Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
November 24, 2020, 11am PST
On-line lecture

Administrative:
1. No Homework assigned this week
Rest & enjoy the holiday.

Lecture: Magnetostatics, Faraday's Law, Quasi-Static Fields. (Jackson chapter 5).
Section 5.1-6 Basic magnetostatics:
The "magnetic field" B and the magnetic-field term in the Lorentz force law.
The curl and divergence of B.
Ohm's law and currents.
Electromotive force; rotational and irrotational sources of currents.
Magnetic potential: Scalar and vector potentials.

BASIC MAGNETOSTATICS
(JACKSON §5.1–.6)

RECALL: "NO FREE MAGNETIC CHARGES" MEANS THE RATIO OF MAGNETIC CHARGE TO ELECTRIC CHARGE IS THE SAME FOR ALL MATERIALS.

WE'RE THEREFORE FREE TO, BY CONVENTION, TAKE THE MAGNETIC CHARGE OF ALL MATERIALS TO BE ZERO ($\rho_M = 0$); LIKEWISE MAGNETIC CURRENTS ARE ZERO ($\vec{J}_M = 0$).

WITH NO MAGNETIC CHARGE, THE BASIC OBJECT IN MAGNETOSTATICS IS THE MAGNETIC DIPOLE $\vec{\mu}$ ($\vec{\mu}$ THE MAGNETIC MOMENT).

THE ORIGINAL DEFINITION OF $\vec{B}$ WAS DERIVED FROM TORQUE $\vec{N}$ OF A MAGNETIC DIPOLE $\vec{\mu}$:

$$\vec{N} = \vec{\mu} \times \vec{B}$$

(THERE'S A SUBTLETY APPLYING DUALITY HERE.)

IN THE 1800'S, CURRENTS WE OBSERVED TO BE ASSOCIATED WITH MAGNETIC FIELDS $\vec{B}$.

RECALL LOCAL CHARGE CONSERVATION:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial}{\partial t} \rho = 0$$

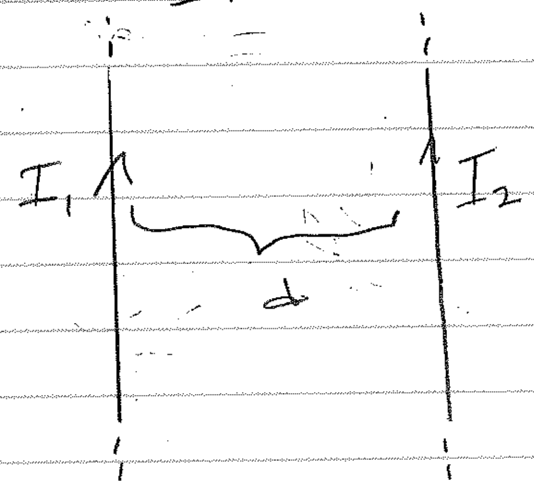
($\vec{J}$ IS AN ELECTRIC CURRENT, ρ IS ELECTRIC CHARGE DENSITY.)

IN STATICS: $\vec{\nabla} \cdot \vec{J} = 0$; WE'LL GET BACK TO THIS.

THE MAGNETIC FIELD $\vec{B}$ (MORE PROPERLY THE "MAGNETIC FLUX DENSITY"! PHYSICISTS, ESPECIALLY THE FUSSY ONES, CALL $\vec{H}$ THE MAGNETIC FIELD).

AMPÈRE AND OTHERS OBSERVED THAT FOR LINE CURRENTS, THE FORCE PER UNIT LENGTH IS:

$$\frac{dF}{dl} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$$



I_1 & I_2 IN THE SAME DIRECTION: ATTRACTIVE;
I_1 & I_2 IN OPPOSITE DIRECTIONS REPULSIVE.

AND $\mu_0 = 4\pi \times 10^{-7}$ N/AMP^2 EXACTLY.

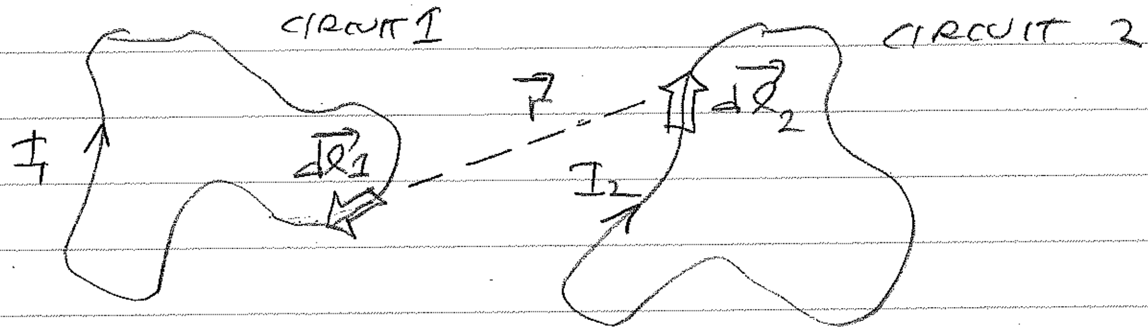
THIS SUGGESTS A LORENTZ FORCE LAW (LOCAL FIELD THEORY)

$$\vec{F} = I\,\vec{\ell} \times \vec{B},$$

THAT IS, CURRENTS SOURCE CIRCULATING FIELDS. IN PARTICULAR THE CIRCULATING FIELD AROUND A STRAIGHT CURRENT-CARRYING WIRE IS

$$\vec{B}(r) = \frac{\mu_0}{2\pi}\frac{I}{r}\hat{\phi}.$$

IN GENERAL FOR TWO CIRCUITS



$$\vec{F} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{\ell}_2 \times \left(d\vec{\ell}_1 \times \frac{\hat{r}}{r^2} \right)$$

WRITTEN IN THIS WAY, IT'S NOT OBVIOUS THIS OBEYS NEWTON'S LAW,

EXERCISE: SHOW THE FORCE $\vec{F}$ CAN BE EXPRESSED IN A FORM WHERE $d\vec{\ell}_1$ AND $d\vec{\ell}_2$ ARE SYMMETRIC.

A: THE FORCE $\vec{F}$ ALSO HAS THE UNCONVENTIONAL FORM

$$\vec{F} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \, \frac{\hat{r}}{r^2}.$$

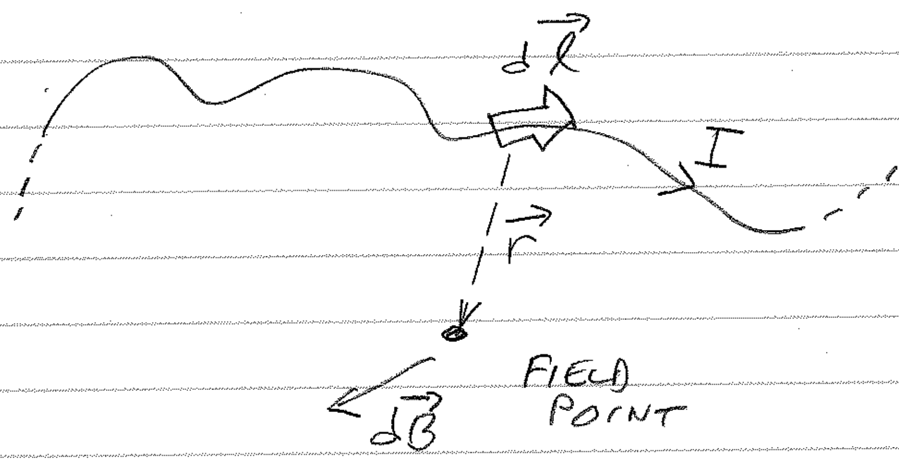
MUCH AS HOW TWO STRAIGHT CURRENTS,

$$\frac{dF}{dl} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}$$

LED TO A CORRESPONDING MAGNETIC FIELD $\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi} \frac{I}{r}$,

THE FORCE LAW $\vec{F} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_2 \times \left(d\vec{l}_1 \times \frac{\hat{r}}{r^2} \right)$

LEADS TO A GENERIC MAGNETIC FIELD



$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \, d\vec{l} \times \left(\frac{\hat{r}}{r^2} \right).$$

THE BIOT-SAVART LAW.

JACKSON MENTIONS: THIS LOOKS SIMILAR TO THE ELECTROSTATIC $d\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} dQ \frac{\hat{r}}{r^2}$

WITH dQ REPLACED BY $I\, d\vec{l} \times$.
BUT THIS IS "DANGEROUS" AND LEADS TO PARADOXES. JACKSON FOOTNOTE P. 176.

VARIOUS FORMS OF THE BIOT-SAVART LAW:

LINE CURRENTS $\vec{B} = \frac{\mu_0}{4\pi} \int I\, d\vec{l} \times \frac{\hat{r}}{r^2}$;

SURFACE CURRENTS: $\vec{B} = \frac{\mu_0}{4\pi} \iint \vec{K} \times \frac{\hat{r}}{r^2}\, dA$;

VOLUME CURRENTS: $\vec{B} = \frac{\mu_0}{4\pi} \iiint \vec{J} \times \frac{\hat{r}}{r^2}\, d\tau$.

LORENTZ FORCE LAW:

$$d\vec{F} = I\, d\vec{l} \times B + dq\, \vec{E}$$

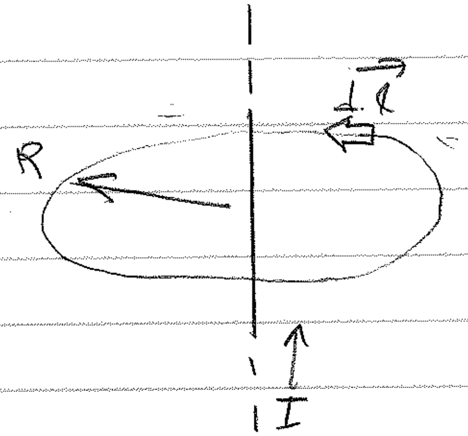
$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

EXERCISE! SHOW THIS FORM OF $\vec{F}$ FOLLOWS FROM $d\vec{F}$.

EVALUATION OF $\vec{\nabla}\times\vec{B}$..

RECALL FOR THE LONG STRAIGHT WIRE

$$\vec{B}(R)=\frac{\mu_0}{2\pi}\frac{I}{R}.$$

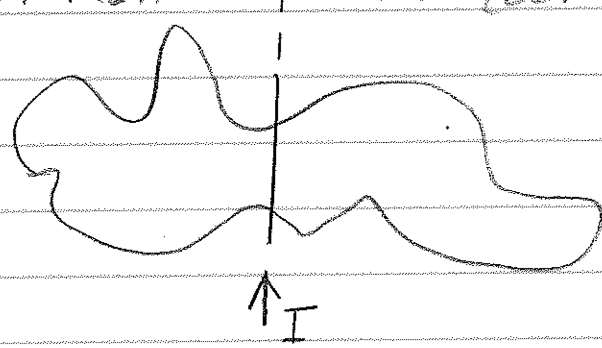


IT'S TRIVIAL TO SHOW, FOR THE CIRCULAR LOOP ABOVE,

$$\oint\vec{B}\cdot d\vec{l}=\mu_0 I$$

IS INDEPENDENT OF R.

BUT WHAT IF, INSTEAD OF A CIRCULAR LOOP, YOU HAVE AN ARBITRARY "SINGLE LOOP" (CURRENT I THREADS THE LOOP A SINGLE TIME



IN THIS CASE

$$\oint \vec{B} \cdot d\vec{l} = \oint \vec{B} \cdot \{ dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z} \}$$

$$= \oint \vec{B} \cdot r\,d\phi\,\hat{\phi} \qquad \text{Q: WHY?}$$

$$= \oint \frac{\mu_0}{2\pi} \frac{I}{R} R\,d\phi = \mu_0 I$$

("SINGLE LOOP")

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$ REGARDLESS OF THE SHAPE OF THE LOOP. ("SINGLE LOOP").

WHAT IF THERE ARE MULTIPLE CURRENTS THREADING THE LOOP?

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \sum_j I_j .$$

WHAT IF BULK CURRENTS THREAD THE LOOP?

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \iint \vec{J} \cdot \hat{n}\, dA$$

NOTICE THE SURFACE IS ARBITRARY SO LONG AS IT CONNECTS TO THE LOOP AT THE EDGES.

WE INVOKE STOKE'S THEOREM WITH ARBITRARY LOOP TO GIVE

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J},$$

THE STATIC FORM OF AMPÈRE'S LAW.

!EVALUATION OF $\vec{\nabla} \cdot \vec{B}$:

START WITH THE BIOT-SAVART LAW:

$$d\vec{B} = \frac{\mu_0}{4\pi} I \, d\vec{l} \times \frac{\hat{r}}{r^2}.$$

APPLY A DIVERGENCE ACTING ON FIELD POINTS:

$$\vec{\nabla} \cdot d\vec{B} = \frac{\mu_0}{4\pi} I \, \vec{\nabla} \cdot \left\{ d\vec{l} \times \frac{\hat{r}}{r^2} \right\}$$

WITH $\vec{\nabla} \cdot \left\{ d\vec{l} \times \frac{\hat{r}}{r^2} \right\}$

$$= \frac{\hat{r}}{r^2} \cdot \vec{\nabla} \times d\vec{l} - d\vec{l} \cdot \vec{\nabla} \times \frac{\hat{r}}{r^2}$$

AND $\vec{\nabla} \times d\vec{l} = 0$ Q: WHY?

AND $\vec{\nabla} \times \frac{\hat{r}}{r^2} = 0$ Q: WHY?,

$\vec{\nabla} \cdot \vec{B} = 0$ ALWAYS.

"COULOMB'S LAW" FOR MAGNETIC FIELDS.

OHM'S LAW AND CURRENTS.

RECALL CONTINUITY $\vec{\nabla} \cdot \vec{J} + \frac{\partial}{\partial t}\rho = 0$

LEADS TO STATICS

$$\vec{\nabla} \cdot \vec{J} = 0 \quad \text{(STATICS)}.$$

$\vec{\nabla} \cdot \vec{J} = 0$ IN STATICS IMPLIES THERE IS NO ACCUMULATION OF CHARGE ANYWHERE.

IN MANY CASES, BUT NOT ALL CASES!

(OHM'S LAW) $\vec{J} = \sigma \vec{E}$ σ CONDUCTIVITY,

THIS IS NOT A FUNDAMENTAL EQUATION OF MAGNETOSTATICS.

Q: COME UP WITH A PHYSICAL SYSTEM WHERE $\vec{J} = \sigma \vec{E}$, σ CONSTANT, IS VIOLATED?

SOMETIMES THIS IS WRITTEN AS

$$\vec{E} = \rho \vec{J}, \quad \rho \text{ RESISTIVITY}$$

$$\rho = 1/\sigma.$$

MISC. COMMENTS!

- ρ IS THE RESISTIVITY $[\Omega \cdot m]$
- σ IS THE CONDUCTIVITY $\left[\frac{1}{\Omega \cdot m}\right]$ $\left[\frac{\text{"MHO"}}{m}\right]$
- AND AGAIN, $\vec{J} = \sigma \vec{E}$ IS VERY COMMON BUT NOT UNIVERSALLY VALID.

(• SHOULD THE MATERIAL BE ISOTROPIC, σ IS A TENSOR.)

- WHAT IS THE STATIC RATE OF EXPENDING ENERGY?

$$dW = \vec{F} \cdot d\vec{l} = q(\vec{E} + \vec{V} \times B) \cdot \vec{V} dt.$$

WITH $q = \rho \, d\mathcal{V}$ AND $\vec{J} = \rho \vec{V}$,
AND $(\vec{V} \times \vec{B}) \cdot \vec{V} = 0$,

$$dW = d\mathcal{V} \, \vec{E} \cdot \vec{J} \, dt$$

$$\frac{d}{dt}\left\{\frac{dW}{d\mathcal{V}}\right\} = \vec{E} \cdot \vec{J}$$

THIS IS
THE VOLUME RATE OF ENERGY THE CURRENTS EXPEND.

NOW, WE NEED TO ASK THE SUBTLE QUESTION OF WHAT CAUSES STATIONARY CURRENTS.

HINTS OF AN ISSUE: THE ELECTROSTATIC (IRROTATIONAL) ELECTRIC FIELD IS CONSERVATIVE, YET THE CURRENTS SET IN MOTION BY THE ELECTRIC FIELD ARE EXPENDING ENERGY; THEY ARE NON-CONSERVATIVE. IT SEEMS STATIONARY CURRENTS REQUIRE THE PRESENCE OF ROTATIONAL ELECTRIC FIELDS. WE CALL THE SOURCES OF SUCH ROTATIONAL ELECTRIC FIELDS ELECTROMOTIVE FORCES. A FURTHER COMMENT: THAT IRROTATIONAL FIELDS ARE CONSERVATIVE FOLLOWS FROM THEM BEING $-\vec{\nabla}\Phi$. A ROTATIONAL FIELD CANNOT BE SO DESCRIBED.

OHM'S LAW THEN READS

$$\vec{J} = \sigma\left(\vec{E}^{I} + \vec{E}^{R}\right)$$

WITH $\vec{E}^{I}$ THE IRROTATIONAL PART AND $\vec{E}^{R}$ THE ROTATIONAL PART.

WE DEFINE THE "ELECTROMOTIVE FORCE" $\mathcal{E}$ AROUND A CLOSED LOOP AS,

$$\mathcal{E} \equiv \oint \left(\vec{E}^{I} + \vec{E}^{R}\right) \cdot d\vec{l}$$

$$= \oint \vec{E}^{R} \cdot d\vec{l} \quad \left\{ = \oint \frac{\vec{J}}{\sigma} \cdot dl \right\}$$

THIS IS A RESTATEMENT THAT STATIONARY CURRENTS ARE DUE TO NON-CONSERVATIVE SOURCES.

WE'LL FIND IN SOME SYSTEMS (e.g., A BATTERY AND RESISTOR) IT MAKES SENSE TO ASSIGN $\mathcal{E}$ TO A SPECIFIC SOURCE. IN SOME OTHER SYSTEMS, IT DOESN'T MAKE SENSE TO LOCALIZE $\mathcal{E}$.

EXERCISE: SUPPOSE NO CURRENTS ARE FLOWING. INTEGRATE AS FOLLOWS ALONG A PATH TRAVERSING THE REGION CONTAINING THE NON-CONSERVATIVE (ROTATIONAL) FIELDS:

$$-\int_1^2 \vec{E}^I \cdot \vec{d\ell} = +\int_1^2 \vec{E}^R \cdot \vec{d\ell} \qquad Q: \text{WHY?}$$

$$= \oint \vec{E}^R \cdot \vec{d\ell} \qquad Q: \text{WHY?}$$

THE VOLTAGE DIFFERENCE BETWEEN POINTS 1 AND 2 IS $-\int_1^2 \vec{E}^I \cdot \vec{d\ell}$,

SO THE "OPEN-CIRCUIT" (NO CURRENT) VOLTAGE DIFFERENCE IS A MEASUREMENT OF THE ELECTROMOTIVE FORCE. THIS IS HOW YOU MEASURE THE $\mathcal{E}$ OF A BATTERY BY MEASURING THE VOLTAGE ACROSS ITS TERMINALS.

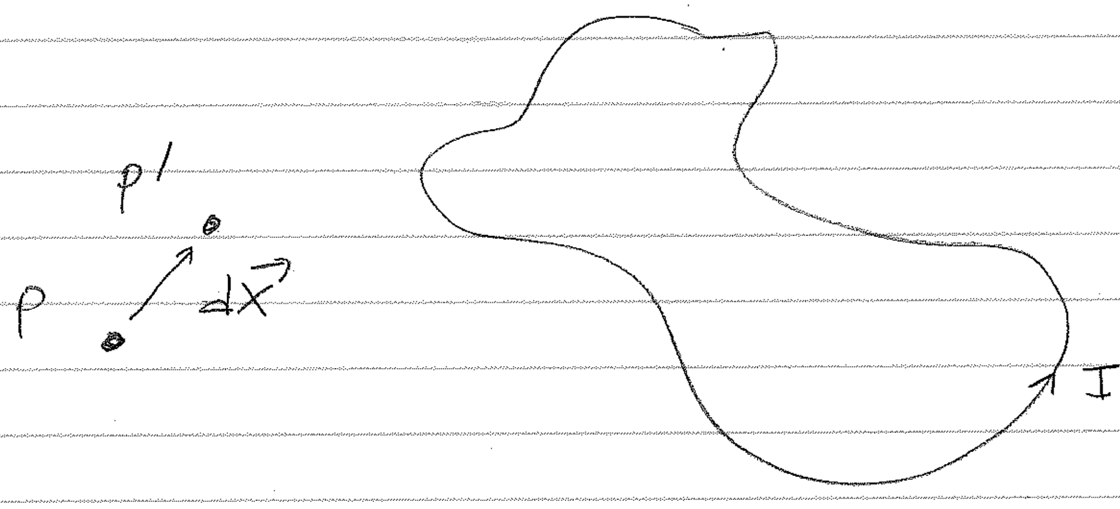
MAGNETIC POTENTIAL

COULD THE MAGNETIC POTENTIAL BE A SCALAR? THAT IS, IS THERE Φ_M SUCH THAT

$$\vec{B} = -\mu_0 \vec{\nabla} \Phi_M \; ?$$

(THE μ_0 HERE IS A CONVENTION.)

CONSIDER THE GEOMETRY OF A CURRENT-CARRYING LOOP AND A FIELD POINT P



NOW, CONSIDER THE CHANGE IN Φ_M AS P GOES TO P':

$$d\Phi_M = -\frac{d\vec{x} \cdot \vec{B}}{\mu_0}$$

WE EXPRESS $\vec{B}$ AS A BIOT-SAVART INTEGRAL:

$$d\Phi_M = -\frac{1}{4\pi} I \oint d\vec{x} \cdot \left(d\vec{\ell} \times \frac{\hat{r}}{r^2} \right)$$

$$= -\frac{1}{4\pi} I \oint \hat{r} \cdot \frac{\left(d\vec{x} \times d\vec{\ell} \right)}{r^2}$$

NOTICE WE'D GET THE SAME EXPRESSION IF P WERE FIXED AND WE MOVE THE LOOP BY $-d\vec{x}$.

WE'VE SEEN THIS BEFORE WHEN WE EVALUATED THE DISCONTINUITY IN Φ ON CROSSING A DIPOLE LAYER:

$$d\Omega = \oint \hat{r} \cdot \frac{\left(d\vec{x} \times d\vec{\ell} \right)}{r^2} ;$$

THIS IS THE CHANGE IN THE SOLID ANGLE OF THE LOOP AS "SEEN" BY P WHEN THE LOOP MOVES BY $-d\vec{x}$.

HENCE

$$d\Phi_M = \frac{I}{4\pi} d\Omega .$$

NOTICE THERE IS A DISCONTINUITY IN Φ_M ON CROSSING THE ASSOCIATED "SHEET" ATTACHED TO THE LOOP. ON ONE SIDE OF THE "SHEET" Ω IS $+2\pi$, JUST ACROSS THE SHEET Ω IS -2π. BUT THE "SHEET" IS LARGELY ARBITRARY, IT ONLY NEED CONNECT TO THE LOOP AT ITS EDGES.

NOW EVALUATE THE LINE INTEGRAL THROUGH THE LOOP

$$\oint \vec{B} \cdot d\vec{l} = -\mu_0 \oint \vec{\nabla}\Phi_M \cdot d\vec{l}$$

$$= \mu_0 \, \Delta\Phi_M = \mu_0 \frac{I}{4\pi} 4\pi$$

$$= \mu_0 I$$

IT FOLLOWS THAT

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}.$$

THIS IS A GEOMETRIC DERIVATION OF THE STATIC AMPÉRE'S LAW.

THIS ILLUSTRATES WHY, IN GENERAL, $\vec{B}$ CANNOT BE DERIVED FROM A SINGLE-VALUED SCALAR POTENTIAL. Φ_M AS A WAY TO DERIVE $\vec{B}$ IS THEREFORE PRACTICAL ONLY ABSENT CURRENTS.

WE WILL SEE THAT MAGNETIC MATERIALS ARE ONE SUCH SYSTEM! HERE Φ_M CAN BE VERY HANDY. BUT IT IS AN UNTRADITIONAL APPROACH.

VECTOR POTENTIAL.

WE START BY POSITING CURRENTS ARE THE ONLY SOURCE OF $\vec{B}$ FIELDS AND $\vec{B}$ COMES FROM A BIOT-SAVART LAW.

RECALL $\vec{\nabla}\cdot\vec{B} = 0$ SO

WITH $\vec{B} = \vec{\nabla}\times\vec{A}$, WE HAVE

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0.$$

NOW TO FIND $\vec{A}$ SUCH THAT $\vec{B} = \vec{\nabla}\times\vec{A}$. START WITH THE BIOT-SAVART LAW

$$\vec{B} = \frac{\mu_0}{4\pi}\iiint \vec{J}\times\frac{\hat{r}}{r^2}\,dv$$

$$= \frac{\mu_0}{4\pi}\iiint \vec{J}\times\vec{\nabla}\frac{1}{r}\,dv$$

WITH $\vec{\nabla}$ ACTING ON FIELD COORDINATES

$$= \vec{\nabla}\times\frac{\mu_0}{4\pi}\iiint\frac{\vec{J}}{r}\,dv$$

HENCE

$$\vec{A} = \frac{\mu_0}{4\pi}\iiint\frac{\vec{J}}{r}\,dv.$$

$$\vec{A} = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}}{r}\, d\tau$$

IS SIMILAR TO $\Phi = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho}{r}\, d\tau$.

$\vec{A}$ IS ALSO MUCH EASIER TO CALCULATE THAN $\vec{B}$. KEEP IN MIND $\vec{A}$ IS IN THE DIRECTION OF THE CURRENTS.

A COUPLE COMMENTS:

- TO $\vec{A}$ YOU CAN ADD ANYTHING WHOSE CURL VANISHES.
- LINE CURRENTS, e.g., WOULD HAVE $\vec{A} = \frac{\mu_0}{4\pi} \oint I \frac{\vec{dl}}{r}$.

THIS IS PROBABLY A GOOD PLACE TO INTRODUCE MAGNETIC MOMENTS. RECALL THE ELECTRIC DIPOLE MOMENT $\vec{p}$:

$$\vec{p} = \iiint \vec{r}\,'\rho(\vec{r}\,')\,dv'$$

SO YOU EXPECT A REASONABLE DEFINITION IS

$$\vec{m} = \iiint \vec{r}\,' \times \vec{J}(\vec{r}\,')\,dv'$$

(OR $\frac{1}{2}\iiint \vec{r}\,' \times \vec{J}(\vec{r}\,')\,dv'$;

JACKSON EQN. 5.54).

NOTE THE MECHANICAL ANALOGY. FOR ANGULAR MOMENTUM $\vec{S}$

$$\vec{S} = \iiint \rho_m \vec{r}\,' \times \vec{v}(\vec{r}\,')\,dv'$$

AND FOR A CHARGE DENSITY ρ MOVING WITH VELOCITY $\vec{v}$

$$\vec{m} = \frac{1}{2}\iiint \rho\,\vec{r}\,' \times \vec{v}(\vec{r}\,')\,dv'$$

RECALL $\vec{\tau} = \vec{m} \times \vec{B}$ (THE STARTING POINT OF MAGNETOSTATICS) AND

$$U = -\vec{m} \cdot \vec{B}$$

(C.F. $U = -\vec{P} \cdot \vec{E}$ IN ELECTROSTATICS).

FROM THIS WE GET THE FAMOUS (GYROMAGNETIC) $|\vec{m}|/|\vec{S}|$ FOR PARTICLES OF MASS m AND CHARGE e:

$$\Gamma = \frac{e}{2m}.$$

IF THERE'S MORE COMPLICATED STRUCTURE THAN A POINT PARTICLE THEN WE WRITE

$$\Gamma = g\frac{e}{2m}; \quad g \text{ THE "} g \text{ FACTOR".}$$

Q: WHY IS THIS HUGELY IMPORTANT?
A: ASK PROF. DAVID HERTZOG.