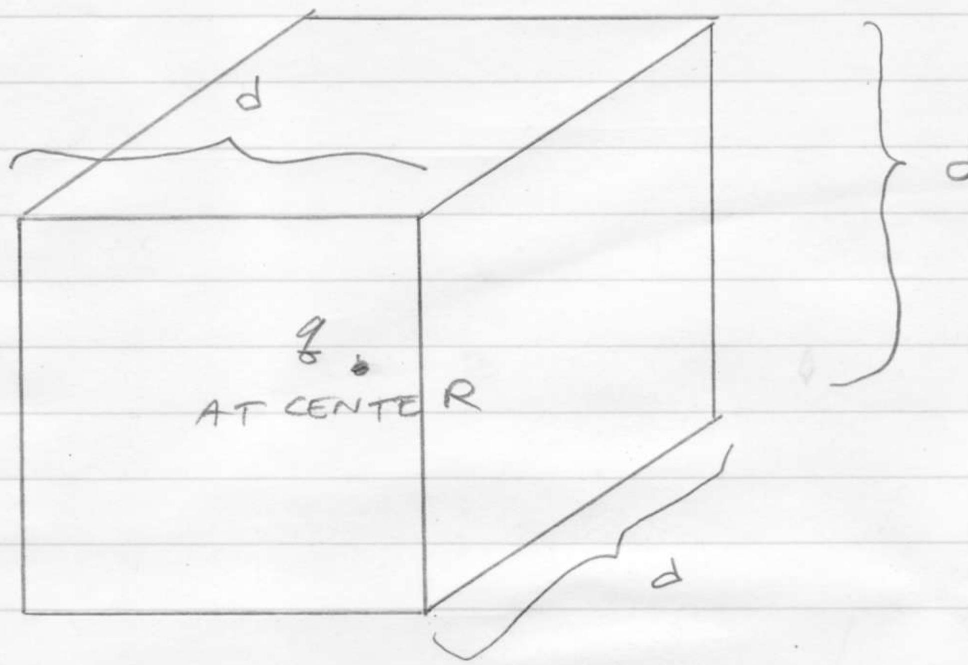
Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
October 29, 2020, 11am
On-line lecture

Administrative:

1. **Homework 4 posted at**
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
2. **Homework 3 grading notes posted at**
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
3. **Ensure you're getting your graded homework back.**
4. **Draft of this lecture posted at**
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
5. **Office hours today after class at 12:30.**

Lecture: Methods of finding potentials in boundary-value problems. (close out Jackson chapters 2 & 3).
Section 3.6. Addition theorem of spherical harmonics; 1/R expansion II.
Section 3.8. Boundary-value problems in cylindrical coordinates II: Bessel and related functions
Section 3.11. Green's function in cylindrical coordinates I.

COMMENTS ON HW3, PROBLEM 1,
GROUNDED CUBICAL SURFACE WITH
CHARGE AT CENTER



THE USUAL EXPANSION IN RECTANGULAR
COORDINATE HAVE

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-C_x^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dY^2}}_{-C_Y^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dZ^2}}_{+C_Z^2} = 0$$

WITH $C_x^2 = \left(\frac{n\pi}{d}\right)^2$, $C_Y^2 = \left(\frac{m\pi}{d}\right)^2$

$C_Z^2 = C_x^2 + C_Y^2$

WITH "PARTIAL" SOLUTION

$$\Phi(x, y, z) = \sum_{n,m}$$

$$\cdot \sin \frac{n\pi}{d} x \quad \sin \frac{n\pi}{d} y$$

$$\cdot \left\{ a_{nm} \cosh C_{z_{n,m}} z + b_{nm} \sinh C_{z_{n,m}} z \right\}.$$

BUT I IMMEDIATELY WROTE THE "PARTIAL" SOLUTION AS

$$\Phi(x, y, z) = \sum_{n,m,l}$$

$$f_{n,m,l} \sin \frac{n\pi}{d} x \sin \frac{n\pi}{d} y \quad \sin \frac{n\pi}{d} z.$$

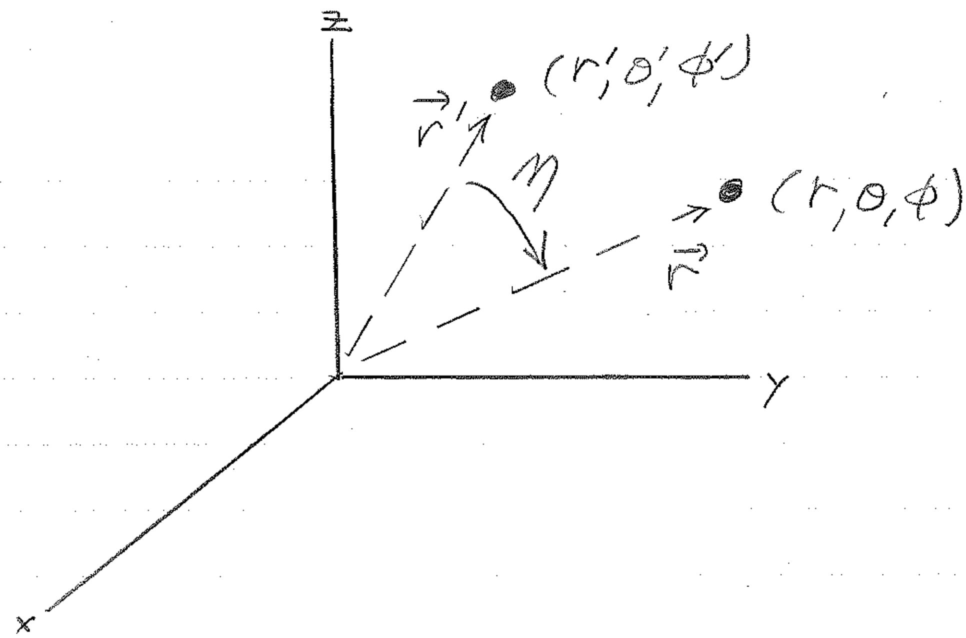
Q: WHY WAS I ABLE TO RIGHT AWAY WRITE THE SOLUTION?

Q! WHAT HAPPENED TO THE COSH & SINH? HOW DID THE DOUBLE SUM BECOME A TRIPLE SUM.

SEE JACKSON SECTION 3.12, ESPECIALLY EQUATION 3.169.

JACKSON 3.6

ADDITION THEOREM FOR $Y_{\ell m}$ AND THE 1/R EXPANSION (w/o AZIMUTHAL SYMMETRY).



ADDITION THEOREM: ASSERT (JACKSON EQN. 3.62)

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{+\ell} Y_\ell^{*m}(\theta', \phi')\, Y_\ell^{m}(\theta, \phi)$$

(THE PROOF IS GIVEN AS JACKSON EQN.S 3.63 – 3.69.

THIS ALLOWS THE 1/R EXPANSION

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum_{\ell}\left(\frac{r'}{r}\right)^{\ell} P_{\ell}(\cos\gamma) \qquad r > r'$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r'}\sum_{\ell}\left(\frac{r}{r'}\right)^{\ell} P_{\ell}(\cos\gamma) \qquad r < r'$$

TO BE WRITTEN AS

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum_{\ell}\left(\frac{r'}{r}\right)^{\ell} \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{+\ell} Y_{\ell}^{m*}(\theta',\phi')\,Y_{\ell}^{m}(\theta,\phi)$$

$$r > r'$$

AND

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r'}\sum_{\ell}\left(\frac{r}{r'}\right)^{\ell} \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{+\ell} Y_{\ell}^{m*}(\theta',\phi')\,Y_{\ell}^{m}(\theta,\phi)$$

THIS EXPRESSES $1/|\vec{r}-\vec{r}\,'|$ IN TERMS OF (r,θ,ϕ) AND (r',θ',ϕ') EXPLICITLY.

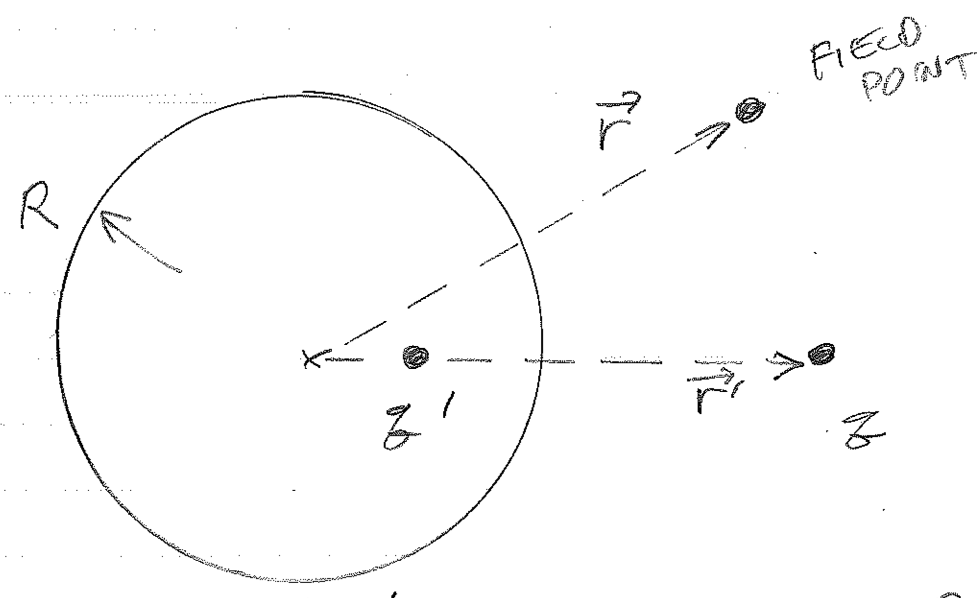
NOTICE WE REMOVED THE REQUIREMENT OF AZIMUTHAL SYMMETRY.

THIS WILL BE USED SHORTLY

RECALL $1/|\vec{r}-\vec{r}'|$ IS THE FREE-SPACE SPHERICAL GREEN'S FUNCTION. WE CAN EXTEND THIS TO THE "EXTERIOR SPHERICAL PROBLEM".

RECALL $G(\vec{r}, \vec{r}')$ IS THE POTENTIAL DUE TO A UNIT CHARGE (BY OUR CONVENTION $4\pi\epsilon_0$) PLUS THE POTENTIAL DUE TO THE INDUCED SURFACE CHARGE.

RECALL THE IMAGE-CHARGE RESULT:



$$r_{q'} = R^2/r' \; ; \; q' = -q\, R/r'$$

AND

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{R/r'}{|\vec{r} - \hat{r}' \, R^2/r'|}$$

EXPAND BOTH TERMS

$G(\vec{r}, \vec{r}')$

$$= \sum_{l,m} \frac{4\pi}{2l+1} \left\{ \frac{r_<^l}{r_>^{l+1}} - \frac{R}{r'} \frac{\left(\frac{R^2}{r'}\right)^l}{r^{l+1}} \right\}$$

$$\cdot \, Y_l^{m*}(\theta', \phi') \, Y_l^m(\theta, \phi)$$

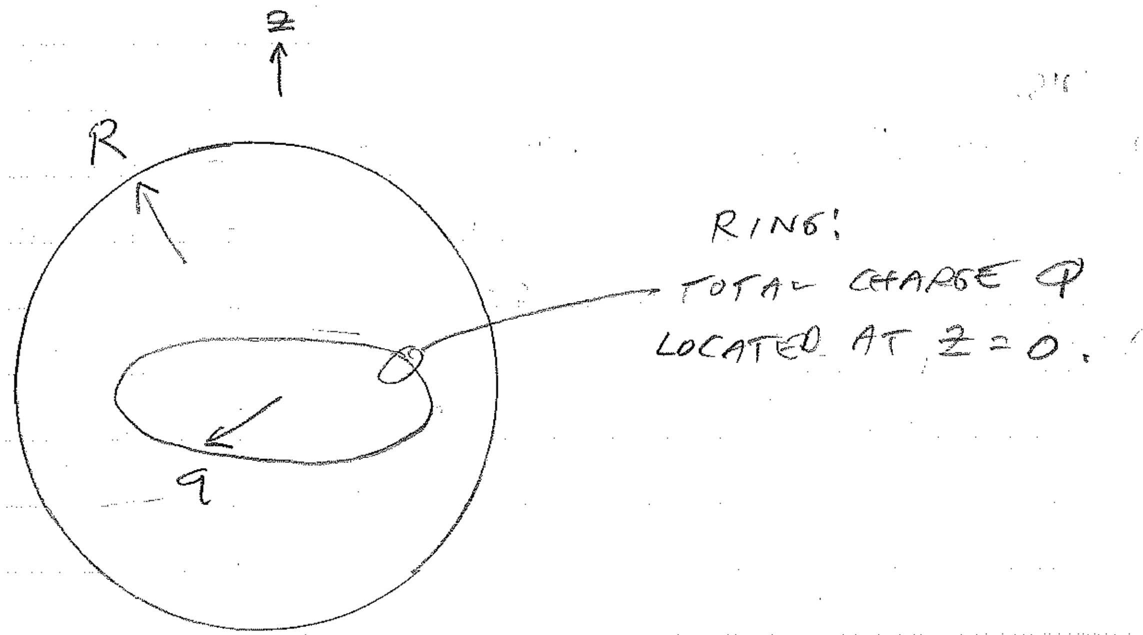
(USING THE $1/R$ EXPANSION)..

$$= \sum_{l,m} \frac{4\pi}{2l+1} \left\{ \frac{r_<^l}{R_>^{l+1}} - \frac{1}{R} \left(\frac{R^2}{rr'}\right)^{l+1} \right\}$$

$$\cdot \, Y_l^{m*}(\theta', \phi') \, Y_l^m(\theta, \phi).$$

Q: IS THIS $G(\vec{r}, \vec{r}')$ SENSIBLE? e.g., DOES IT SATISFY BOUNDARY CONDITIONS FOR $r, r' \to R$ AND $r, r' \to \infty$?; DOES IT RESPECT RECIPROCITY; IS IT A SOLUTION TO LAPLACE'S EQUATION

EXAMPLE: CHARGED RING INSIDE GROUNDED SPHERE (THE "INTERIOR" PROBLEM, SEE JACKSON P. 123). (THERE'S A SIMPLER APPROACH SUGGESTED ON THE HOMEWORK).



- FIRST, GO FROM THE "EXTERIOR" TO THE "INTERIOR" GREEN'S FUNCTION.

THE TERM $\frac{1}{R}\left(\frac{R^2}{rr'}\right)^{\ell+1}$

BECOMES $\dfrac{R(rr')^{\ell}}{(R^2)^{\ell+1}}$

- EXPRESS THE RING'S CHARGE AS A CHARGE DENSITY

$$\rho(\vec{r}) = \frac{Q}{2\pi a^2}\,\delta(r-a)\,\delta(\cos\theta)$$

Q: CHECK THE VOLUME INTEGRAL GIVES Q.

THE POTENTIAL AT AN INTERIOR FIELD POINT IS

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint \rho(\vec{r}\,')\,G(\vec{r},\vec{r}\,')\,dV'$$

(THERE IS NO SURFACE TERM FOR THE GROUNDED SPHERE.).

- THE SYSTEM HAS AZIMUTHAL SYMMETRY, SO $m=0$. WE'RE LEFT WITH

$$Y_\ell^0(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\,P_\ell^0(\cos\theta)$$

(SEE JACKSON EQN. 3.53) AND $P_\ell^0 = P_\ell$ (SEE JACKSON EQN. 3.49).

- THE δ-FUNCTION MAKES THE VOLUME INTEGRAL FOR $\Phi(\vec{r})$ EASY:

$$\Phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \sum_\ell P_\ell(0)$$

$$\cdot \left\{ \frac{r_<^\ell}{r_>^{\ell+1}} - \frac{R\, r_<^\ell r_>^\ell}{(R^2)^{\ell+1}} \right\} P_\ell(\cos\theta)$$

WITH $r_<$ AND $r_>$ REFERING TO r AND a.

- NOTICE $P_\ell(0)$ APPEARS.
P_ℓ, ℓ EVEN IS EVEN,

P_ℓ, ℓ ODD IS ODD.

SO $P_\ell(0)$ HAS ONLY ℓ EVEN CONTRIBUTIONS.

WE'RE BASICALLY DONE, JACKSON INTRODUCES A TRICK

$$P_{2\ell}(0) = \frac{(-1)^\ell (2\ell-1)!!}{2^\ell \ell!}, \text{ so}$$

$$\Phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \sum_\ell \frac{(-1)^\ell (2\ell-1)!!}{2^\ell \ell!}$$

$$\cdot \left\{ \frac{r_<^\ell}{r_>^{\ell+1}} - \frac{R\, r_<^\ell r_>^\ell}{(R^2)^{2\ell+1}} \right\} P_{2\ell}(\cos\theta) \quad \left(\begin{array}{c}\text{J.} \\ \text{EQN} \\ 3.131\end{array}\right)$$

LAPLACE'S EQUATION IN CYLINDRICAL COORDINATES: BESSEL'S EQUATION AND BESSEL FUNCTIONS.

SEPARATE VARIABLES IN SPHERICAL COORDINATES

$$\nabla^2 \Phi(\rho, \phi, z);$$

$$\Phi(\rho, \phi, z) = R(\rho)\,\Phi(\phi)\,Z(z)$$

THIS GIVES EQUATIONS

$$\rho \frac{d}{d\rho}\left(\rho \frac{d}{d\rho} R\right) + (k^2\rho^2 - n^2) R = 0;$$

$$\frac{d^2 \Phi}{d\phi^2} + n^2 \Phi = 0;$$

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0.$$

(JACKSON EQNS 3.73-5.).

K AND n ARE SEPARATION CONSTANTS.

PROPERTIES OF n AND K.

IF SOLUTIONS ARE DESIRED WHICH ARE SINGLE-VALUED IN ϕ, THEN SOLUTIONS ARE PERIODIC IN ϕ, AND THEREFORE n IS A REAL INTEGER

IF K IS REAL (AND WE MAY RETURN TO K IMAGINARY), THEN THE RADIAL SOLUTIONS

$$\mathbb{R}(\rho) \sim J_n(K\rho),\ N_n(K\rho)$$

ARE BESSEL FUNCTIONS OF THE FIRST AND SECOND KINDS.

SOME GENERAL COMMENTS:

- THERE ARE MANY GREAT TEXTS ON THE PROPERTIES OF BESSEL FUNCTIONS.
- J_n AND N_n BOTH GO TO 0 FOR $K\rho \longrightarrow \infty$.
- J_n IS THE REGULAR SOLUTION, N_n IS SINGULAR FOR $K\rho \rightarrow 0$.

FOR OUR CONVENTIONS n AND k BOTH REAL, n INTEGER:

$$\mathbb{R}(\rho) \sim J_n(k\rho),\ N_n(k\rho) \quad (k \neq 0)$$

Q: WHAT OF $\mathbb{R}(\rho)$ FOR $k=0$?
A: IN THAT CASE THE RADIAL EQUATION IS

$$\rho \frac{d}{d\rho}\left(\rho \frac{d}{d\rho} \mathbb{R}\right) - n^2 \mathbb{R} = 0.$$

WE'VE SEEN THIS BEFORE IN LECTURE 22 OCT '20 WHEN WE STUDIED FIELDS NEAR CORNERS. THIS HAS SOLUTIONS

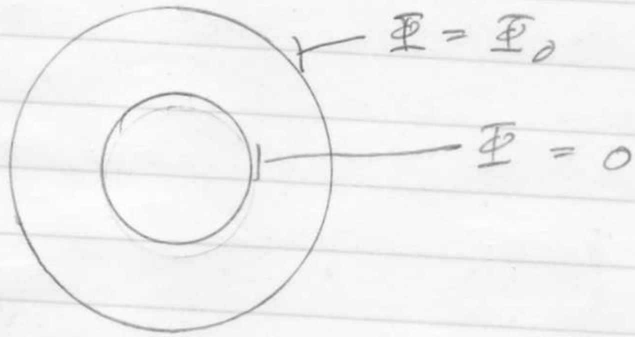
$$\mathbb{R}(\rho) \sim \rho^{+n},\ \rho^{-n} \quad (k=0).$$

$$\Phi(\phi) \sim \cos n\phi,\ \sin n\phi \quad (k \neq 0)$$
$$\sim \text{CONSTANT} \quad (k=0)$$

$$Z(z) \sim e^{+kz},\ e^{-kz} \quad (k \neq 0)$$
$$(\sim \cosh kz,\ \sinh kz)$$

$$\sim \text{CONSTANT} \quad (k=0)$$

Q! SUPPOSE YOU HAVE CONCENTRIC CYLINDERS AT FIXED POTENTIALS!



A: $\Phi(\phi) =$ CONSTANT. HENCE $k=0$.

SO YOU MIGHT SUPPOSE $R(\rho) \sim \rho^{+n}, \rho^{-n}$.

BUT ALSO NOTICE $n=0$
Q: WHY IS $n=0$?

HENCE BESSEL'S EQUATION

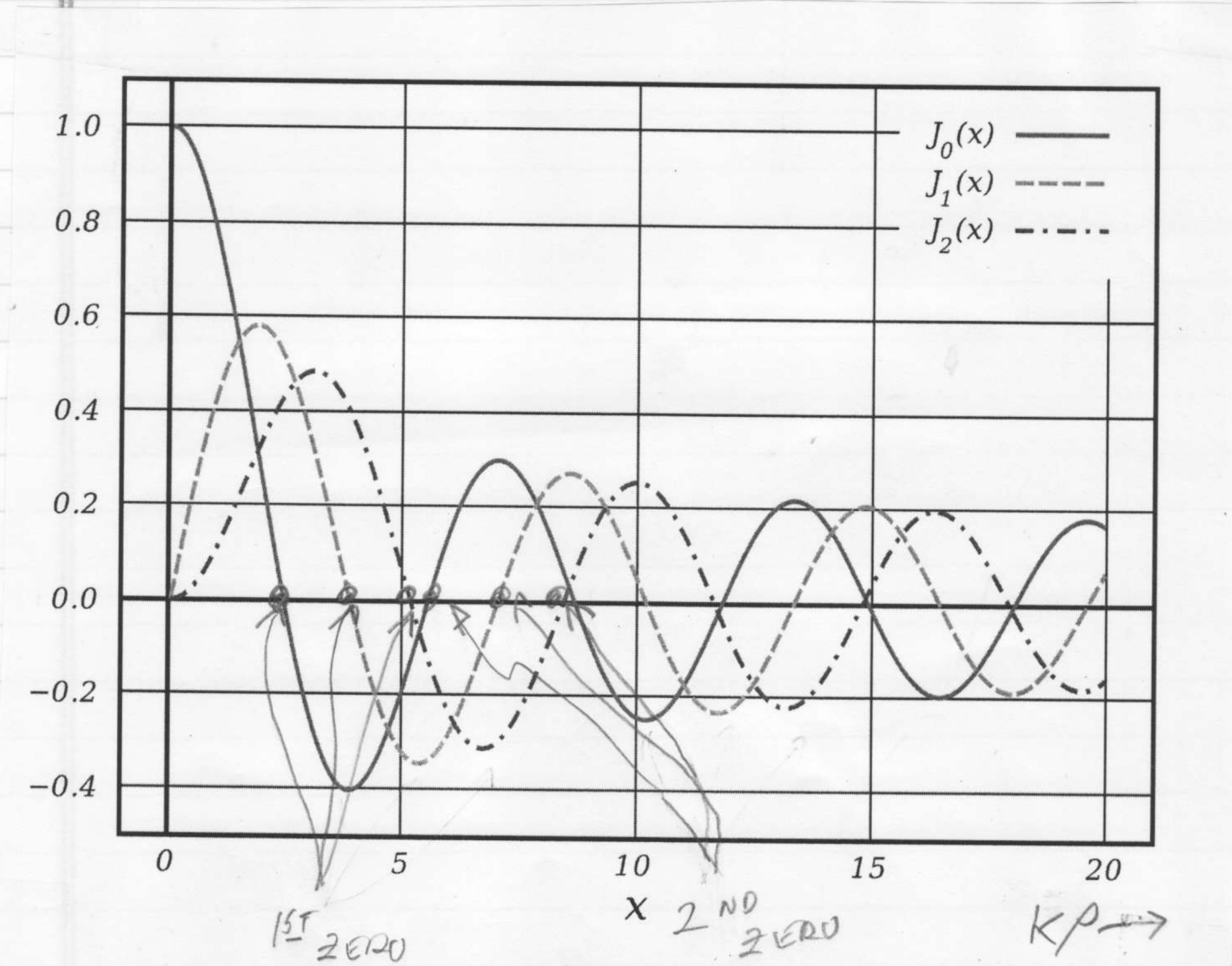
$$\rho \frac{d}{d\rho}\left(\rho \frac{d}{d\rho} R\right) + (k^2\rho^2 - n^2) R = 0$$

REDUCES TO

$$\rho \frac{d}{d\rho}\left(\rho \frac{d}{d\rho} R\right) = 0.$$

THIS CAN BE DIRECTLY INTEGRATED TO GIVE $\Phi(\rho) \sim \ln \rho$, AS YOU FOUND IN UNDERGRADUATE E&M.

J_0 IS SPECIAL: IT'S THE ONLY J_n NON-ZERO AT THE ORIGIN, HERE'S THE FIRST FEW J_n's!

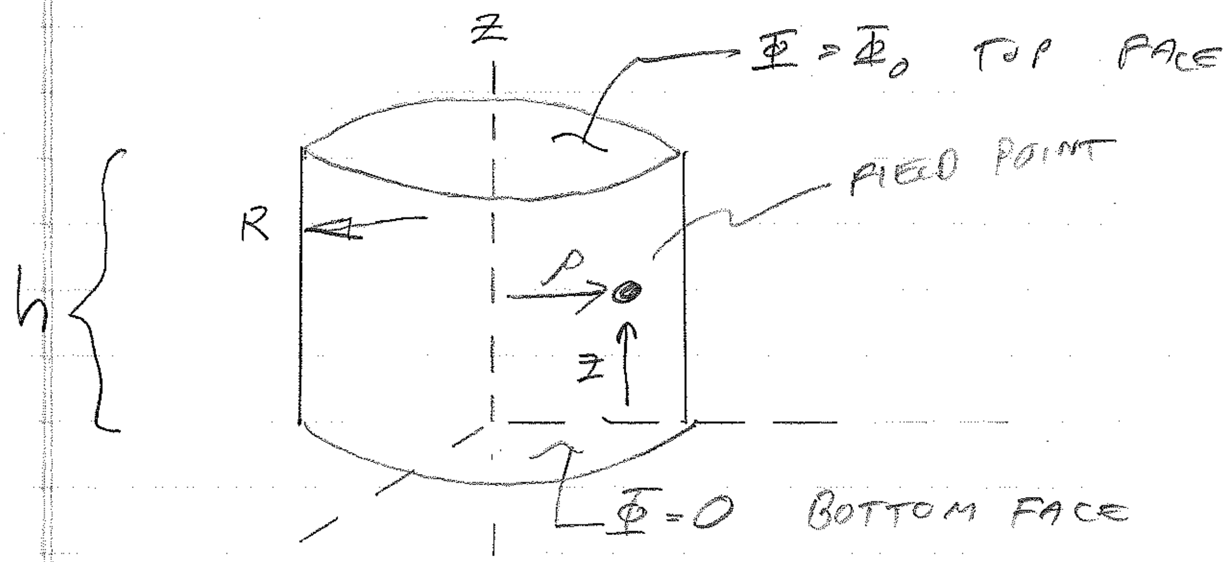


THESE ZEROS ARE DENOTED (JACKSON P. 114) X_{nm} FOR THE m^{th} ZERO OF THE n^{th} BESSEL FUNCTION

INTERESTING TIDBITS OF BESSEL FUNCTIONS!

- SUMS AND DIFFERENCES OF J_n AND iN_n ARE HANKEL FUNCTIONS!
- $J_0(0) = 1$, ALL OTHER $J_n(0) = 0$ $n \neq 0$.
- THERE ARE LARGE AND SMALL $k\rho$ ASYMPTOTIC FORMS OF J_n AND N_n (SEE JACKSON EQN.S. 3.89 – 91.).
- AS YOU MIGHT SURMISE, J_n AND N_n SATISFY CLOSURE AND ORTHONORMALITY (SEE JACKSON EQN.S. 3.94 – 95.). NOTE ESPECIALLY JACKSON EQN 3.95; ORTHONORMALITY BRINGS IN A BESSEL FUNCTION IN THE NORMALIZATION

EXAMPLE: MODES IN A CLOSED-ENDED CYLINDER (J, EXAMPLE 3.8 CONSIDERABLY SIMPLIFIED).



START APPLYING BOUNDARY CONDITIONS.

- $\Phi(z=0) = 0 \Rightarrow Z(z) \sim \sinh kz$
  WITH k T.B.D.

- $\Phi(k\rho)$ FINITE FOR $k\rho \to 0$
  $\Rightarrow \mathbb{R}(k\rho) \sim J_n(k\rho)$.

- $\Phi(kR) = 0 \Rightarrow \mathbb{R}(k\rho) \sim J_n\left(\frac{x_{nm}}{R}\rho\right)$.

  NOTICE WE JUST FOUND k

  $k = k_{nm} = x_{nm}/R$.

- AZIMUTHAL SYMMETRY

$\Rightarrow \Phi(\phi) \sim$ CONSTANT

BUT $K \neq 0$ IN

$\Phi(\phi) \sim \cos n\pi, \quad \sin n\pi$

SO $n = 0$. (SO ONLY J_0).

OUR PARTIAL SOLUTION (BEFORE EVALUATING EXPANSION COEFFICIENTS) AND APPLYING LAST BOUNDARY CONDITION);

$$\Phi(\rho, \phi, z) = \sum_{m=1}^{\infty} J_0\left(\frac{x_{0m}}{R}\rho\right) \sinh \frac{x_{0m}}{R} z \; a_m$$

THE BOUNDARY CONDITION AT $z = h$:

$$\Phi(\rho, \phi, z=h) =$$

$$\Phi_0 = \sum_{m=1}^{\infty} J_0\left(\frac{x_{0m}}{R}\rho\right) \sinh \frac{x_{0m}}{R} h \quad a_m$$

APPLY ORTHONORMALITY

MULTIPLY BOTH SIDES BY $J_0\left(\frac{x_{0m}}{R}\rho\right)$;

INTEGRATE $\int_0^R (\dots) \rho \, d\rho$.

ONE RIGHT SIDE HAS

$$\int_0^R \rho \, J_0\left(\frac{x_{0m}}{R}\right) J_0\left(\frac{x_{0m}}{R}\right) d\rho$$

$$= \frac{R^2}{2}\left[J_1(x_{0m})\right]^2$$

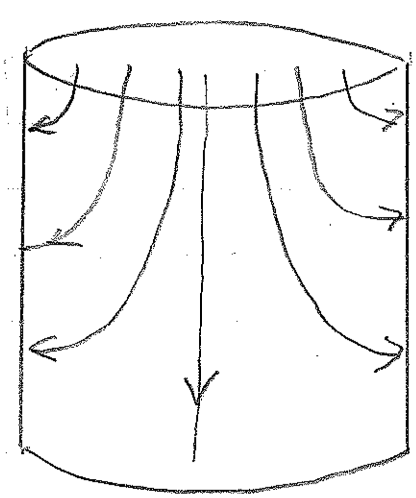
(JACKSON EQN 3.95).

THE LEFT SIDE HAS

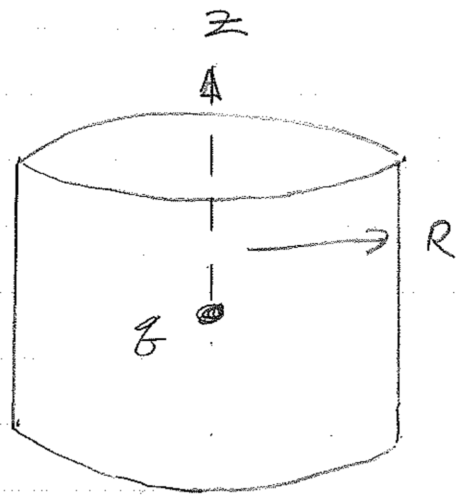
$$\int_0^R \rho \, J_0\left(\frac{x_{0m}}{R}\rho\right) d\rho$$

THIS LAST HAS NO CLOSED FORM; BUT IT IS A NUMBER. JACKSON LEAVES IT IN THIS FORM (SEE JACKSON EQN. 3.97)

SHAPE OF $\vec{E}$-FIELD



EXAMPLE: POINT CHARGE AT THE CENTER OF A GROUNDED CYLINDER.



- AS BEFORE, $\Phi(\rho, z, \phi)$ INDEPENDENT OF Φ RESTRICTS n TO $n=0$ AND RADIAL SOLUTIONS TO $J_0(k\rho)$.

THE SOLUTIONS HAVE FORM

$$\Phi_-(\rho, z) = \sum_m a_m e^{+k_m z} J_0(k_m \rho) \quad z<0;$$

$$\Phi_+(\rho, z) = \sum_m b_m e^{-k_m z} J_0(k_m \rho) \quad z>0.$$

CONTINUITY OF Φ_- WITH Φ_+ AT $z=0$ REQUIRES $a_m = b_m$.

APPLY MORE BOUNDARY CONDITIONS.

$$\Phi_{\mp}(\rho = R) = 0 \Rightarrow J_0(K_m R) = 0$$

so $K_m = \frac{x_{0m}}{R}$.

CONSIDERING THE PLANE AT $z=0$ A "SURFACE CHARGE", WE HAVE

$$\frac{q\,\delta(\rho)}{\varepsilon_0} = \left.\frac{\partial \Phi_-}{\partial z}\right|_{z=0} - \left.\frac{\partial \Phi_+}{\partial z}\right|_{z=0}$$

$$= 2\sum_m K_m a_m J_0(K_m \rho)$$

K_m AS ABOVE.

FIND THE EXPANSION PARAMETERS FROM ORTHONORMALITY:

- MULTIPLY BOTH SIDES BY $J_0(K_{m'}\rho)$ AND INTEGRATE $\int_0^R (\cdots)\rho\, d\rho$.

THE RIGHT SIDE AGAIN GIVES

$$\int_0^R \left[J_0(k_m \rho)\right]^2 \rho \, d\rho = \frac{R^2}{2} \left[J_1(k_m R)\right]^2.$$

THE LEFT SIDE GIVES

$$J_0(0) \quad \text{FROM} \quad \delta(\rho).$$

RECALLING $J_0(0) = 1$,

$$\Phi_-(\rho, z) = \frac{q}{2\pi R^2 \varepsilon_0} \sum_m e^{+\frac{x_{0m}}{R} z} \frac{J_0\left(\frac{x_{0m}}{R} z\right)}{\frac{x_{0m}}{R}\left[J_1(x_{0m})\right]^2}$$

$$\Phi_+(\rho, z) = \frac{q}{2\pi R^2 \varepsilon_0} \sum_m e^{-\frac{x_{0m}}{R} z} \frac{J_0\left(\frac{x_{0m}}{R} z\right)}{\frac{x_{0m}}{R}\left[J_1(x_{0m})\right]^2}$$

THIS IS HOW YOU APPROACH THE PROBLEM OF FINDING THE GREEN'S FUNCTION FOR THE INSIDE OF A CYLINDER.

THIS IS A "RESTRICTED" GREEN'S FUNCTION FOR THE SOURCE UNIT CHARGE AT THE CYLINDER'S CENTER

Q: HOW WOULD YOU EXTEND THE ABOVE RESULT TO FIND THE RESTRICTED GREEN'S FUNCTION FOR THE CHARGE SOMEWHERE ON THE INTERIOR CYLINDER'S AXIS?

Q: HOW WOULD YOU EXTEND THE RESULT JUST ABOVE TO THE GREEN'S FUNCTION FOR THE SOURCE CHARGE ANYWHERE INSIDE THE CYLINDER?