Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
October 27, 2020, 11am
On-line lecture

Administrative:

1. Homework 4 posted on
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
2. Homework 3 solutions posted on
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
3. No homework assigned next week.

Lecture: Methods of finding potentials in boundary-value problems. (Jackson chapters 2 & 3).

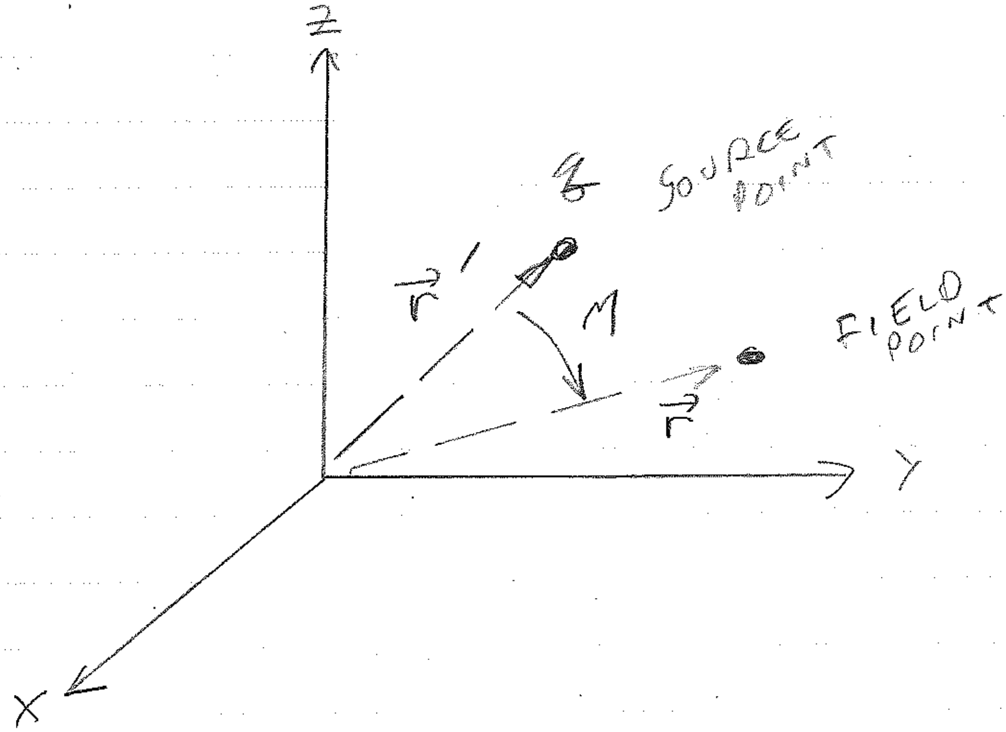
Section 3.3: Azimuthal symmetry: Potentials & 1/R expansion & examples.

Section 3.5: Associated Legendre polynomials and spherical harmonics.

Section 3.7 Expansion in cylindrical coordinates I: Azimuthal symmetry & example.

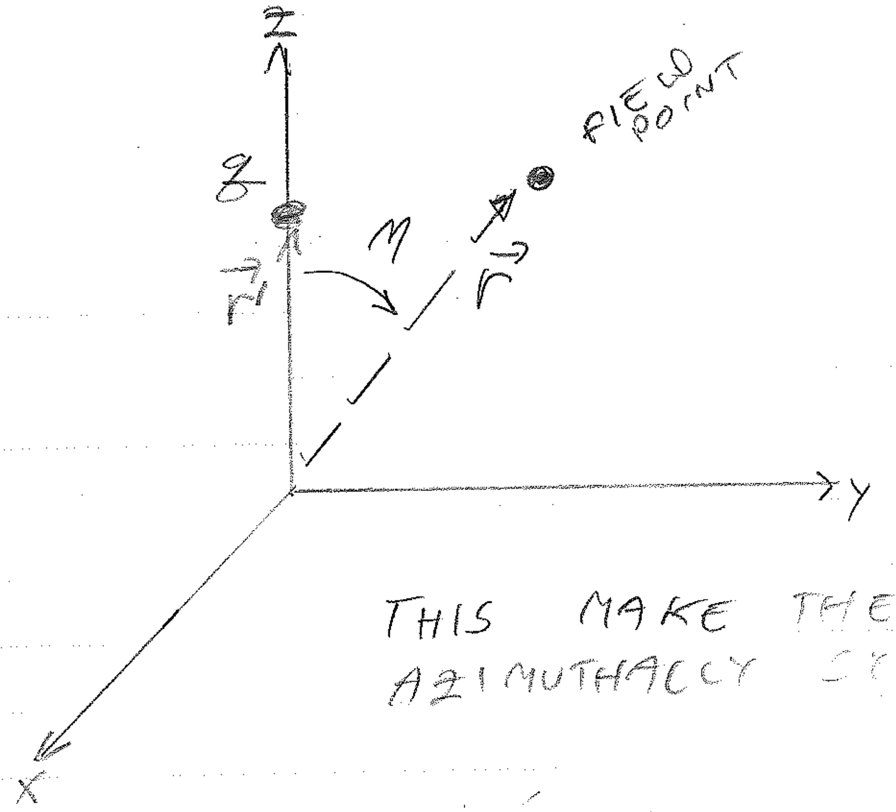
1/R, EXPANSION AS AN APPLICATION OF P_ℓ's APPLIED TO PROBLEMS WITH AZIMUTHAL SYMMETRY (JACKSON EQN. 3.38).

IT STARTS WITH A POINT CHARGE q OFFSET FROM THE ORIGIN.



HOW DO YOU EXPRESS THE POTENTIAL AT THE FIELD POINT IN TERMS OF r, r' AND γ?

WE FIRST ROTATED THE CHARGE TO THE z-AXIS;



THIS MAKE THE POTENTIAL AZIMUTHALLY SYMMETRIC.

WE EXPRESS $1/|\vec{r}-\vec{r}'|$ IN TWO WAYS:

$$1/|\vec{r}-\vec{r}'| = \sum_{l}\left[a_l r^l + \frac{b_l}{r^{l+1}}\right] P_l(\cos\eta)$$

AND

$$1/|\vec{r}-\vec{r}'| = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\eta}}$$

YOU SHOULD NOTICE THE RADIAL AND ANGULAR PARTS FACTORIZE.

NOW, WE'LL APPLY THIS TO A SIMPLER GEOMETRY: SUPPOSE THE FIELD POINT IS AS WELL ON THE z-AXIS. IF WE FIND THE RADIAL PART OF THIS SOLUTION, WE NEED ONLY MULTIPLY IT BY $P_\ell(\cos\gamma)$ TO FIND THE MORE GENERAL SOLUTION.

FOR BOTH POINTS ON THE z-AXIS, ALL $P_\ell(1) = 1$ (SEE JACKSON EQN 3.15 WITH $x = 1$).

WE THEN STUDIED TWO CASES:

1. $r > r'$ (FIELD POINT FARTHER AWAY)

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\,\frac{1}{\sqrt{1+(r'/r)^2 - 2\,r/r'}}$$

$$= \frac{1}{r}\left\{1 - \tfrac{1}{2}\left[\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\right] + \cdots\right\}$$

$$= \frac{1}{r}\sum_{\ell}\left(\frac{r'}{r}\right)^{\ell}.$$

2. $r < r'$ (FIELD POINT CLOSER)

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r'}\,\frac{1}{\sqrt{1+(r/r')^2 - 2\,r/r'}}$$

$$\vdots$$

$$= \frac{1}{r'}\sum_{\ell}\left(\frac{r}{r'}\right)^{\ell}.$$

THIS IS THE RADIAL PART OF THE POTENTIAL: IF THE FIELD POINT WERE TO MOVE OFF AXIS, THE RADIAL PART IS UNCHANGED.

NOW ALLOW THE FIELD POINT TO BE OFF AXIS. ALL WE NEED DO IS MULTIPLY THE RADIAL PART (FROM P. 4) BY P_ℓ!

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_\ell \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\gamma) \quad r > r'.$$

OR

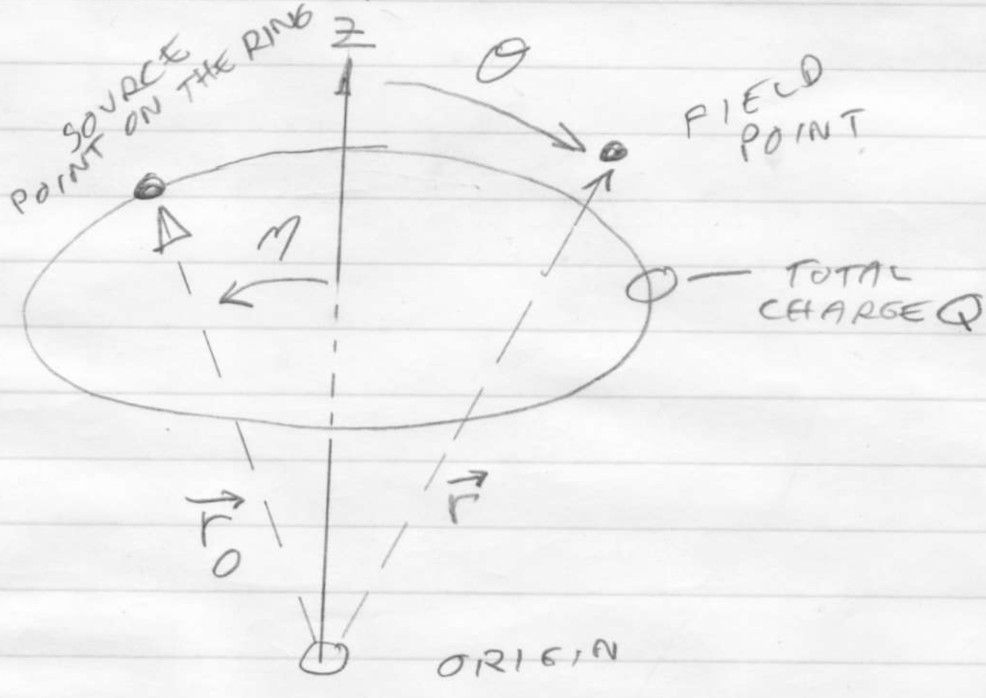
$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r'}\left(\frac{r}{r'}\right)^\ell P_\ell(\cos\gamma) \quad r < r'.$$

MORE COMPACTLY, THIS IS WRITTEN (JACKSON EQN ON P. 103)

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^\ell P_\ell(\cos\gamma).$$

THIS GENERALIZES! IF YOU SOMEHOW FIND THE POTENTIAL ALONG THE SYMMETRY AXIS, YOU CAN THEN FIND THE OFF-AXIS SOLUTION BY WEIGHTING EACH TERM IN THE SUM BY $\cos\theta$.

EXAMPLE: UNIFORMLY-CHARGED RING.
(SEE JACKSON P. 103).
YOU DID THIS AS A 1ST-YEAR UNDERGRADUATE FOR FIELD POINTS ALONG THE z-AXIS. LET'S MOVE THE RING AWAY FROM THE ORIGIN AND CONSIDER FIELD POINTS OFF THE SYMMETRY AXIS.



OBVIOUSLY, EVERYWHERE ON THE RING η IS CONSTANT AND r_0 IS CONSTANT.

FIRST, FIND THE POTENTIAL ON THE z-AXIS.

1. $r_0 > z$

$$\Phi(z, \theta = 0) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r_0} \sum_l \left(\frac{z}{r_0}\right)^l P_l(\cos\eta)$$

EACH PIECE OF THE RING CONTRIBUTES EQUALLY TO THE ON-AXIS POTENTIAL

2. $r_0 < z$

$$\Phi(z, \theta=0) = \frac{q}{4\pi\varepsilon_0} \frac{1}{z} \sum_l \left(\frac{r_0}{z}\right)^l P_l(\cos\eta).$$

(WE CAN PUT $1/z$ BACK IN THE SUM, AND PULL OUT r_0 OUT OF THE SUM:

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{r_0} \sum_l \left(\frac{r_0}{z}\right)^{l+1} P_l(\cos\eta).$$

I DONT KNOW WHY JACKSON WRITES IT THUS.)

TO FIND THE OFF-AXIS FIELD, MULTIPLY BY $P_\ell(\cos\theta)$:

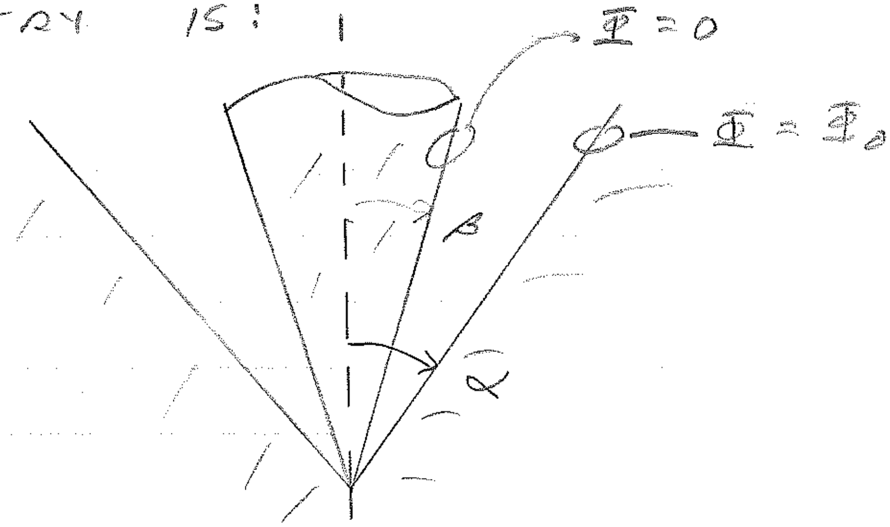
1. $r_0 > r$

$$\Phi(r,\theta) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_0}\sum_\ell \left(\frac{r}{r_0}\right)^\ell P_\ell(\cos\eta)P_\ell(\cos\theta).$$

2. $r_0 < r$

$$\Phi(r,\theta) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\sum_\ell \left(\frac{r}{r_0}\right)^\ell P_\ell(\cos\eta)P_\ell(\cos\theta),$$

EXAMPLE: COAXIAL CONES AT DIFFERENT POTENTIALS. THE GEOMETRY IS:



Q: BEFORE WE START, WHAT'S THE NATURE (SHAPE) OF THE POTENTIAL AND FIELDS?

A: THIS PROBLEM HAS NO LENGTH SCALE. IF, e.g., YOU MULTIPLY LENGTHS BY 2, YOU HAVE THE SAME PROBLEM. HENCE

$$\Phi(r, \theta) \rightarrow \Phi(\theta); \quad \vec{E}(\theta) \rightarrow \hat{\theta}.$$

APPLY AZIMUTHAL SYMMETRY.

$$\Phi(r,\theta) = \sum_{\ell} \left(a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

BUT $\Phi(r,\theta) = \Phi(\theta)$ MEANS ONLY $\ell = 0$ SURVIVES (NO r-DEPENDENCE).

HENCE

$$\Phi(r,\theta) = a_0 P_0(\cos\theta) = \text{CONSTANT.}$$

OH-OH, THIS SOLUTION DOESN'T SATISFY THE BOUNDARY CONDITIONS AT $\theta = \alpha, \beta$. IT'S NOT A SOLUTION.

Q: WHAT HAPPENED?

A: NOTICE $\theta = 0, \pi$ ARE EXCLUDED. THIS ALLOWS THE IRREGULAR SOLUTIONS Q_ℓ.

$$\Phi(r,\theta) = \sum_{\ell} \left(a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}} \right) Q_\ell(\cos\theta).$$

AGAIN, ONLY $\ell = 0$ SURVIVES, SO

$$\Phi(r,\theta) = a_0 Q_0(\cos\theta)$$

(11)

UNFORTUNATELY, JACKSON DOESN'T TABULATE LEGENDRE FUNCTIONS OF THE $2^{\underline{ND}}$ KIND. BUT ABRAMOWITZ AND STEGUN DO!

8.4. Explicit Expressions

$(x = \cos\theta)$

8.4.1 $\quad P_0(z) = 1 \qquad P_0(x) = 1$

8.4.2

$$Q_0(z) = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right) \qquad Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$= xF(\tfrac{1}{2}, 1; \tfrac{3}{2}; x^2)$$

8.4.3 $\quad P_1(z) = z \qquad P_1(x) = x = \cos\theta$

8.4.4

$$Q_1(z) = \frac{z}{2}\ln\left(\frac{z+1}{z-1}\right) - 1 \qquad Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

8.4.5

$$P_2(z) = \tfrac{1}{2}(3z^2 - 1) \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$

$$= \tfrac{1}{4}(3\cos 2\theta + 1)$$

8.4.6

$$Q_2(z) = \frac{1}{2}P_2(z)\ln\left(\frac{z+1}{z-1}\right) - \frac{3z}{2} \qquad Q_2(x) = \left(\frac{3x^2-1}{4}\right)\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}$$

$$Q_0(x) = \frac{1}{2}\ln\frac{x+1}{x-1}$$

LOOKING UP AN IDENTITY IN ABRAMOWITZ AND STEGUN:

$$\frac{1}{2}\ln\frac{x+1}{x-1} = -\ln\tan\tfrac{1}{2}\theta.$$

NOW APPLY THE BOUNDARY CONDITION
AT $\theta = \alpha, \beta$.

$$\Phi(\theta) = \Phi_0 \frac{\ln \dfrac{\tan\frac{1}{2}\beta}{\tan\frac{1}{2}\theta}}{\ln \dfrac{\tan\frac{1}{2}\beta}{\tan\frac{1}{2}\alpha}} \quad \text{EXACTLY.}$$

Q! DOES $\Phi(\theta)$ SATISFY THE
BOUNDARY CONDITIONS?

ASSOCIATED ("GENERALIZED")
LEGENDRE POLYNOMIALS AND
$Y_\ell^m(\theta,\phi)$ SPHERICAL HARMONICS
(JACKSON §3.5).

RECALL THE SEPARATION OF VARIABLES IN SPHERICAL COORDINATES

WITH $\Phi(r,\theta,\phi) = \frac{U(r)}{r} \mathbb{P}(\theta) \mathbb{Q}(\phi)$,

WE HAD FUNCTIONS

$$\mathbb{Q}(\phi) \sim e^{+im\phi}, e^{-im\phi};$$

$$\frac{U(r)}{r} \sim r^\ell, \frac{1}{r^{\ell+1}};$$

$$\mathbb{P}(\cos\theta) = P_\ell^m(\cos\theta),$$

THE "GENERALIZED LEGENDRE POLYNOMIALS".

THESE ARE SOLUTIONS TO
(JACKSON EQN.S 3.4, 3.6, 3.7).

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2;$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P = 0;$$

$$\frac{d^2 U}{dr^2} - \frac{\ell(\ell+1)}{r^2}U = 0.$$

SPHERICAL HARMONICS Y_ℓ^m.
(SEE JACKSON EQN 3.53.)
COMBINE ANGULAR PARTS

$$Y(\theta,\phi) = P(\theta)\Phi(\phi).$$

THE FIRST TWO EQUATIONS COMBINE TO

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dY}{d\theta}\right)$$

$$+ \frac{1}{\sin^2\theta}\frac{d^2 Y}{d\phi^2} + \ell(\ell+1)Y = 0.$$

NOTICE THE $1/\sin^2\theta$ TERM.

TECHNICALLY, BECAUSE OF THE $1/\sin^2\theta$, THE EQUATION FOR Y HAS SINGULARITIES AT $\theta = 0, \pi$. THE FORM OF THE SEPARATION CONSTANT $\ell(\ell+1)$ WITH ℓ AN INTEGER AROSE BY REQUIRING A REGULAR SOLUTION AT $\theta = 0$ AND π. HOWEVER, IF $\theta = 0$ AND π ARE EXCLUDED, ℓ NEED NOT BE AN INTEGER.

SELECTED PROPERTIES OF ASSOCIATED LEGENDRE POLYNOMIALS.

- $P_\ell^m(x)$ IS REGULAR AT $x = \pm 1$; "LEGENDRE FUNCTION OF

  P_ℓ^m IS THE "ASSOCIATED LEGENDRE FUNCTION OF THE FIRST KIND".

  $Q_\ell^m(x)$ IS IRREGULAR AT $x = \pm 1$. Q_ℓ^m IS THE "ASSOCIATED LEGENDRE FUNCTION OF THE SECOND KIND".

- P_ℓ^{+m} AND P_ℓ^{-m} ARE RELATED BY A CUMBERSOME FUNCTION OF ℓ AND m; SEE JACKSON EQN. 3.51.

- ORTHOGONALITY; e.g., WITH m FIXED:

$$\int_{-1}^{+1} P_l^m(x)\, P_{l'}^m(x)\, dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\,\delta_{l,l'}$$

THE NORMALIZED SPHERICAL HARMONIC IS JACKSON EQN 3.53:

$$Y_\ell^m(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\sqrt{2\ell+1}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}$$

$$\cdot P_\ell^m(\cos\theta)\, e^{im\phi}$$

SELECTED PROPERTIES OF Y_ℓ^m

- $Y_\ell^{-m} = (-1)^m Y_\ell^{+m\,*}$,

- ORTHOGONALITY

$$\int_{\phi=0}^{2\pi}\int_{\cos\theta=-1}^{+1} Y_{l'}^{m'*}(\theta',\phi')\,Y_l^{m}(\theta',\phi')\,d\cos\theta'\,d\phi' = \delta_{l,l'}\,\delta_{m,m'}.$$

Q: WHAT DOES "COMPLETENESS" LOOK LIKE FOR Y_l^m?

A: $$\sum_l \sum_{m=-l}^{+l} Y_l^{m*}(\theta',\phi')\,Y_l^m(\theta,\phi) = \delta(\phi-\phi')\cdot\delta(\cos\theta-\cos\theta').$$

EXAMPLE Y_l^m: SEE JACKSON P. 109

$l=0$: $Y_0^0 = \dfrac{1}{\sqrt{4\pi}}$

$l=1$: $Y_1^{+1} = -\sqrt{3/8\pi}\,\sin\theta\, e^{+i\phi}$

$Y_1^0 = \sqrt{3/4\pi}\,\cos\theta$

$Y_1^{-1} = +\sqrt{3/8\pi}\,\sin\theta\, e^{-i\phi}$

Q: HOW DID I GET FROM Y_1^{+1} TO Y_1^{-1}?

THE Y_ℓ^m ARE "SPHERICAL HARMONICS", THE RADIAL SOLUTIONS ARE STILL

$$U(r)/r \sim r^\ell, \frac{1}{r^{\ell+1}}.$$

MOST TEXTS SIMPLY DO A SEPARATION INTO $U(r)/r \cdot Y(\theta,\phi)$, THEN CONTINUE SEPARATING AS $Y(\theta,\phi) = P(\theta)\Phi(\phi)$.

BACK TO CYLINDRICAL SYMMETRY IN 2D.
(WE LOOKED AT THIS 2 LECTURES AGO IN THE CONTEXT OF FIELDS NEAR CORNERS.)

SEPARATION IN 2D:

$$\nabla^2 \Phi(\rho, \phi) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\Phi\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\Phi = 0.$$

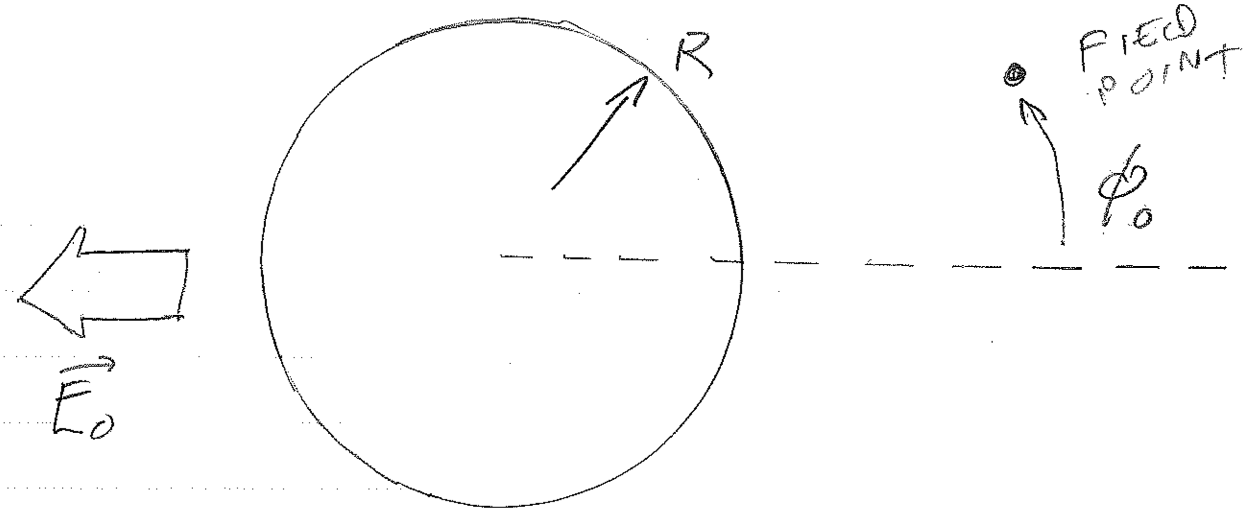
SOLUTIONS HAVE FORM

$$\Phi(\rho, \phi) = \mathbb{R}(\rho)\,\Phi(\phi), \quad \text{WITH}$$

$$\mathbb{R}(\rho) = \rho^n, \ 1/\rho^n \ ; \quad n = 0, 1, \ldots$$

$$\Phi(\phi) \sim \cos n\phi, \ \sin n\phi.$$

EXAMPLE: NEUTRAL CONDUCTING CYLINDER WITH AXIS AT RIGHT ANGLES TO A UNIFORM $\vec{E}$-FIELD. GEOMETRY



AT LARGE DISTANCES, $\Phi \sim \rho \cos\phi$

Q: WHY?

BEFORE WE PULL OUT ALL THE "ORTHONORMAL" COMPUTATIONAL MACHINERY, LET'S ASK WHAT n's WILL CONTRIBUTE. ON THE SURFACE, $\Phi = 0$. THIS SURFACE POTENTIAL IS THE SUM OF THE EXTERNAL APPLIED FIELD ($\sim \cos\phi$) PLUS THE CYLINDER'S FIELD. NOTICE THE EXTERNAL FIELD HAS ANGULAR DEPENDENCE $\sim \cos\phi$, SO TERMS WITH $\cos n\phi$, $n \neq 1$ WON'T SATISFY THE BOUNDARY CONDITION ON

THE SPHERE SINCE BOTH CONTRIBUTIONS MUST CANCEL EVERYWHERE ON THE SPHERE. THIS IS A HUGE TIME SAVER. IF YOU DON'T TRUST IT, USE ORTHOGONALITY THEN APPLY BOUNDARY CONDITIONS.

SO, $\Phi(\rho, \phi) = -E_0 \rho \cos\phi + \frac{a_1}{\rho} \cos\phi.$

Q: WHAT HAPPENED TO THE $b_1 \rho \cos\phi$ SOLUTION?

A: IT GIVES INCORRECT BOUNDARY CONDITIONS AT BIG DISTANCES. IMPLICIT IS FAR FROM THE SPHERE, THE $1/\rho$ - POTENTIAL IS MUCH SMALLER THAN THE APPLIED POTENTIAL.

WE REQUIRE $\Phi(\rho = R, \phi) = 0$, SO

$$\Phi(\rho, \phi) = \underbrace{-E_0 \rho}_{\text{POTENTIAL DUE TO EXTERNAL FIELD}} \left\{ 1 - \underbrace{\frac{R^2}{\rho^2}}_{\text{POTENTIAL OF AN EXTENDED DIPOLE CONSISTING OF PARALLEL POSITIVE \& NEGATIVE LINE CHARGES.}} \right\} \cos\phi$$

SURFACE CHARGE.

$$\sigma \sim -\varepsilon_0 \left[\frac{d}{d\rho} \Phi(\rho, \phi)\right]_S$$

$$\text{''} \sim \cos\phi.$$

ANOTHER "$\cos\phi$" (OR $\cos\theta$)
CHARGE DISTRIBUTION.

SUPPOSE YOU "PASTED" THE ABOVE
SURFACE CHARGE σ ON THE
CYLINDER. WHAT'S THE FIELD
EVERYWHERE.

OUTSIDE: "EXTENDED DIPOLE".

INSIDE: CONSTANT.
Q: WHY?