Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
November 19, 2020, 11am
On-line lecture

Administrative:
1. The draft of this lecture is posted at
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
2. Office hours are today after class at 12:30.
5. Exam statistics are posted.

Lecture: Multipoles, dielectrics. (Jackson chapter 4).
Section 4.7 Electrostatic energy in dielectric media.
Application (Special lecture): Dielectric liquids. This is a challenging “beyond-Jackson” topic.

ELECTROSTATIC ENERGY IN DIELECTRIC MEDIA (JACKSON §4.7)

RECAP:

WE ADDED A SMALL AMOUNT OF "FREE" CHARGE $\delta\rho$ TO A DIELECTRIC SYSTEM. THIS INDUCED A CHANGE IN $\vec{\nabla}\cdot\delta\vec{D}$, WHICH MEANS THE WORK THE EXTERNAL AGENT HAD TO DO TO ADD $\delta\rho$ IS

$$\delta W = \iiint \vec{E}\cdot\delta\vec{D}\,dV.$$

THIS IS AS FAR AS YOU CAN GO WITHOUT KNOWING THE CONSTITUTIVE RELATION $\vec{E}(\vec{D})$.

FOR A LINEAR DIELECTRIC ε $\delta\vec{D} = \varepsilon\,\delta\vec{E}$, WE CAN KEEP ADDING CHARGE ρ (AND THEREBY $\vec{D}$) AND FIND THE TOTAL WORK REQUIRED.

$$U = \int_0^D \delta W = \int_0^D \iiint \vec{E} \cdot \delta \vec{D} \, dv$$

$$U = \frac{1}{2} \iiint \vec{E} \cdot \vec{D} \, dv.$$

THIS APPLIES SPECIFICALLY TO A LINEAR DIELECTRIC.

APPLICATION OF THESE IDEAS:
THOMSON'S THEOREM.

"CHARGES REDISTRIBUTE THEMSELVES ON A CONDUCTOR SO AS TO MINIMIZE THE ENERGY."

WE LOOKED AT THIS BY CONSIDERING A VIRTUAL DISPLACEMENT OF A BIT OF CHARGE $\delta\rho$ ALONG THE SURFACE OF THE CONDUCTOR; THE TOTAL CHARGE ON THE CONDUCTOR IS UNCHANGED.

WE THEN WENT BACK TO

$$\delta U = \iiint \vec{E} \cdot \delta \vec{D} \, dv$$

WHICH CAME FROM

$$\delta U = \iiint \Phi \vec{\nabla} \cdot \delta \vec{D} \, dv$$

$$= \iiint \Phi \, \delta\rho \, dv.$$

THE CONDUCTOR IS AN EQUIPOTENTIAL

SO $\delta U = \Phi \iiint \delta\rho \, dv = 0$

SINCE THE TOTAL CHARGE IS UNCHANGED ON A CONDUCTOR.

WE MOVED CHARGE $\delta\rho$ IN A VIRTUAL DISPLACEMENT FROM THE EQUILIBRIUM CHARGE POSITION. THAT IS

$$\delta U / \delta \rho = 0$$

SO U AT EQUILIBRIUM IS A STATIONARY POINT IN REGARDS TO THE POSITION OF THE CHARGE, THUS ESTABLISHING THE THEOREM.

THOMSON'S THEOREM IS RELATED TO THE 2ND UNIQUENESS THEOREM: IN A SYSTEM OF CONDUCTORS, THE ELECTRIC FIELD IS UNIQUELY DETERMINED IF THE TOTAL CHARGE ON EACH CONDUCTOR IS SPECIFIED.

THESE IN TURN, e.g., ADDRESS THE QUESTION OF WHY A CHARGE PLACED ON A CONDUCTING SPHERE ARRANGES ITSELF UNIFORMLY IN EQUILIBRIUM.

HOMEWORK 7 CONTAINS SEVERAL PROBLEMS COMBINING ELECTROSTATICS AND THERMODYNAMICS/STAT. MECH.

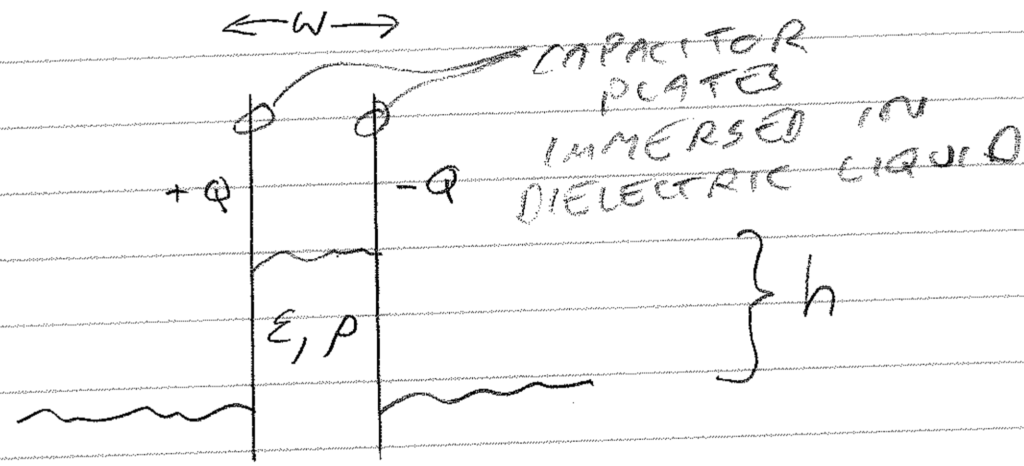
ONE OF THESE PROBLEMS IS A DERIVATION OF THE DILUTE-GAS FORM OF THE CLAUSIUS-MOSSITTI RELATION: $\frac{\varepsilon(T)}{\varepsilon_0} = 1 + c/T$.

A SECOND PROBLEM ASKS HOW MUCH HEAT IS RELEASED (OR ABSORBED) WHEN AN ELECTRIC FIELD THREADS A DIELECTRIC.
Q: FROM DUALITY AND PERSONAL EXPERIENCE, WHY MIGHT YOU ANTICIPATE SUCH EFFECTS?

ANOTHER PATH IS EXPLORING DIELECTRIC LIQUIDS, FOLLOWING.

DIELECTRIC LIQUIDS. FORCES.

WE STARTED THIS DISCUSSION BY RECALLING A FIRST-YEAR PROBLEM: HOW HIGH DOES THE DIELECTRIC-LIQUID COLUMN RISE.



CAN FIND h BY ENERGY BALANCE

SO THERE MUST BE FORCES PUSHING (OR PULLING) THE COLUMN UP. HOW IS THIS FORCE CREATED? WHERE DOES IT ARISE IN THIS SYSTEM?

THIS IS CLOSELY RELATED TO ANOTHER QUESTION:

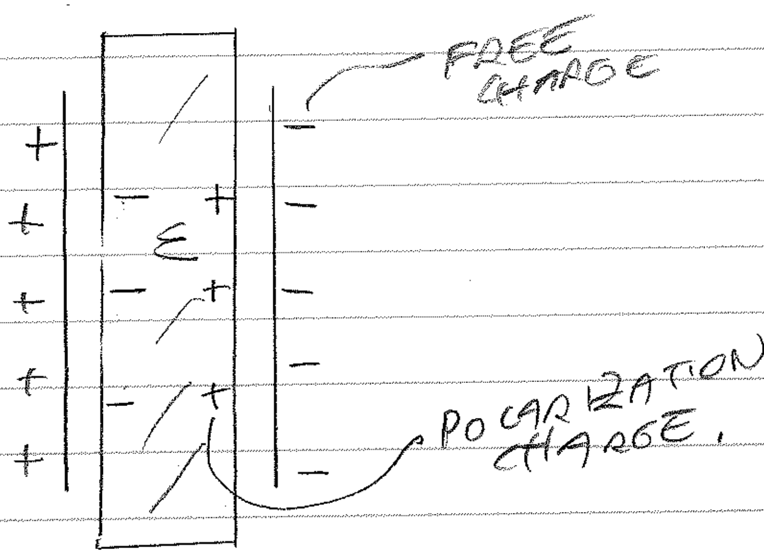
IDEAL PARALLEL PLATE CAPACITOR WITH LIQUID OR SOLID DIELECTRIC.

Q: DOES THE ATTRACTIVE FORCE BETWEEN THE PLATES DIFFER BETWEEN THE TWO CASES.

A: (WRONG ANSWER). IT SHOULDN'T. FIELD THEORY SAYS THE FORCE ON A PLATE IS LOCAL: WE NEED ONLY CONSIDER THE CHARGES ON THE PLATE AND THE ELECTRIC FIELD AT THE POSITION OF THOSE CHARGES.

THE CHARGES ON THE PLATES ARE THE SAME IN BOTH CASES. IS THE ELECTRIC FIELD AT THE PLATES THE SAME IN BOTH CASES?

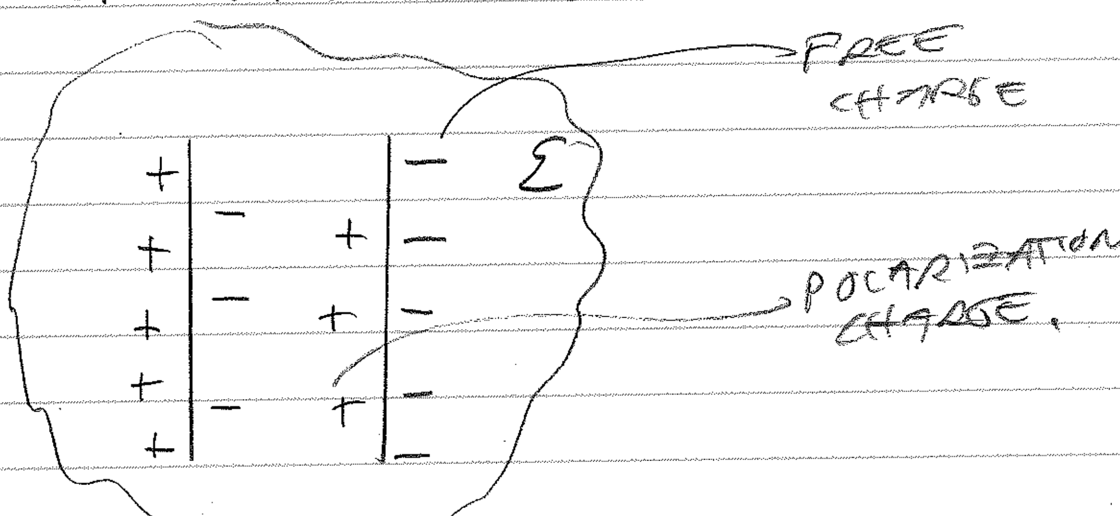
CASE 1: SOLID DIELECTRIC



THE SHEETS OF + AND - POLARIZATION CHARGE DON'T PRODUCE AN E FIELD AT THE PLATES

Q: WHY

CASE 2: LIQUID DIELECTRIC



THE CONFIGURATION OF CHARGES IS THE SAME AS FOR CASE 1.

SO FIELD THEORY OF ELECTROSTATICS SUGGESTS THE ATTRACTIVE FORCE BETWEEN THE PLATES IS THE SAME. BUT THIS IS WRONG; THE ATTRACTIVE FORCE IN THE CASE OF THE LIQUID DIELECTIC IS LESS.

OUR ARGUMENTS TO THIS POINT ARE CORRECT. HENCE, THERE ARE NON-ELECTROSTATIC EFFECTS.

WE NEED TO ADDRESS THE QUESTION OF THE SOURCE OF MECHANICAL STRESS IN DIELECTRICS.

IN A SENSE, THE ORIGIN OF MECHANICAL STRESS IS SIMPLE:

$$\frac{d\vec{F}}{dv} = \rho \vec{E}.$$

BUT, JUST AS WE COULD ALWAYS, e.g., FIND A POTENTIAL FROM

$$\Phi = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} dv',$$

THIS IS IN PRACTICE IMPRACTICAL SINCE WE DON'T HAVE EASY ACCESS TO ALL ρ IN THE DIELECTRIC.

SO, WE NEED AN ALTERNATE APPROACH. HERE'S A PATH (FROM LANDAU & LIFSCHITZ).

INTRODUCE A VELOCITY FIELD $\vec{v}(\vec{r})$ WITHIN THE DIELECTRIC. THEN THE RATE ENERGY IS LOST DUE TO THIS MOTION IS

$$\frac{dU}{dt} = -\iiint [\,\cdots\,] \cdot \vec{v}\, dv$$

WHERE $[\,\cdots\,]$ IS $\frac{d\vec{F}}{dv}$.

THE STRATEGY IS TO GET $\frac{dU}{dt}$ INTO THIS FORM AND IDENTIFY [.....].

WE'LL NEED ANOTHER PIECE FROM FLUID MECHANICS: IN HYDROSTATIC EQUILIBRIUM $\frac{d\vec{F}}{dV} = \vec{\nabla} P$, P THE PRESSURE.

NOW, TO PROCEED. LIQUID OR NOT, RECALL FOR A LINEAR DIELECTRIC

$$\delta U = \tfrac{1}{2}\,\delta \iiint \vec{E}\cdot\vec{D}\,dV$$

LET'S ALLOW CHANGES IN "FREE" CHARGE $\delta\rho$ AND CHANGES IN THE DIELECTRIC CONSTANT $\delta\varepsilon$. THESE ARE THE RESULT OF VIRTUAL DISPLACEMENTS OF CHARGES AND DIELECTRIC (AS FOR THOMSON'S THEOREM, THE TOTAL CHARGE DOES NOT CHANGE).

WITH $\delta\rho$ AND $\delta\varepsilon$,

$$\delta U = \frac{1}{2}\delta\iiint \vec{E}\cdot\vec{D}\,dv \quad \text{BECOMES}$$

$$= \frac{1}{2}\iiint \vec{D}\,\delta\left(\frac{1}{\varepsilon}\right)\cdot\vec{D}\,dv + \iiint \vec{E}\cdot\delta\vec{D}\,dv$$

$$= -\frac{1}{2}\iiint \vec{E}\,\delta\varepsilon\cdot\vec{E}\,dv + \iiint \vec{E}\cdot\delta\vec{D}\,dv.$$

THE FIRST TERM ARISES FROM CHANGES IN THE DIELECTRIC CONSTANT, THE SECOND FROM VIRTUAL DISPLACEMENTS OF FREE CHARGE.

WE EVALUATED THE SECOND TERM IN THE LAST LECTURE:

$$\iiint \vec{E}\cdot\delta\vec{D}\,dv = -\iiint \vec{\nabla}\Phi\cdot\delta\vec{D}\,dv$$

$$= \iiint \Phi\,\vec{\nabla}\left(\delta\vec{D}\right)dv + \text{SURFACE TERM}$$

$$\longrightarrow \iiint \Phi\,\delta\rho\,dv \quad \text{FOR A LOCALIZED CHARGE.}$$

HENCE (WITH $\frac{1}{dt}$)

$$\frac{dU}{dt} = \iiint\left\{\Phi\frac{d\rho}{dt} - \frac{1}{2}E^2\frac{d\varepsilon}{dt}\right\}dv$$

WE HAVE CHARGE ρ AND MASS ρ_m CONTINUITY:

$$\vec{\nabla}\cdot(\rho\vec{V}) + \frac{\partial\rho}{\partial t} = 0 \quad \text{AND}$$

$$\vec{\nabla}\cdot(\rho_m\vec{V}) + \frac{\partial\rho_m}{\partial t} = 0.$$

SINCE THERE'S MOTION OF THE DIELECTRIC AND CHARGE, THE TOTAL DERIVATIVE PICKS UP A VELOCITY TERM (A "CONVECTIVE DERIVATIVE"):

$$\frac{D\varepsilon}{Dt} = \frac{\partial\varepsilon}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial\varepsilon}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\varepsilon}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial\varepsilon}{\partial t}$$

THAT IS, ε AT A POINT COULD CHANGE BECAUSE THE POINT IS MOVING, THEREBY BRING IN A DIFFERENT ε, OR ε ITSELF MAY BE CHANGING.

MORE COMPACTLY

$$\frac{D\varepsilon}{Dt} = -\vec{\nabla}\varepsilon\cdot\vec{V} + \frac{\partial\varepsilon}{\partial t}, \quad \text{ALSO}$$

$$\frac{D\rho}{Dt} = -\vec{\nabla}\rho\cdot\vec{V} + \frac{\partial\rho}{\partial t}.$$

WE'VE, AT LEAST FORMALLY, FOUND $\partial\rho/\partial t$ AND $\partial\varepsilon/\partial t$ IN dV/dt

AS
$$\frac{\partial\varepsilon}{\partial t} = -\vec{\nabla}\varepsilon\cdot\vec{v} + \frac{D\varepsilon}{Dt} \quad \text{AND}$$

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\rho\cdot\vec{v} + \frac{D\rho}{Dt}.$$

BUT, HERE WE'RE STUCK. WE DON'T KNOW $D\varepsilon/Dt$ OR $D\rho/dt$. BUT SUPPOSE WE KNOW THE DIELECTRIC'S "EQUATION OF STATE", THAT IS $\varepsilon(\rho_m)$. IN THIS CASE THE CONVECTIVE DERIVATIVE READS
$$\frac{D\varepsilon}{Dt} = \frac{\partial\varepsilon}{\partial\rho_m}\cdot\frac{D\rho_m}{Dt},$$

WHERE $\partial\varepsilon/\partial\rho_m$ COMES FROM ASKING AN ENGINEER.

RECALL $D\rho_m/Dt$ IS ALSO
$$\frac{D\rho_m}{Dt} = \vec{\nabla}\rho_m\cdot\vec{v} + \frac{\partial\rho_m}{\partial t}.$$

HENCE, $\frac{D\varepsilon}{Dt} = \frac{\partial\varepsilon}{\partial\rho_m}\frac{D\rho_m}{Dt}$ READS

$$\frac{D\varepsilon}{Dt} = \frac{d\varepsilon}{d\rho_m}\left\{\frac{d\rho_m}{dt} + \vec{\nabla}\rho_m \cdot \vec{v}\right\}.$$

FROM CONTINUITY $\frac{d\rho_m}{dt} = -\vec{\nabla}\cdot(\rho_m\vec{v})$,

$$\frac{D\varepsilon}{Dt} = \frac{d\varepsilon}{d\rho_m}\left\{-\vec{\nabla}\cdot(\rho_m\vec{v}) + \vec{\nabla}\rho_m \cdot \vec{v}\right\}$$

$$= -\frac{d\varepsilon}{d\rho_m}\rho_m(\vec{\nabla}\cdot\vec{v})$$ FROM INTEGRATION BY PARTS.

ALSO RECALL $\frac{D\varepsilon}{Dt} = \vec{\nabla}\varepsilon\cdot\vec{v} + \frac{d\varepsilon}{dt}$, SO

$$\frac{d\varepsilon}{dt} = -\vec{\nabla}\varepsilon\cdot\vec{v} - \frac{d\varepsilon}{d\rho_m}\rho_m(\vec{\nabla}\cdot\vec{v}).$$

ALSO RECALL

$$\frac{dU}{dt} = \iiint\left\{\Phi\frac{d\rho}{dt} - \frac{1}{2}E^2\frac{d\varepsilon}{dt}\right\}dv, \text{ SO}$$

$$\frac{dU}{dt} = \iiint\left\{\Phi\vec{\nabla}(\rho\vec{v}) + \frac{1}{2}E^2\frac{d\varepsilon}{d\rho_m}\rho_m(\vec{\nabla}\cdot\vec{v})\right.$$

$$\left. + \left(\frac{1}{2}E^2\vec{\nabla}\varepsilon\right)\cdot\vec{v}\right\}dv.$$

THIS IS NOT QUITE IN THE FORM

$$\frac{dU}{dt} = \iiint[\ldots]\cdot\vec{v}\,dv.$$

THE FIRST TERM $\iiint \Phi \vec{\nabla}(\rho \vec{v})\, dv$ CAN BE REWRITTEN WITH THE IDENTITY

$$\vec{\nabla} \cdot (\Phi \rho \vec{v}) = \Phi \vec{\nabla}(\rho \vec{v}) + \rho \vec{v} \cdot \vec{\nabla}\Phi .$$

AFTER ELIMINATING THE SURFACE TERM THERE'S A REMAINING TERM WITH $\cdot \vec{v}$, AS DESIRED.

THE SECOND TERM $\frac{1}{2}\iiint E^2 \frac{d\varepsilon}{d\rho_m} \rho_m (\vec{\nabla} \cdot \vec{v})\, dv$ LIKEWISE, ON ELIMINATING THE SURFACE TERM, LEAVES

$$-\frac{1}{2} \vec{\nabla}\left(E^2 \frac{d\varepsilon}{d\rho_m} \rho_m\right) \cdot \vec{v} .$$

WE THEREFORE HAVE

$$\frac{dU}{dt} = \iiint \left[-\rho \vec{E} + \frac{1}{2} E^2 \nabla \varepsilon - \frac{1}{2} \vec{\nabla}\left(E^2 \frac{d\varepsilon}{d\rho_m} \rho_m\right) \right] \cdot \vec{v}\, dv$$

WE IDENTIFY THE TERM IN $[\cdots]$ AS THE VOLUME FORCE $\frac{d\vec{F}}{dv}$.

THE FIRST TERM $\rho\vec{E}$ IS THE ORDINARY ELECTROSTATIC VOLUME FORCE.

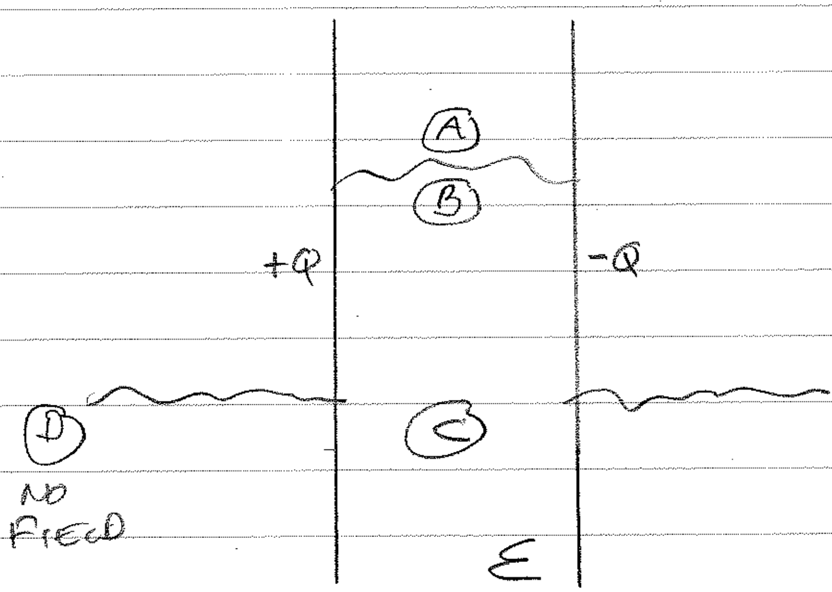
THE SECOND TERM $-\frac{1}{2}E^2\vec{\nabla}\varepsilon$ ARISES FROM HAVING AN INHOMOGENEOUS DIELECTRIC IN AN ELECTRIC FIELD.

THE THIRD TERM $\frac{1}{2}\vec{\nabla}\left(E^2\frac{\partial\varepsilon}{\partial\rho_m}\rho_m\right)$ (THE "ELECTROSTRICTION" TERM) IS (TYPICALLY) DUE TO AN INHOMOGENEOUS ELECTRIC FIELD.

Q: CAN THE ELECTROSTRICTION TERM CONTRIBUTE TO A TOTAL FORCE ON A BLOCK OF DIELECTRIC?
A: NO. IT'S A PURE GRADIENT.
Q: SO, THEN, UNDER WHAT CIRCUMSTANCES DOES IT VANISH?

EVEN THOUGH THE TOTAL CONTRIBUTION OF THE ELECTROSTRICTION TERM VANISHES, IT CAN STILL INTRODUCE EFFECTS.

BACK TO THE PROBLEM AT HAND:
CAPACITOR PLATES IMMERSED IN
AN INCOMPRESSIBLE LIQUID DIELECTRIC.



LET'S FIND THE PRESSURE DIFFERENCE
FROM (A) TO (D).

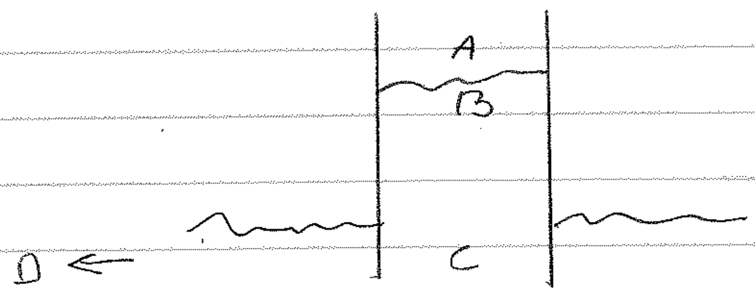
Q: WHICH VOLUME-FORCE TERMS
CONTRIBUTE?

A: ONLY THE TERM $\sim \vec{\nabla}\varepsilon$
CONTRIBUTES.

$$\frac{dF}{dV} = \vec{\nabla} P = -\frac{1}{2} E^2 \vec{\nabla} \varepsilon, \qquad \text{SO}$$

$$P_A - P_D = \frac{1}{2} \int_A^D E^2 \vec{\nabla} \varepsilon \cdot d\vec{l}$$

THE ONLY PLACE IN THE INTEGRAL WHERE THERE'S A CONTRIBUTION TO $\vec{\nabla}\varepsilon$ IS AT THE LIQUID–VACUUM BOUNDARY, AB. SO



$$P_A - P_D = \frac{1}{2}\int_A^B E^2 \vec{\nabla}\varepsilon \cdot d\vec{l}.$$

SINCE PROBLEMS OFTEN INVOLVE CAPACITOR PLATES AND THE LIKE, IT'S USEFUL TO SEPARATE $\vec{E}$ INTO "NORMAL" AND TANGENTIAL COMPONENTS AT THE BOUNDARY.

$$P_A - P_D = \frac{1}{2}\int_A^B \left(E_t^2 + E_n^2\right)\frac{d\varepsilon}{dl}\, dl.$$

RECALL, THE TANGENTIAL COMPONENTS OF $\vec{E}$ ARE CONTINUOUS ACROSS THE BOUNDARY. THE NORMAL COMPONENTS OF $\varepsilon_A E_{N,A}$ $(= \varepsilon_B E_{N,B})$ ARE CONTINUOUS ACROSS THE BOUNDARY.

HENCE

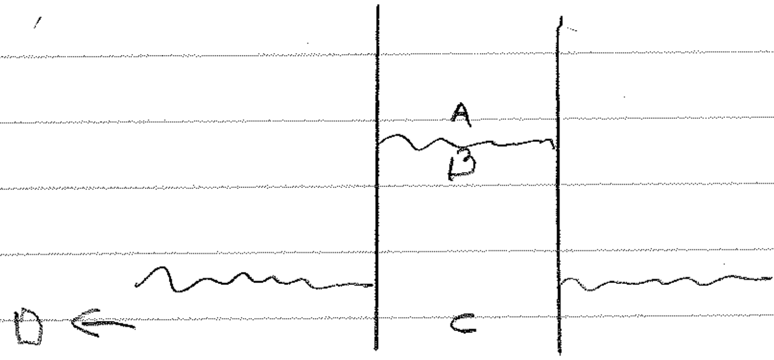
$$P_A - P_D = \frac{1}{2}\left[E_{t,B}^2(\varepsilon-\varepsilon_0) + \varepsilon^2 E_{n,B}^2 \int_A^B \frac{d\varepsilon'}{\varepsilon'^2}\right]$$

$$= \frac{1}{2}(\varepsilon-\varepsilon_0)\left[E_{t,B}^2 + \frac{\varepsilon}{\varepsilon_0} E_{n,B}^2\right]$$

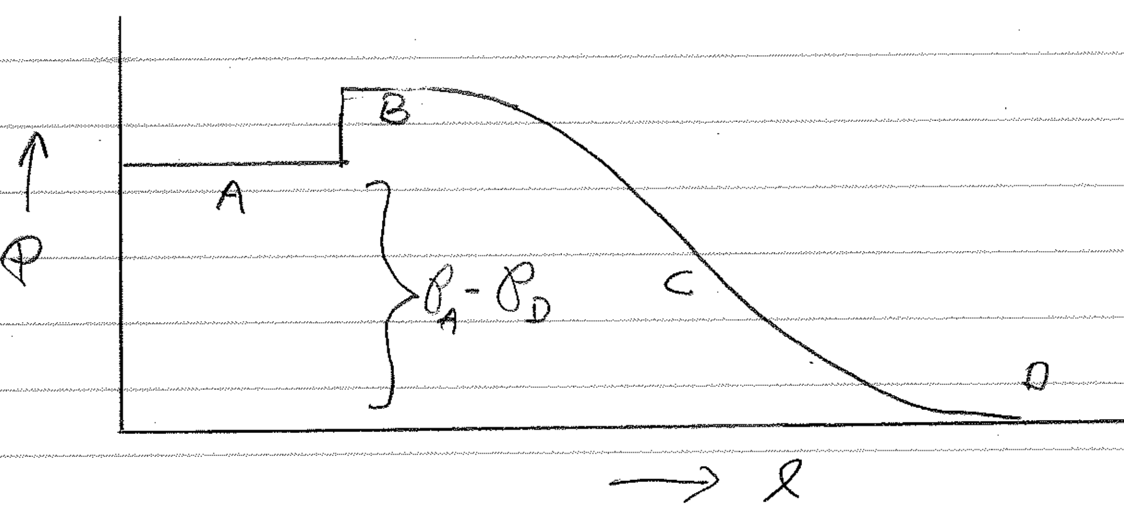
NOTICE THE FIELDS ABOVE ARE REFERENCED TO THOSE INSIDE THE DIELECTRIC. THIS MAKES THIS EXPRESSION MORE GENERAL

IN THIS PARTICULAR PROBLEM, THERE'S NO E_n COMPONENTS.

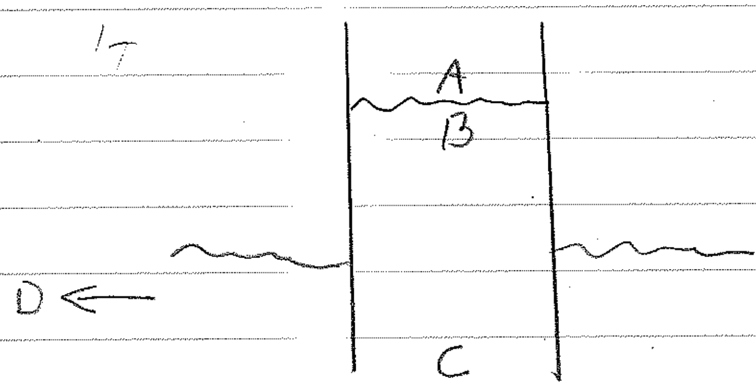
$P_A - P_D$ IS ACCURATELY DESCRIBED BY THIS! WE'VE TURNED THIS INTO A PROBLEM OF HYDROSTATICS. HOWEVER, WE DON'T YET KNOW HOW THE PRESSURE CHANGES ALONG THE PATH $A \to D$; HERE'S WHERE WE'D NEED TO INCLUDE THE ELECTROSTRICTION TERM.



INTERESTINGLY, THE PRESSURE CHANGE FROM A TO B IS OPPOSITE IN SIGN FROM THE PRESSURE CHANGE FROM A $\rightarrow$ D. IN DETAIL:



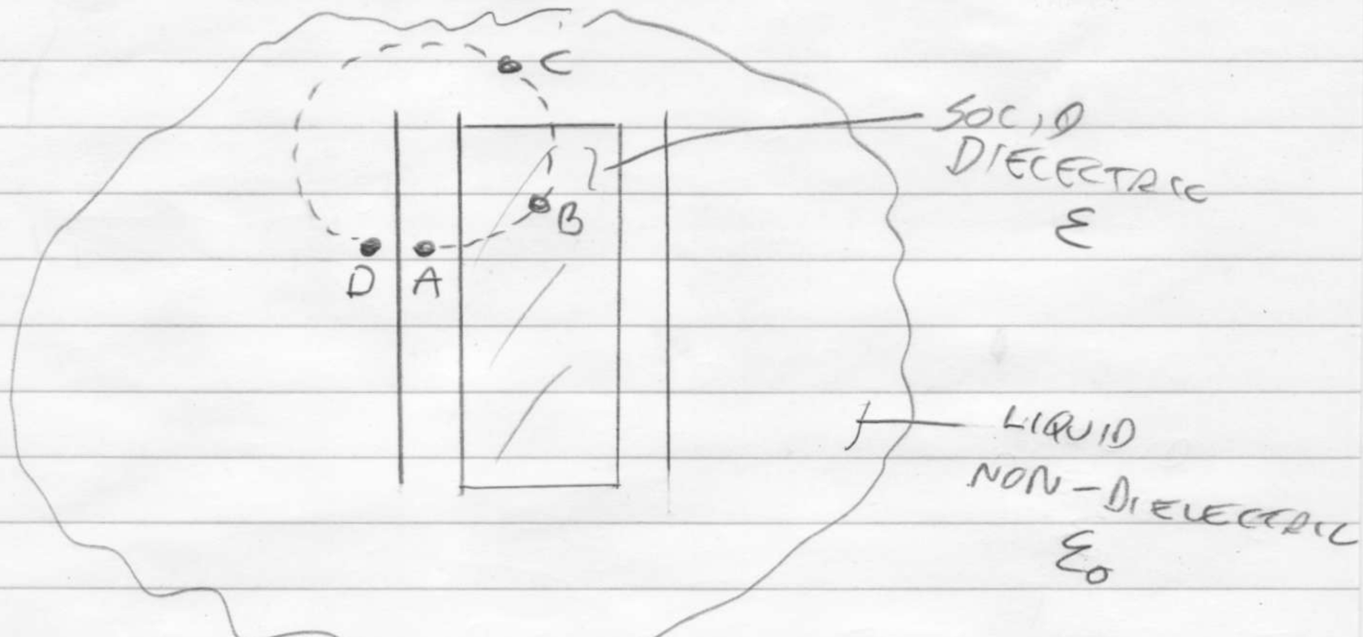
WHERE ARE THE FORCES EXERTED THAT ACTUALLY FORCE THE LIQUID UPWARDS?



THE PRESSURE THAT FORCES THE LIQUID UP ARISES AT C. THIS IS WHERE THE FIELD IS HOMOGENEOUS AND THE $\frac{1}{2}\vec{\nabla}\left(E^2 \frac{d\varepsilon}{d\rho_m}\rho_m\right)$ ELECTRO-STRICTION COMES INTO PLAY.

THE PHYSICAL REASON FOR THIS IS THE POLAR MOLECULES COMPRISING THE LIQUID HAVE LOWER ENERGY IN THE HIGHER FIELD REGION, AND THEREFORE THE MOLECULES ARE DRAWN INTO REGIONS OF HIGHER FIELD. THE FORCES AT THE AB BOUNDARY TEND TO PUSH THE LIQUID DOWN.

BACK TO ATTRACTIVE FORCES BETWEEN CAPACITOR PLATES. LET US IMAGINE WE IMMERSE THE PLATES PLUS SOLID DIELECTRIC IN A NON-POLARIZABLE LIQUID.



LET'S FIND THE HYDROSTATIC PRESSURE DIFFERENCE $P_A - P_D$ (OBVIOUSLY WITHOUT THE LIQUID THIS IS ZERO). FOR THE PATH ABOVE, THE ONLY CONTRIBUTION TO $P_A - P_D$ COMES FROM THE AB INTERFACE.

Q: WHY?

AT THIS AB INTERFACE, $\vec{E}$ IS "NORMAL". THUS

$$P_A - P_D = \frac{1}{2}(\varepsilon - \varepsilon_0)\frac{\varepsilon}{\varepsilon_0} E_{n,B}^2$$

$$= \frac{1}{2}\left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}\right) D_{n,B}^2$$

$$P_A - P_D = \frac{1}{2}\left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}\right)\sigma^2$$

THIS IS AN ADDITIONAL HYDROSTATIC PRESSURE, NOT PRESENT IN THE CASE OF THE WHOLLY SOLID DIELECTRIC. THIS SERVES TO DECREASE THE FORCE BETWEEN THE PLATES.

EXERCISE: SHOW THAT THE FORCE PER UNIT AREA ON THE PLATES IS $\frac{1}{2}\frac{\sigma^2}{\varepsilon}$.

EXERCISE: SUPPOSE TWO POINT CHARGES ARE IMMERSED IN A DIELECTRIC LIQUID. IS THE COULOMB FORCE BETWEEN THEM REDUCED?