Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
October 20, 2020, 11am
On-line lecture

Administrative:
1. Homework 3 posted on
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
2. Homework 2 solutions posted on
`faculty.washington.edu/ljrberg/AUT20_PHYS513`

Lecture: Methods of finding potentials in boundary-value problems. (Jackson chapter 2).
Section 2.8: Orthogonal functions and expansions (incl. separation of variables)
Separation of variables in rectangular coordinates.

JACKSON §2.8 ORTHOGONAL FUNCTIONS AND EXPANSIONS.

THE FOCUS HERE IS AGAIN SOLUTIONS TO LAPLACE'S EQUATION $\nabla^2\Phi = 0$. THESE SOLUTIONS ARE EXPLICITLY SUMS OF ORTHONORMAL FUNCTIONS, THE PARTICULAR CHOICE OF FUNCTIONS CHOSEN TO MATCH THE SYMMETRIES OF THE PROBLEM.

QUICK REVIEW. (YOU'VE SEEN THIS BEFORE, MAYBE IN QUANTUM MECHANICS): OVER SOME INTERVAL $[a, b]$ WITH INDEPENDENT VARIABLE ξ, THERE'S A SET OF FUNCTIONS $\{U_n(\xi)\}$ WHICH ARE NORMALIZABLE (SO-CALLED "SQUARE INTEGRABLE") AND ORTHOGONAL. THAT IS

(ORTHOGONAL) $$\int_a^b U_m^*(\xi)\, U_n(\xi)\, d\xi = 0 \qquad m \neq n;$$

NORMALIZABLE $$\int_a^b U_n^*(\xi)\, U_n(\xi)\, d\xi = 1;$$

OR COMPACTLY

$$\int_a^b U_n^*(\xi)\, U_m(\xi) = \delta_{n,m}.$$

AN ARBITRARY FUNCTION $f(\xi)$ CAN BE EXPANDED THUS

$$f(\xi) = \sum_n a_n U_n(\xi),$$

THAT IS, $\{U_n\}$ ARE "COMPLETE".

EXPLICITLY, a_n IS GIVEN BY

$$a_n = \int_a^b U_n^*(\xi) f(\xi)\, d\xi$$

Q: SHOW THIS.

WE ALSO HAVE

$$\sum_n U_n^*(\xi') U_n(\xi) = \delta(\xi - \xi')$$

(THAT IS, "CLOSURE")

Q: SHOW THIS.

EXAMPLE: SINES AND COSINES ON $x \in [-a/2, +a/2]$.

$$A_n(x) = \frac{1}{\sqrt{a}} \cos(2\pi n \, x/a) \quad n = 0, 1, \ldots$$

$$B_n(x) = \frac{1}{\sqrt{a}} \sin(2\pi n \, x/a) \quad n = 1, 2, \ldots$$

Q: SHOW A_n AND B_n ARE ORTHONORMAL OVER $[-a/2, +a/2]$.

SUPPOSE YOU'D LIKE TO EXPAND $f(x)$ IN SINES AND COSINES:

$$f(x) = \sum_n a_n A_n(x) + \sum_n b_n B_n(x)$$

WITH

$$a_n = \frac{1}{\sqrt{a}} \int_{-a/2}^{+a/2} f(x) \cos(2\pi n \, x/a) \, dx$$

AND

$$b_n = \frac{1}{\sqrt{a}} \int_{-a/2}^{+a/2} f(x) \sin(2\pi n \, x/a) \, dx$$

WHAT IF THE INTERVAL WERE INFINITE?

WE NEED TO TRANSITION FROM FUNCTIONS OVER THE INTERVAL $[-a/2, +a/2]$

$$U_n(x) = \frac{1}{\sqrt{a}} e^{\pm i 2\pi n x/a} \qquad n = 0, 1, \ldots$$

WITH $$a_n = \frac{1}{\sqrt{a}} \int_{-a/2}^{+a/2} e^{i 2\pi n x'/a} f(x')\, dx'$$

TO FUNCTIONS OVER THE INTERVAL ABOVE WHERE $a \to \infty$:

$$2\pi n/a \to k;$$

$$\sum_n \longrightarrow \int_{-\infty}^{+\infty} dn \longrightarrow \frac{a}{2\pi} \int_{-\infty}^{+\infty} dk;$$

$$a_n \longrightarrow \sqrt{\frac{2\pi}{a}}\, a(k)$$

WITH THESE,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(k) e^{+ikx} \, dk; \quad \text{AND}$$

$$a(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} \, dx.$$

THESE ARE THE "FOURIER INTEGRAL" AND THE "INVERSE FOURIER INTEGRAL". A GREAT REVIEW OF THIS IS "NUMERICAL RECIPES".

ORTHOGONALITY IS

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(k-k')x} \, dx = \delta(k-k').$$

CLOSURE IS

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-x')} \, dk = \delta(x-x').$$

(SOMETIMES, AS IN THE EXAMPLE TODAY, WE KEEP THE DISCRETE FORM ...)

SEPARATION OF VARIABLES.

THERE'S A CONSIDERABLE LITERATURE ON FINDING SOLUTIONS ψ TO

$$\mathcal{H}\psi = F,$$

AND, IN PARTICULAR, $\mathcal{H} = \nabla^2 + k^2$.

E.g., $k = 0$ FOR LAPLACE'S & POISSON'S EQUATION;

k REAL FOR THE WAVE (HELMHOLTZ) EQUATION;

$k = k(\vec{r})$ FOR SCHROEDINGER'S EQUATION.

THERE ARE IN GENERAL AN EMBARRASSINGLY LARGE NUMBER OF SOLUTIONS. WE'RE NOT SO INTERESTED IN FINDING ALL SOLUTIONS, WE'RE MORE INTERESTED IN FINDING PARTICULAR SOLUTIONS TO A PARTICULAR PROBLEM.

THIS BRINGS US BACK TO BOUNDARY CONDITIONS.

BOUNDARY CONDITIONS.

THE LARGE NUMBER OF SOLUTIONS DIFFER BY THE NATURE OF THE BOUNDARY CONDITIONS. (N.B., THE "INITIAL CONDITIONS" ARE BOUNDARY CONDITIONS FOR THE SURFACE AT $t=0$.)

SURFACES CAN BE "OPEN" OR "CLOSED." BOUNDARY CONDITIONS CAN BE DIRICHLET, NEUMANN OR MIXED ("CAUCHY" BOUNDARY CONDITIONS IN THIS CONTEXT).

FOR NOW, WE RESTRICT OURSELVES TO CLOSED OR "QUASI-CLOSED" SURFACES OR INTERIOR REGIONS WITH $\rho = 0$ (SO $\nabla^2 \Phi = 0$). (THERE ARE WAYS TO EXTEND THIS TO $\nabla^2 \Phi \neq 0$.)

THE CORRESPONDING ORTHONORMAL SOLUTIONS TO Φ ARE "HARMONIC" FUNCTIONS. THERE'S A VAST LITERATURE ON HARMONIC FUNCTIONS.

THESE HARMONIC FUNCTIONS, AND SOLUTIONS TO $\nabla^2 \Phi$ HAVE OBVIOUS PROPERTIES. FOR INSTANCE, IF Φ_1 AND Φ_2 ARE SOLUTIONS TO $\nabla^2 \Phi$, THEN $\Phi = \Phi_1 + \Phi_2$ IS A SOLUTION, etc.

YOU MIGHT GUESS THERE ARE AN INFINITE NUMBER OF COORDINATE SYSTEMS THAT HAVE UTILITY. BUT ONLY ABOUT 11 ARE "SEPERABLE", WHERE $\nabla^2 \Phi$ HAS SOLUTIONS

$$\Phi(\lambda_1, \lambda_2, \lambda_3) = \Lambda_1(\lambda_1)\Lambda_2(\lambda_2)\Lambda_3(\lambda_3)$$

THIS IS A VERY SPECIAL PROPERTY OF RECTANGULAR, SPHERICAL, CYLINDRICAL, ... SEE MORSE AND FESHBACH CHAPTER 5.

WE'LL EXPLOIT THESE "SEPERABLE, ORTHOGONAL" COORDINATE SYSTEMS,

EXAMPLE: SEPARATION INTO RECTANGULAR COORDINATES. HOPEFULLY, A FAMILIAR EXAMPLE.

$$\nabla^2 \Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \Phi = 0.$$

RECTANGULAR COORDINATES ARE SEPERABLE. THAT IS,

$$\Phi(x, y, z) = X(x)\, Y(y)\, Z(z).$$

WE'LL IMPLICITLY EXPLOIT UNIQUENESS: IF THIS DOES PROVIDE A SOLUTION, IT IS THE SOLUTION.

1. DIVIDE BY XYZ;
2. NOTICE TOTAL DERIVATIVES:

$$\frac{1}{X}\frac{d^2}{dx^2}X(x) + \frac{1}{Y}\frac{d^2}{dy^2}Y(y) + \frac{1}{Z}\frac{d^2}{dz^2}Z(z) = 0.$$

THIS HAS AN INTERESTING STRUCTURE; THE THREE SEPARATE, INDEPENDENT, ORDINARY DIFFERENTIAL EQUATIONS SUM TO ZERO FOR ALL VALUES OF x, y AND z. HENCE, EACH TERM IS SEPARATELY CONSTANT.

THAT IS,

$$-C_x^{\,2} - C_y^{\,2} + C_z^{\,2} = 0$$

1. C_x, C_y AND C_z ARE THE "SEPARATION CONSTANTS".
2. WE CHOOSE SIGNS TO EASE THE SOLUTION.

   FOR INSTANCE, IN 1D:

   $$\frac{1}{X}\frac{d^2}{dx^2}X(x) = -k^2$$

   HAS SOLUTION $\cos kx$, WHILE

   $$\frac{1}{X}\frac{d^2}{dx^2}X(x) = +k^2$$

   HAS SOLUTION $\cosh kx$.

3. NOT ALL SEPARATION CONSTANTS ARE INDEPENDENT! ONE DOF IS LOST SINCE THEIR SUM IS ZERO.

IN RECTANGULAR COORDINATES, THE SOLUTIONS ARE

$$X(x) \sim \cos C_x x, \quad \sin C_x x;$$

$$Y(y) \sim \cos C_y y, \quad \sin C_y y;$$

$$Z(z) \sim \cosh C_z z, \quad \sinh C_z z.$$

OR, EQUIVALENTLY

$$X(x) \sim e^{\pm i C_x x};$$

$$Y(x) \sim e^{\pm i C_y y};$$

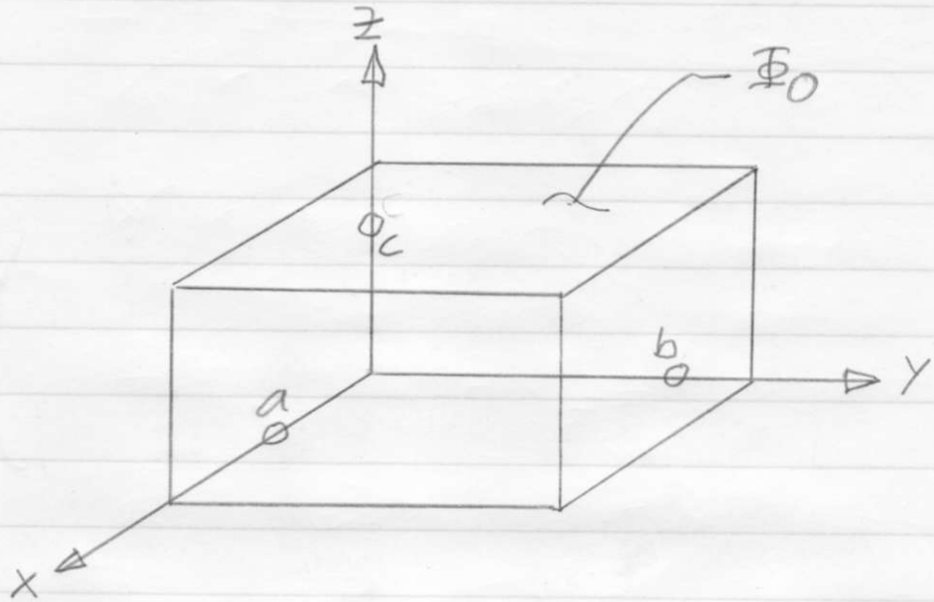
$$Z(z) \sim e^{\pm C_z z}.$$

HENCE

$$\Phi(x, y, z) \sim e^{\pm i C_x x} e^{\pm i C_y y} e^{\pm C_z z}.$$

NOW, WE NEED TO APPLY BOUNDARY CONDITIONS.

BOUNDARY CONDITIONS IN OUR EXAMPLE:



THE RECTANGULAR SURFACE AT $Z = c$ HAS POTENTIAL Φ_0, OTHER SURFACES HAVE POTENTIAL ZERO.

X: $\Phi(X=0, a; Y, Z) = 0$.

$\mathbb{X}(x) \sim \cos C_x X, \quad \sin C_x X$.

NOTICE THE BOUNDARY CONDITION AT $X = 0$ DOESN'T ALLOW COS SOLUTIONS. HENCE $\mathbb{X}(x) \sim \sin C_x X$. WE REQUIRE $\sin C_x X \to 0$ AT $X = a$. HENCE $C_{x,n} = n\pi/a$.

THIS DISCRETIZATION IS CHARACTERISTIC OF BOUNDED DOMAINS.

NOW, FOR THE $\mathcal{Y}$ DIFFERENTIAL EQUATION, IT'S BASICALLY THE SAME AS FOR $\mathcal{X}$. HENCE

$$\mathcal{Y}(y) \sim \sin C_y y$$

WITH $C_{y,m} = m\pi/b$.

$\mathcal{Z}$:

RECALL $C_x^2 + C_y^2 - C_z^2 = 0$. SO

$$C_{z;n,m}^2 = \left(n\frac{\pi}{a}\right)^2 + \left(m\frac{\pi}{b}\right)^2.$$

$$\mathcal{Z}(z) \sim \cosh C_z z, \quad \sinh C_z z;$$

WE REQUIRE $\Phi(x, y, z=0) = 0$, WHICH DOESN'T ALLOW COSH SOLUTIONS.

(WE'LL GET TO THE LAST BOUNDARY CONDITION AT $z = c$ SOON.)

WE HAVE A PARTIAL (MISSING THE LAST BOUNDARY CONDITION) SOLUTION

$$\Phi(x, y, z) = \sum_{n,m} A_{n,m}$$

$$\cdot \sin n\frac{\pi}{a}x \cdot \sin m\frac{\pi}{b}y \cdot \sinh C_{z_{n,m}} z.$$

BACK TO THE LAST BOUNDARY CONDITION:

$$\Phi_0 = \sum_{n',m'} A_{n',m'}$$

$$\cdot \sin n'\frac{\pi}{a}x \cdot \sin m'\frac{\pi}{b}y \cdot \sinh C_{z_{n',m'}} C$$

WE SEEK THE "EXPANSION COEFFICIENTS" $A_{n',m'}$. WE CAN USE ORTHOGONALITY TO "PICK OUT" JUST ONE COEFFICIENT. THAT IS:

1. MULTIPLY BOTH SIDES OF $\Phi = \ldots$ BY $\sin n\frac{\pi}{a}x \cdot \sin m\frac{\pi}{b}y$;

2. INTEGRATE BOTH SIDES BY

$$\int_0^a \int_0^b (\ldots) \cdot dx\, dy$$

THIS GIVES

$$\Phi_0 \underbrace{\int_0^a \sin n\frac{\pi}{a}\,dx}_{\substack{\frac{2}{\pi}\frac{a}{n}\ (n\ \text{ODD}) \\ 0\ (n\ \text{EVEN})}} \cdot \underbrace{\int_0^b \sin m\frac{\pi}{d}\,dy}_{\substack{\frac{2}{\pi}\frac{b}{m}\ (m\ \text{ODD}) \\ 0\ (m\ \text{EVEN})}}$$

$$= \sinh C_{z;\,n',m'} \overbrace{\cdot \int_0^a \sin n'\frac{\pi}{a} \sin n\frac{\pi}{a}\,dx}^{\sim \delta_{n,n'}}$$

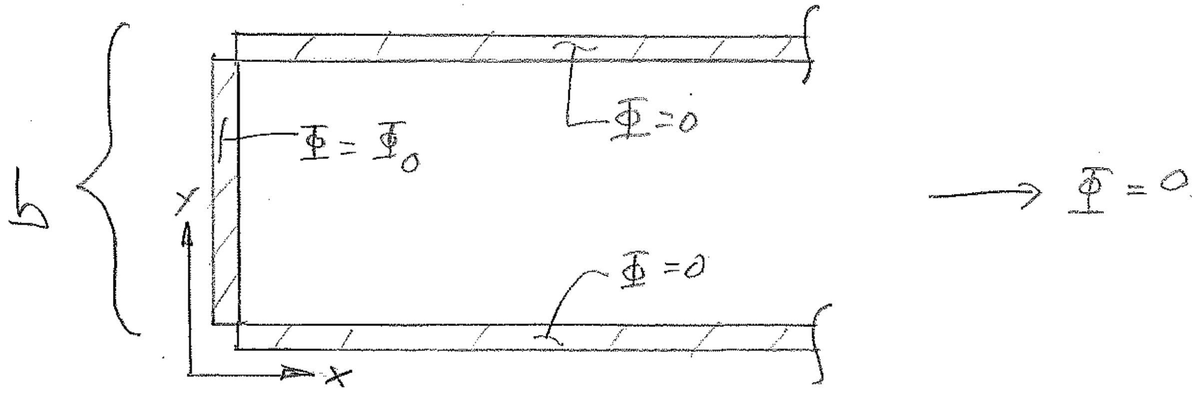
$$\underbrace{\cdot \int_0^b \sin m'\frac{\pi}{b} \sin m\frac{\pi}{b}\,dy}_{\sim \delta_{m,m'}}$$

HENCE,

$$A_{n,m} = \frac{2}{a}\,\frac{2}{b}\,\frac{1}{\sinh C_{z;\,n,m}}\;C$$

$$\times \Phi_0 \underbrace{\int_0^a \sin C_{x,n}x\,dx}_{\substack{\frac{2}{\pi}\frac{a}{n}\ (n\ \text{ODD}) \\ 0\ (n\ \text{EVEN})}} \cdot \underbrace{\int_0^b \sin(C_{x,n}Y)\,dY}_{\substack{\frac{2}{\pi}\frac{b}{n}\ (n\ \text{ODD}) \\ 0\ (n\ \text{EVEN})}}$$

EXAMPLE: OPEN BOUNDARY IN 2D.



$$\Phi(x,y) = X(x)\, Y(y);$$

$$\nabla^2 \Phi \rightarrow$$

$$\underbrace{\frac{1}{X}\frac{d^2}{dx^2}X(x)}_{+c^2} + \underbrace{\frac{1}{Y}\frac{d^2}{dy^2}Y(y)}_{-c^2} = 0$$

1. ONLY 1 IDEPENDENT SEPARATION CONSTANT.

2. WHY THE CHOICE OF SIGN? WE ANTICIPATE THE Y-DIRECTION HAS NO OSCILLITORY COMPONENTS.

$\mathcal{Y}(y) \sim \cos cy, \quad \sin cy.$

$\mathcal{Y}(y=0) = 0 \longrightarrow \mathcal{Y}(y) \simeq \sin cy.$

$\mathcal{Y}(y=b) = 0 \longrightarrow \mathcal{Y}_n(y) \sim \sin n\frac{\pi}{b}y.$

$\mathcal{X}(x) \sim e^{+cy}, \quad e^{-cy}.$

$\mathcal{X}(x \to \infty) = 0 \longrightarrow$ NO e^{+cy} TERM.

LIKE THE PREVIOUS EXAMPLE, THE LAST BOUNDARY CONDITION TAKES MORE WORK. WE HAVE PARTIAL SOLUTION

$$\Phi(x,y) = \sum_n M_n \sin n\frac{\pi}{b}y \cdot e^{-n\frac{\pi}{b}x}.$$

WE HAVE LAST BOUNDARY CONDITION

$$\Phi_0 = \Phi(x=0,y) = \sum_n M_n \sin n\frac{\pi}{b}y.$$

WE WANT THE VALUES OF M_n.

IN A MANNER SIMILAR TO WHAT WE DID IN THE PREVIOUS EXAMPLE:

1. MULTIPLY BOTH SIDES BY $\sin n'\frac{\pi}{b}y$;

2. INTEGRATE BOTH SIDES $\int_0^b (\cdots)\,dy$.

$$\int_0^b \Phi_0 \sin n'\frac{\pi}{b}y \quad \begin{cases} \Phi_0\, 2b/n\pi & (n \text{ ODD}) \\ 0 & (n \text{ EVEN}) \end{cases}$$

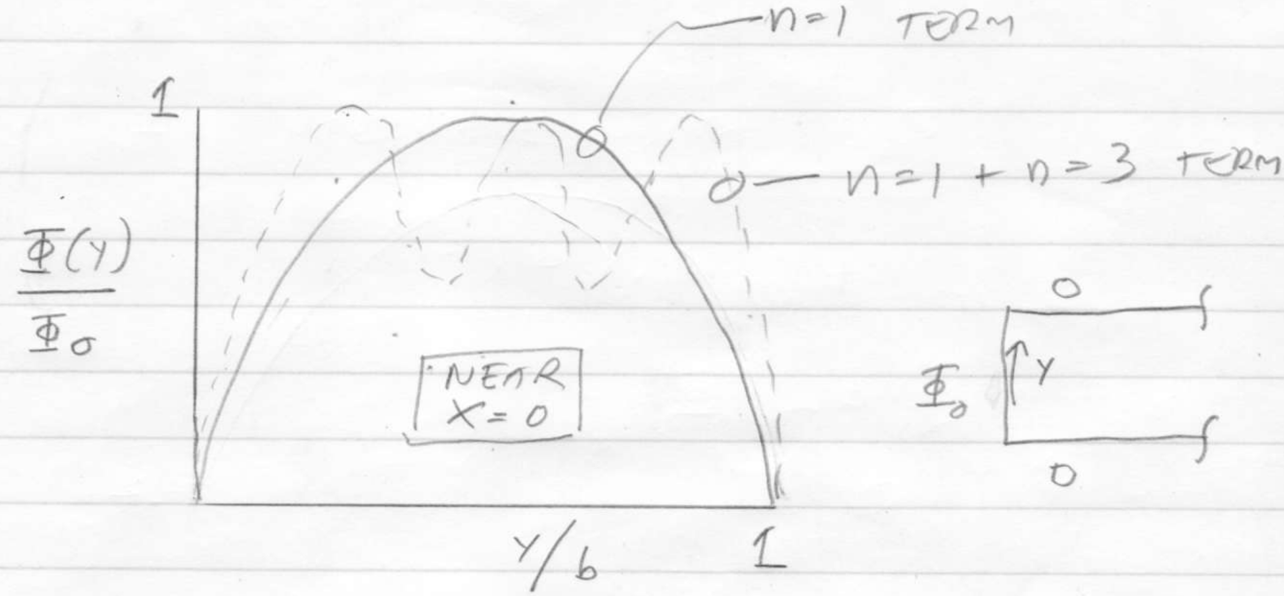
$$= \underbrace{\int_0^b \sum_n \eta_n \sin n\frac{\pi}{b}y \cdot \sin n'\frac{\pi}{b}y \, dy}_{\begin{array}{ll} 0 & (n \neq n') \\ \eta_n \frac{b}{2} & (n = n') \end{array}}$$

$$\eta_n = 4\Phi_0/n\pi \quad (n \text{ ODD})$$

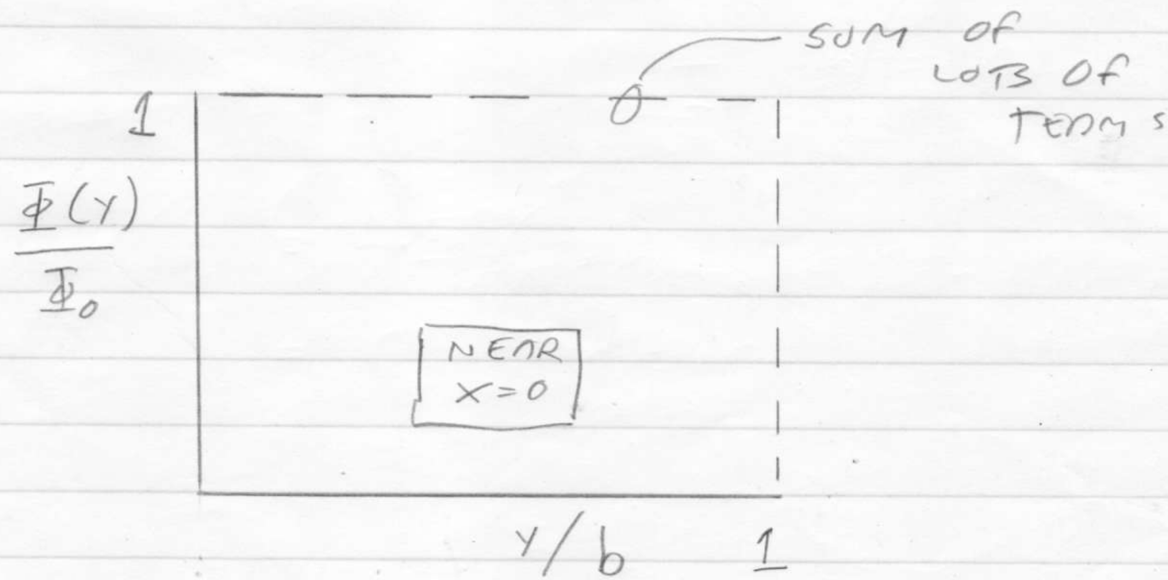
$$= 0 \quad (n \text{ EVEN}).$$

$$\Phi(x,y) = \sum_{\substack{n \\ n \text{ ODD}}} \left(\frac{4\Phi_0}{n\pi}\right) \sin n\frac{\pi}{b}y \cdot e^{-n\frac{\pi}{b}x}$$

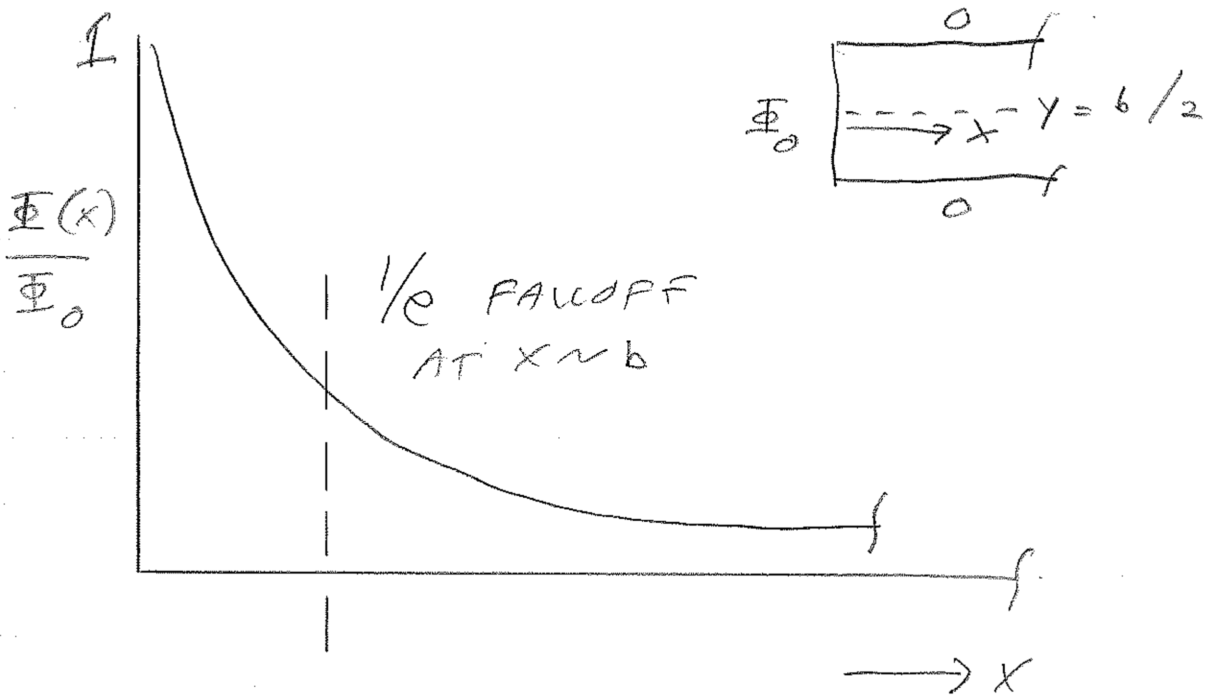
GRAPHICAL SOLUTIONS



⇓ SUMMING LOTS OF TERMS



IN THE MID-PLANE ~ $Y = b/2$:



Q: SUPPOSE YOU WANT TO SHIELD A PHOTOTUBE FROM A MAGNETIC FIELD. HOW LONG SHOULD THE SHIELD STICK OUT FROM THE FACE OF THE TUBE?