Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
October 22, 2020, 11am
On-line lecture

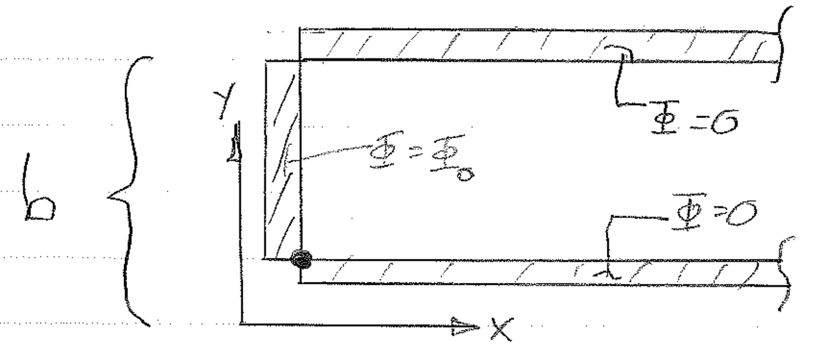
Administrative:

1. Homework 3 posted at
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
2. Homework 2 grading notes posted at
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
3. Ensure you're getting your graded homework back.
2. Draft of this lecture posted at
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
3. Office hours today after class at 12:30.

Lecture: Methods of finding potentials in boundary-value problems. (Jackson chapters 2 & 3).

Section 2.10. 2D problems in rectangular coordinates.
Section 2.11. Fields & charge densities in 2D corners and edges.
Section 3.3 Azimuthal symmetry: boundary-value problems and the 1/R expansion.
(NB., we'll get to 3D cylindrical geometries in section 3.8).

RECALL THE RESULT OF THE EXAMPLE!

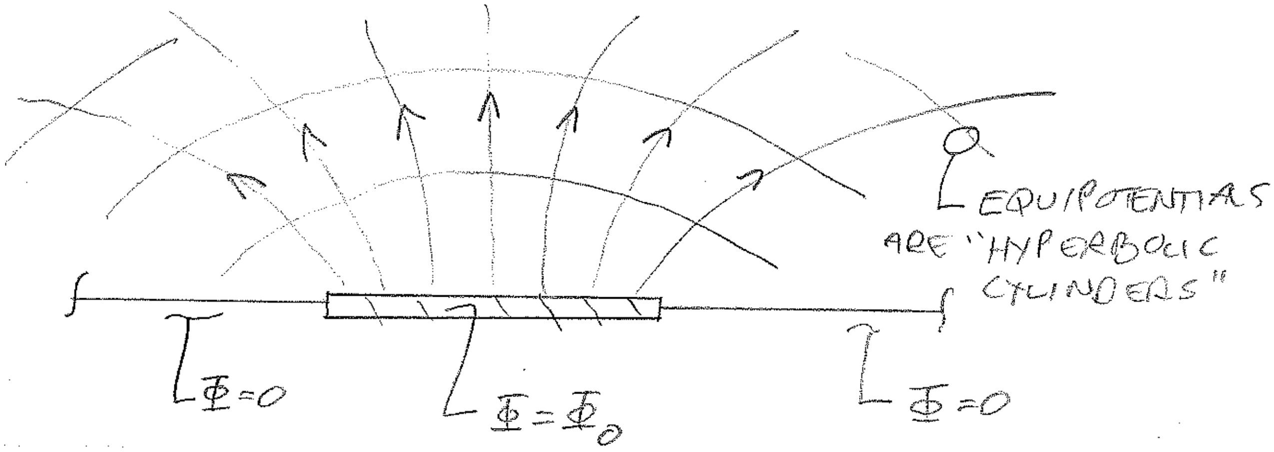


VIA SEPARATION OF VARIABLES, THIS HAS SOLUTION

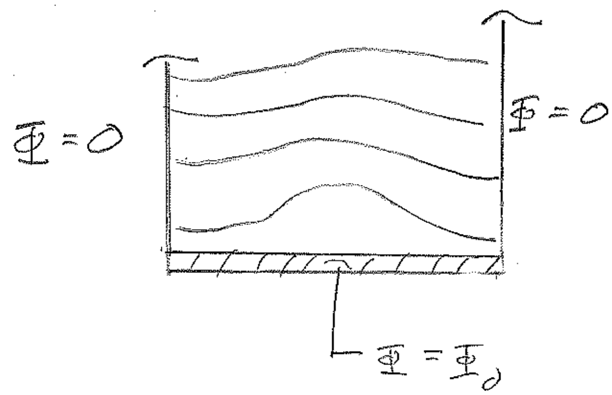
$$\Phi(x,y) = \Phi_0 \sum_{\substack{n \\ n \text{ ODD}}} \frac{4}{n\pi} \cdot \sin n \frac{\pi}{b} y \cdot e^{-n \frac{\pi}{b} x}$$

IT'S BEYOND THIS CLASS, BUT THIS IS ALSO SOLVABLE WITH A "SCHWARTZ TRANSFORMATION", FROM CONFORMAL-TRANSFORMATION IN COMPLEX ANALYSIS.

THE SYSTEM WITH A KNOWN SOLUTION IS

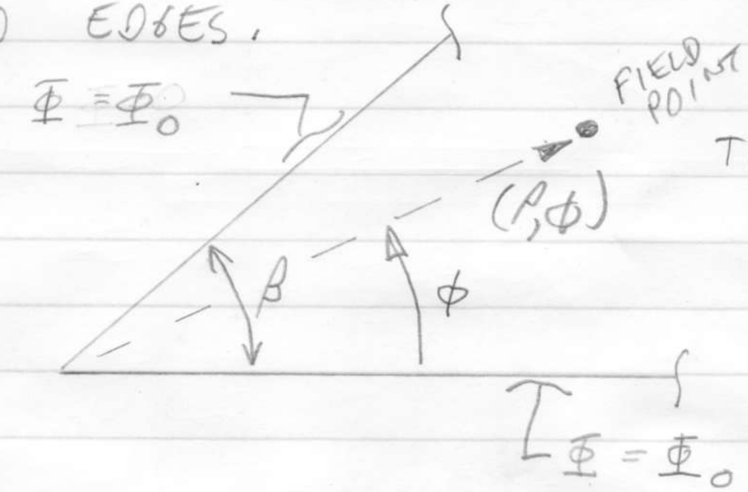


A SCHWARTZ TRANSFORMATION TURNS THIS INTO



JACKSON §2.11 CYLINDRICAL COORDINATES I

A 2D EXAMPLE: FIELDS NEAR CORNERS AND EDGES.



TWO INTERSECTING PLANES

WE WRITE THE LAPLACIAN IN CYLINDRICAL COORDINATES (SUPPRESSING THE d^2/dz^2 TERM)

$$\nabla^2 \Phi = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\Phi\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\Phi = 0$$

1. SEPARATE VARIABLES
$$\Phi(\rho,\phi) = R(\rho)\Phi(\phi),$$

2. MULTIPLY BY ρ^2/Φ

$$\underbrace{\frac{1}{\mathbb{R}}\,\rho\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\mathbb{R}\right)}_{+\nu^2} + \underbrace{\frac{1}{\Phi}\frac{d^2}{d\phi^2}\Phi}_{-\nu^2} = 0$$

REGARDING SEPARATION CONSTANTS, SEE EXAMPLE LAST LECTURE.

THE EQUATIONS HAVE SOLUTIONS

$$\nu = 0;\quad \mathbb{R}(\rho) = a_0 + b_0 \ln\rho$$

$$\Phi(\phi) = A_0 + B_0\phi$$

$$\nu \neq 0:\quad \mathbb{R}(\rho) = a\rho^{+\nu} + b\rho^{-\nu}$$

$$\Phi(\phi) = A\cos\nu\phi + B\sin\nu\phi.$$

NOW, START TO APPLY BOUNDARY CONDITIONS

$$\Phi(\rho, \phi = 0) = \Phi_0;$$

$$\Phi(\rho, \phi = \beta) = \Phi_0$$

APPLYING BOUNDARY CONDITIONS IS A LITTLE TRICKY. (SEE JACKSON P.77 FOR A DISCUSSION OF THIS.)

b_0 (IN $b_0 \ln \rho$) $= 0$ (NO $\ln 0$, CAN'T HAVE ρ DEPENDENCE);

B_0 (IN $B_0 \phi$) $= 0$ (HAVE TO HAVE SAME VALUE AT $\phi = 0, \beta$);

b (IN $b \rho^{-\nu}$) $= 0$ (CAN'T HAVE $1/\rho^{\nu}$)

A (IN $A \cos \nu\phi$) $= 0$ (A TERM DEPENDENT ON ρ MUST VANISH AT $\phi = 0, \beta$).

THE BOUNDARY CONDITIONS REQUIRE $\sin \nu\beta = 0$, SO $\nu = m\pi/\beta$.

WE HAVE PARTIAL SOLUTION

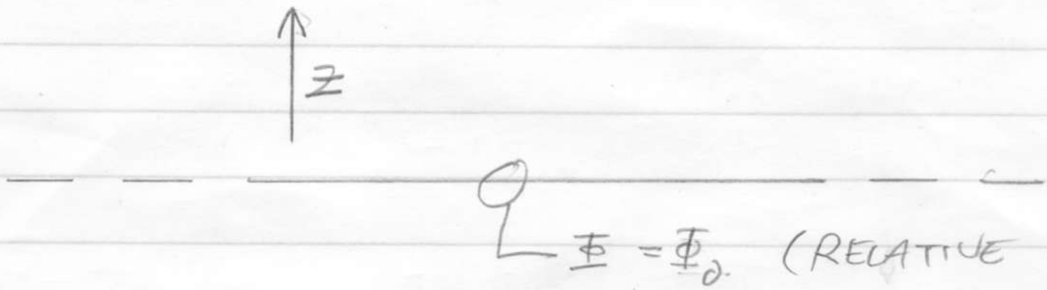
$$\Phi(\rho, \phi) = \Phi_0 + \sum_{m=1}^{\infty} a_n \rho^{m\frac{\pi}{\beta}} \cdot \sin m\frac{\pi}{\beta}\phi.$$

Q: DOES THIS SATISFY THE BOUNDARY CONDITIONS?

A: FOR $\phi = 0, \beta$, YES

BUT WE HAVE AN ISSUE FOR $\rho \to \infty$.

SINCE THE TWO PLANES EXTEND TO $\rho \to \infty$, WE CAN'T JUST SET $\Phi(\rho \to \infty) = 0$. THIS ISSUE IS SIMILAR IN SPIRIT TO THE POTENTIAL AT ∞ DUE TO AN INFINITE-PLANE AT FINITE POTENTIAL:



SO, CAN YOU SET $\Phi(z \to \infty)$ TO 0? NO, BUT YOU CAN SET $\Phi(z = z_0)$ TO A REFERENCE POTENTIAL. (SO "$\Phi = \Phi_0$" HAS Φ_0 RELATIVE TO A REFERENCE POTENTIAL)

TO STUDY CORNERS AND EDGES, WE'RE NOT SO INTERESTED IN LARGE ρ, WE'RE MORE INTERESTED IN THE BEHAVIOR OF SOLUTIONS AS $\rho \to 0$.

NOTICE AS $\rho \to 0$, THE $m=1$ TERM DOMINATES THE SERIES. ($\rho^{m\pi/\beta}$).
SO,

$$\Phi(\rho \to 0, \phi) \approx \Phi + a_1 \rho^{\pi/\beta} \sin \tfrac{\pi}{\beta}\phi$$

THE POTENTIAL NEAR THE CORNER GIVES FIELDS NEAR THE CORNER:

$$E_\rho = -\frac{d}{d\rho}\Phi = -a_1 \frac{\pi}{\beta} \rho^{\frac{\pi}{\beta}-1} \cdot \sin \tfrac{\pi}{\beta}\phi;$$

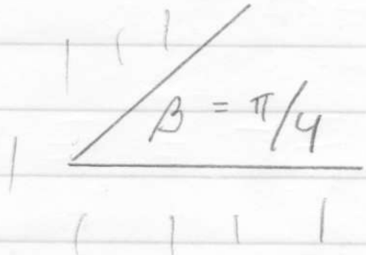
$$E_\phi = -\frac{1}{\rho}\frac{d}{d\phi}\Phi = -a_1 \frac{\pi}{\beta} \rho^{\pi/\beta-1} \cdot \cos \tfrac{\pi}{\beta}\phi.$$

THE SURFACE CHARGE ON THE PLATES IS RELATED TO THE NORMAL DERIVATIVE OF THE FIELD (WHICH IN TURN IS RELATED TO $E\phi$).

$$\frac{\sigma(\rho, \phi=0, \beta)}{\varepsilon_0} = E_\phi(\rho, \phi=0, \beta)$$
$$= -a_1 \frac{\pi}{\beta} \rho^{\pi/\beta - 1}.$$

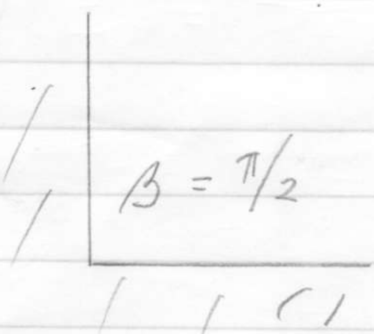
NOTICE E_ρ, E_ϕ AND σ CONTAIN $\rho^{\pi/\beta-1}$, AND $\pi/\beta-1$ CAN BE POSITIVE, NEGATIVE OR ZERO, DEPENDING ON β.

SPECIAL CASES.



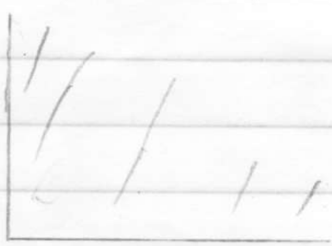
$\beta = \pi/4$ $\qquad \frac{\pi}{\beta} - 1 = 3 \qquad \sigma \sim \rho^3$

NEAR CORNER



$\beta = \pi/2 \qquad \frac{\pi}{\beta} - 1 = 1 \qquad \sigma \sim \rho^1$

$\beta = \pi \qquad \frac{\pi}{\beta} - 1 = 0 \qquad \sigma \sim \rho^0$



$\frac{\pi}{\beta} - 1 = -\frac{1}{3} \qquad \sigma \sim \rho^{-1/3}$

$\beta = 3/2\,\pi$

$\beta = 2\pi \qquad \frac{\pi}{\beta} - 1 = -\frac{1}{2} \qquad \sigma \sim \rho^{-1/2}$

THIS EXPLAINS THE PHYSICS OF A LIGHTNING ROD.

WE'LL RETURN TO CYLINDRICAL COORDINATES IN 3D; THIS BRINGS IN BESSEL'S EQUATION.

JACKSON § 3.1
SEPARATION OF VARIABLES IN SPHERICAL COORDINATES.
START WITH AZIMUTHAL SYMMETRY, THEN LATER RELAX THE SYMMETRY REQUIREMENT.

THIS SEPARATION IS NON-TRIVIAL.

$$\nabla^2 \Phi = 0$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\Phi\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Phi = 0.$$

IT ISN'T OBVIOUS HOW TO DO THIS SEPARATION. WE PROCEED IN STEPS. FORTUNATELY YOU ONLY HAVE TO DO THIS ONCE.

WE ASSUME A FORM

$$\Phi(r,\theta,\phi) = \frac{U(r)}{r} \mathbb{P}(\theta) \mathbb{Q}(\phi).$$

FOCUS FOR NOW ON THE LAST TERM $\frac{1}{r^2 \sin^2\theta} \frac{d^2}{d\phi^2} \Phi$.

1. REPLACE Φ WITH $\frac{U}{r} \mathbb{P}\mathbb{Q}$;

2. MULTIPLY BY $\frac{r^2 \sin^2\theta}{U \mathbb{P}\mathbb{Q}}$

THE LAST TERM IS

$$\frac{1}{\mathbb{Q}} \frac{d^2}{d\phi^2} \mathbb{Q}$$

[THE FULL EQUATION IS JACKSON EQN. 3.3

$$r^2 \sin^2\theta \left\{ \frac{1}{U} \frac{d^2 U}{dr^2} + \frac{1}{\mathbb{P} r^2 \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\mathbb{P}}{d\theta} \right) \right\} + \frac{1}{\mathbb{Q}} \frac{d^2}{d\phi^2} \mathbb{Q} = 0 \Bigg]$$

THIS HAS TO HOLD FOR ALL r, θ, ϕ. THE LAST TERM CONTAINS ϕ, THE OTHER TERMS DO NOT.

SO, THE LAST TERM IS CONSTANT

$$\frac{1}{\Phi}\frac{d^2}{d\phi^2}\Phi = -m^2.$$

OF COURSE, THE REMAINING TWO TERMS ARE A MESS:

$$r^2 \sin^2\theta \left\{ \frac{1}{U}\frac{d^2 U}{dr^2} + \frac{1}{P r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) \right\}$$

$$- m^2 = 0.$$

THE FIRST TWO TERMS CONTAIN r AN θ; WE CONTINUE THE SEPARATION.

1. FACTOR OUT $\frac{1}{Pr^2}$ OUT OF $\{ \cdots \}$.
2. DIVIDE EVERYTHING BY $\sin^2\theta$.

$$\frac{r^2}{U}\frac{d^2 U}{dr^2} + \frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)$$

$$- \frac{m^2}{\sin^2\theta} = 0$$

THE FIRST TERM $\frac{r^2}{U}\frac{d^2}{dr^2}U$ IS PURELY A FUNCTION OF r; THERE ARE NO OTHER TERMS CONTAINING r; THE EQUATION MUST HOLD FOR ALL r, θ, ϕ; SO THIS TERM IS A CONSTANT:

$$\frac{r^2}{U}\frac{d^2}{dr^2}U = \ell(\ell+1).$$

THIS LEAVES US WITH (AFTER MULTIPLYING BY P)

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}P\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P = 0.$$

SUMMARIZING: WE HAVE 3 EQUATIONS:

$$\frac{1}{\Phi}\frac{d^2}{d\phi^2} + m^2 = 0,$$

$$\frac{d^2U}{dr^2} - \ell(\ell+1)\frac{U}{r^2} = 0,$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}P\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P = 0.$$

THESE HAVE SOLUTIONS

$$\Phi(\phi) = e^{\pm i m \phi}$$

$$U(r) = r^{\ell}, \frac{1}{r^{\ell+1}}$$

THE $\mathbb{P}(\theta)$ EQUATION IS THE "GENERALIZED LEGENDRE EQUATION". IT HAS TWO SETS OF SOLUTIONS

$P_\ell^m(\cos\theta)$ IS REGULAR EVERYWHERE.

THE IRREGULAR SOLUTION $Q_\ell^m(\cos\theta)$ IS SINGULAR FOR $\cos\theta = 0$ (AND THEREFORE Q_ℓ^m IS LESS-FREQUENTLY SEEN).

SEE JACKSON §3.2. UNFORTUNATELY JACKSON DOES NOT DISCUSS THE IRREGULAR SOLUTIONS Q_ℓ^m.

THIS EQUATION FOR $P(\theta)$ IS OFTEN WRITTEN (WITH $x = \cos\theta$):

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x}\right]P = 0.$$

WE'LL START EXPLORING SOLUTIONS FOR THE SIMPLE CASE OF AZIMUTHAL SYMMETRY WITH θ OVER THE ENTIRE RANGE $[0, \pi]$.

WITH AZIMUTHAL SYMMETRY, $\Phi(\phi) = e^{\pm im\phi}$ IS CONSTANT,

SO $m = 0$. SETTING $m = 0$ IN THE GENERALIZED LEGENDRE EQUATION GIVES THE "LEGENDRE EQUATION" (JACKSON EQN. 3.10), WITH SOLUTIONS THE LEGENDRE POLYNOMIALS

$P_\ell(x)$, THE REGULAR AND COMMON SOLUTION, AND

$Q_\ell(x)$ THE IRREGULAR AND LESS COMMON SOLUTION.

IT HAPPENS THAT $P_\ell^0(x) = P_\ell(x)$, BUT TAKE CARE FOR $m \neq 0$. (SEE JACKSON EQN 3.49, FOR EXAMPLE.)

EXPLICIT FORMS FOR $P_\ell(x)$

$P_0(x) = 1$ EVEN

$P_1(x) = x$ ODD

$P_2(x) = \frac{1}{2}(3x^2 - 1)$ EVEN

$\vdots$ $\vdots$

SEE THIS TABULATED IN JACKSON EQN 3.15).

TO USE $P_\ell(x)$ IN BOUNDARY VALUE PROBLEMS, YOU'LL NEED TO KNOW THE ORTHOGONALITY NORMALIZATION AND INTEGRATION OF $P_\ell(x)$ OVER A RANGE. SEE JACKSON § 3.2 FOR THIS.

FOR EXAMPLE, FOR $x \in [-1, +1]$,

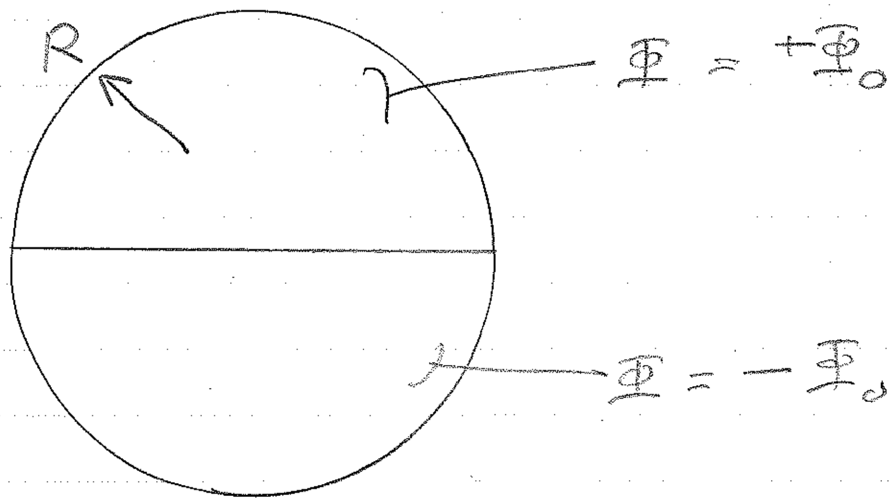
$$\int_{-1}^{+1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1} \delta_{\ell \ell'}.$$

NOTICE THAT $P_\ell(x)$ AS USUALLY DEFINED ARE NOT NORMALIZED.

WE CAN NOW EXHIBIT THE AZIMUTHALLY-SYMMETRIC GENERAL SOLUTION ASSUMING $x=0$ IS IN THE RANGE:

$$\Phi(r,\phi) = \sum_{\ell=0}^{\infty} \left[a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta)$$

EXAMPLE: HEMISPHERES AT DIFFERENT POTENTIALS. FIND THE POTENTIAL INSIDE.



APPLY BOUNDARY CONDITIONS TO THE SOLUTION

$$\Phi(r,\theta) = \sum_\ell \left[a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta).$$

$\Phi(r \to 0)$ IS FINITE $\longrightarrow b_\ell = 0.$

$$\Phi(r=R,\theta) = \sum_\ell a_\ell R^\ell P_\ell(\cos\theta).$$

WE AGAIN WANT THE EXPANSION COEFFICIENTS a_ℓ: WE GET THEM BY ORTHOGONALITY OF P_ℓ.

EXTRACT $a_{\ell'}$!

1. MULTIPLY BY $P_{\ell'}(\cos\theta)$;
2. INTEGRATE $\int_{-1}^{+1} (\ldots)\, d\cos\theta$.

$$\int_{-1}^{+1} \Phi(r=R,\theta)\, P_{\ell'}(\cos\theta)\, d\cos\theta$$

$$= \int_{-1}^{+1} \sum_{\ell} a_{\ell} R^{\ell} \underbrace{P_{\ell}(\cos\theta)\, P_{\ell'}(\cos\theta)}_{\int_{-1}^{+1} P_{\ell}(\cos\theta) P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell'}}\, d\cos\theta$$

$$\int_{-1}^{0} \Phi_0\, P_{\ell'}(\cos\theta)\, d\cos\theta$$

$$\pm \int_{0}^{+1} \Phi_0\, P_{\ell'}(\cos\theta)\, d\cos\theta$$

$$= a_{\ell'} R^{\ell'} \frac{2}{2\ell'+1}$$

NOTICE THAT BECAUSE THE TWO INTEGRALS HAVE OPPOSITE SIGN, WE ONLY KEEP ODD $P_\ell(\cos\theta)$

(ℓ' IS ODD; SEE JACKSON EQN 3.15).

HENCE (WITH $\ell' \to \ell$) FOR ℓ ODD:

$$a_\ell = \frac{2\ell+1}{2R^\ell}\, 2\Phi_0 \int_0^1 P_\ell(\cos\theta)\, d\cos\theta.$$

($=0$ FOR ℓ EVEN).

THIS INTEGRAL IS DO-ABLE, THOUGH COMPLICATED: JACKSON EQN. 3.26:

$$a_\ell = \frac{2\Phi_0}{R^\ell}\left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(2\ell+1)\left[(2\ell-2)!!\right]}{2\left[\left(\frac{\ell+1}{2}\right)!\right]}$$

(ℓ ODD)

Q: HOW WOULD THIS DISCUSSION CHANGE IF WE WANTED THE POTENTIAL OUTSIDE?

A: $r^\ell \to \frac{1}{r^{\ell+1}}$ WAY BACK IN APPLYING BOUNDARY CONDITIONS.

THE $1/r$ EXPANSION AS AN APPLICATION OF P_ℓ. JACKSON EQN. 3.38.

SUPPOSE YOU HAVE A POINT CHARGE q, WHAT'S THE POTENTIAL?

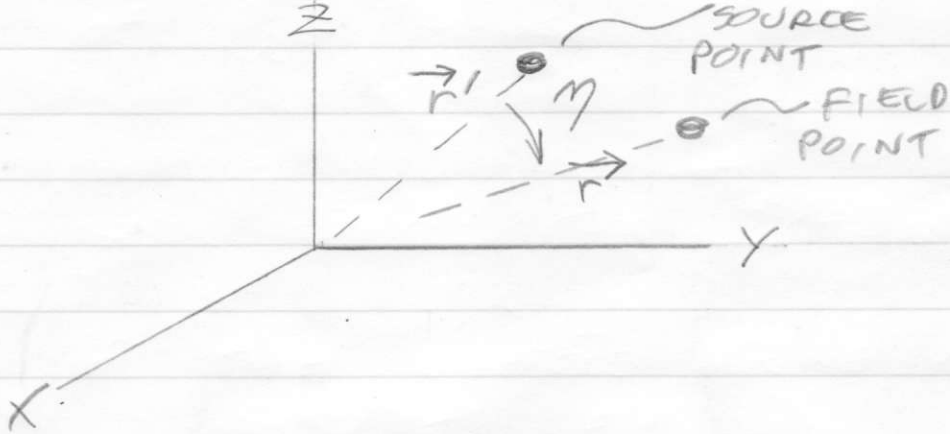
IF THE CHARGE WERE AT THE ORIGIN, IT'S EASY:

$$\Phi(r) = \frac{1}{r}$$ (LET'S LEAVE ε_0 OUT FOR NOW).

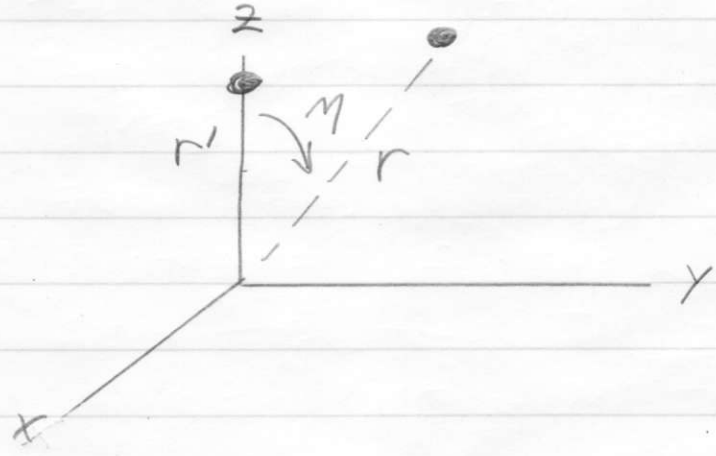
BUT SUPPOSE THE CHARGE ISN'T AT THE ORIGIN. HOW DO YOU EXPRESS THE POTENTIAL IN TERMS OF r' (THE DISTANCE OF THE CHARGE FROM THE ORIGIN), r (THE DISTANCE OF THE FIELD POINT FROM THE ORIGIN), AND γ (THE ANGLE BETWEEN THEM)?

THIS TURNS OUT TO BE VERY USEFUL, IF YOU CAN DEVINE THE SOLUTION ALONG ONE AXIS, YOU CAN GENERATE THE SOLUTION ALONG ANOTHER AXIS.

WE START WITH GEOMETRY:



NOW ROTATE THE SOURCE TO THE Z-AXIS: THIS DOESN'T CHANGE r', r OR η.



NOW, WE RELATE THIS GEOMETRY TO WHAT WE JUST DID REGARDING SPHERICAL SOLUTIONS TO LAPLACE'S EQUATION WITH AZIMUTHAL SYMMETRY.

NOTICE THE POTENTIAL DUE TO THE CHARGE AT $\vec{r}'$ SATISFIES LAPLACE'S EQUATION, EXCEPT AT THE POSITION OF THE CHARGE. SINCE THE CHARGE LIES ON THE Z-AXIS, THE POTENTIAL HAS AZIMUTHAL SYMMETRY. THEREFORE THE POTENTIAL $1/|\vec{r}-\vec{r}'|$ CAN BE EXPANDED:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell} \left(a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}}\right) P_\ell(\cos\gamma).$$

NOW LET'S LOOK AT A SIMPLER GEOMETRY: SUPPOSE THE FIELD POINT AS WELL WERE ON THE Z-AXIS. THEN $\cos\gamma = 1$; THEN $P_\ell(1) = 1$ FOR ALL ℓ. (SEE JACKSON EQN 3.15 WITH $x = 1$), AND

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell} \left[a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}}\right] \quad (\gamma = 0).$$

BUT WE ALSO HAVE THE IDENTITY

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}}$$

STILL WITH $\vec{r}$ AND $\vec{r}'$ ON THE Z-AXIS:
WE LOOK AT TWO CASES:

1. $r > r'$ (FIELD POINT FARTHER AWAY)

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} \frac{1}{\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}}}$$

$$= \frac{1}{r} \frac{1}{\left\{1 - \frac{1}{2}\left[\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\right] + \frac{3}{8}\left[\left[\cdots\right]^2\right] + \cdots\right\}}$$

$$= \frac{1}{r}\left\{1 + \left(\frac{r'}{r}\right) + \left(\frac{r'}{r}\right)^2 + \cdots\right\}$$

$$= \frac{1}{r}\sum_{\ell}\left(\frac{r'}{r}\right)^{\ell},$$

2. $r < r'$ (FIELD POINT CLOSER).
THE STEPS ARE SIMILAR EXCEPT $1/r'$ IS FACTORED OUT:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r'}\sum_{\ell}\left(\frac{r}{r'}\right)^{\ell}.$$

IF THE FIELD POINT WERE TAKEN OFF THE Z-AXIS, THE ABOVE EXPRESSIONS FOR THE RADIAL DEPENDENCE ARE UNCHANGED. THAT'S BECAUSE THE POTENTIAL IS SEPERABLE INTO INDEPENDENT RADIAL AND ANGULAR FACTORS.

THAT IS:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{\ell}\left(\frac{r'}{r}\right)^{\ell} P_{\ell}(\cos\gamma) \qquad (r > r')$$

AND

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r'}\left(\frac{r}{r'}\right)^{\ell} P_{\ell}(\cos\gamma) \qquad (r < r')$$

MORE COMPACTLY

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r_>}\sum_{\ell}\left(\frac{r_<}{r_>}\right)^{\ell} P_{\ell}(\cos\gamma).$$

(SEE JACKSON EQN. ON P. 103.)