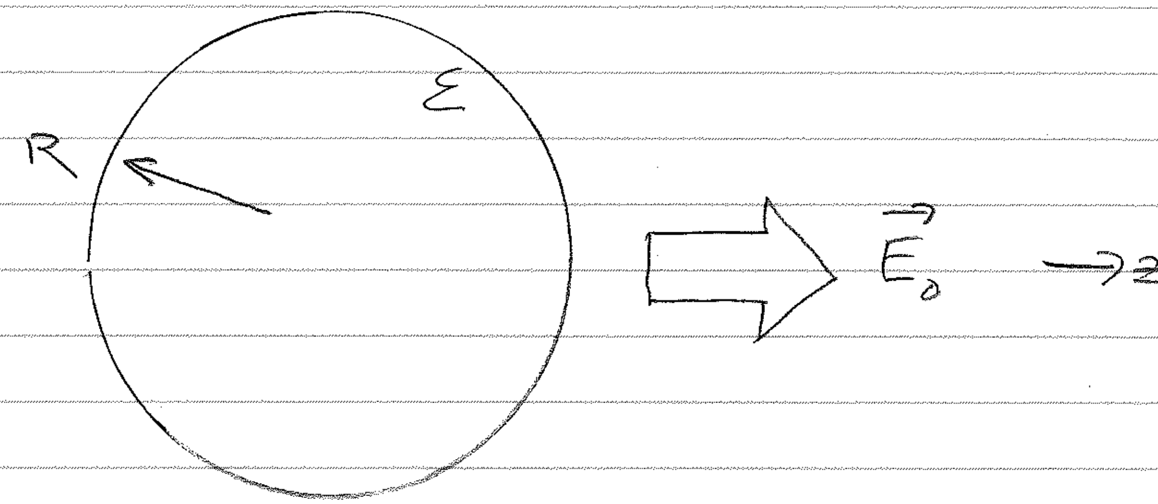
Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
November 17, 2020, 11am PST
On-line lecture

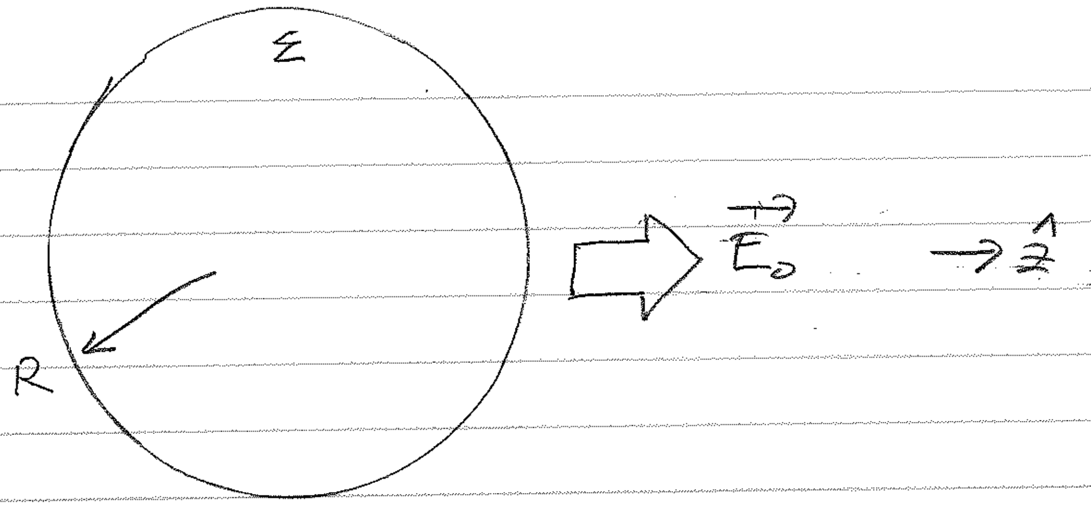
Administrative:
1. Homework 7 posted on
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
2. Exam 1 returned. Exam statistics will be posted.

Lecture: Multipoles, dielectrics. (Jackson chapter 4).
Section 4.4: Boundary-value problems involving dielectrics.
Example: Dielectric sphere in uniform electric field.
Section 4.7 Electrostatic energy in dielectric media.
Example: Thomson's theorem.
Example: Dielectric liquida.

EXAMPLE: DIELECTRIC SPHERE IN A UNIFORM APPLIED $\vec{E}$-FIELD



WHAT DO WE EXPECT? IF $\varepsilon = \varepsilon_0$, THERE'S NO SURFACE POLARIZATION CHARGE. IF $\varepsilon \gg \varepsilon_0$, THERE'S A "$\cos\theta$" SURFACE POLARIZATION CHARGE. MAYBE AS ε INCREASES, THE "$\cos\theta$" SURFACE GROWS UNTIL IT APPROACHES THE $\varepsilon \gg \varepsilon_0$ CASE. WE'LL SEE.



LIKE THE RELATED GROUNDED-SPHERE PROBLEM, THE FIELD AT $r \to \infty$ APPROACHES A CONSTANT $\vec{E}_0$

$$\Phi(r \to \infty) = -E_0 z = -E_0 r \cos\theta.$$

THERE'S A BOUNDARY AT $r = R$, SO WE HAVE "INSIDE" AND "OUTSIDE" POTENTIALS. THESE HAVE GENERIC FORM FOR AZIMUTHAL SYMMETRY

$$\Phi_{r<R} = \sum_\ell a_\ell r^\ell P_\ell(\cos\theta)$$

$$\Phi_{r>R} = \sum_\ell b_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)$$

Q: WHAT HAPPENED TO THE Q_ℓ (IRREGULAR) SOLUTIONS?

APPLY MORE BOUNDARY CONDITIONS (WE ALREADY APPLIED Φ IS FINITE FOR $r \to 0$ AND $r \to \infty$) AND WE ANTICIPATED AT $r = R$ $\Phi \sim \cos\theta$.

APPLY $\Phi_{r<R}(r=R) = \Phi_{r>R}(r=R)$

(Φ CONTINUOUS ACROSS BOUNDARY)

APPLY $\hat{n} \cdot \vec{D}_{r>R}(r=R) = \hat{n} \cdot D_{r<R}(r=R)$

(NORMAL COMPONENT OF $\vec{D}$ ACROSS BOUNDARY):

$$\varepsilon \frac{\partial}{\partial r} \Phi_{r<R}(r=R) = \varepsilon_0 \frac{\partial}{\partial r} \Phi_{r>R}(r=R)$$

RESTATE WHAT WAS SAID BEFORE; SINCE THE $r=R$ BOUNDARY CONDITION MUST APPLY FOR ALL θ, THE BOUNDARY CONDITIONS ARE MET ORDER-BY-ORDER IN l (Q: WHY?) THIS IS A HUGE SIMPLIFICATION: ONLY $l=1$ WILL SURVIVE ON APPLYING ORTHOGONALITY.

$$\Phi_{r<R}(r=R) \longrightarrow a_1 r P_1(\cos\theta)$$

$$\Phi_{r>R}(r=R) \longrightarrow b_1 \frac{1}{r^2} P_1(\cos\theta) - E_0 r P_1(\cos\theta)$$

Q! WHERE DID $-E_0 r P_1(\cos\theta)$ COME FROM?

NOW FOR THE $r=R$ BOUNDARY CONDITIONS

(CONTINUOUS POTENTIAL)

$$a_1 R = b_1 \frac{1}{R^2} - E_0 R$$

(CONTINUOUS $\hat{n} \cdot \vec{D}$)

$$\varepsilon a_1 = b_1 \left(\frac{-2}{R^3}\right) - E_0$$

WE CAN SOLVE FOR a_1 AND b_1!

$$\Phi_{r<R}(r,\theta) = \frac{-3}{\varepsilon/\varepsilon_0 + 2} E_0 r \cos\theta$$

$$\Phi_{r>R}(r,\theta) = \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} E_0 \frac{R^3}{r^2} \cos\theta - E_0 r \cos\theta$$

SOME LIMITING CASES!

- $\varepsilon \to \varepsilon_0$ (NO SPHERE!)

$$\Phi_{r<R}(r,\theta) = -E_0 r \cos\theta;$$

$$\Phi_{r>R}(r,\theta) = -E_0 r \cos\theta.$$

Q: WHY IS THIS EXPECTED?

- $\varepsilon \to \infty$

$$\Phi_{r<R} = 0;$$

$$\Phi_{r>R}(r,\theta) = E_0 \left(\frac{R^3}{r^2} - r\right)\cos\theta$$

Q: WHY IS THIS EXPECTED?

SOME QUESTIONS?

- WHAT'S THE "CHARACTER" OF THE FIELDS?

- HOW TO FIND THE POLARIZATION $\vec{P}$?

- HOW TO FIND THE SURFACE CHARGE σ? IS IT "FREE" OR "POLARIZATION" CHARGE?

- IS THE "INSIDE" FIELD IN MAGNITUDE GREATER OR LESSER THAN $\vec{E}_0$?; WHICH WAY DOES THE INTERIOR FIELD POINT?

TO CONTINUE: FIELDS

$$\vec{E}_{r>R} = -\vec{\nabla}\Phi_{r>R}(r,\theta)$$

$$= \left\{-\hat{r}\frac{\partial}{\partial r}\Phi_{r>R}(r,\theta) - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\Phi_{r>R}(r,\theta)\right\}$$

Q: WHAT DO YOU THINK THIS WILL LOOK LIKE?

$$\vec{E}_{r<R} = -\vec{\nabla}_{r<R}(r,\theta)$$

$$= \frac{\varepsilon}{\varepsilon/\varepsilon_0 + 2}\vec{E}_0 \qquad \text{A CONSTANT.}$$

SURFACE CHARGE VIA $\sigma_p = \vec{P}\cdot\hat{n}$

SINCE $\vec{P} = (\varepsilon-\varepsilon_0)\vec{E} = -(\varepsilon-\varepsilon_0)\vec{\nabla}\Phi$

$$\sigma = 3\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)E_0\cos\theta,$$

A POLARIZATION CHARGE.

THE "EXTERIOR" FIELD, AGAIN.

$$\Phi_{r>R}(r,\theta) = \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} E_0 \frac{R^3}{r^2} \cos\theta$$

$$- E_0 r \cos\theta$$

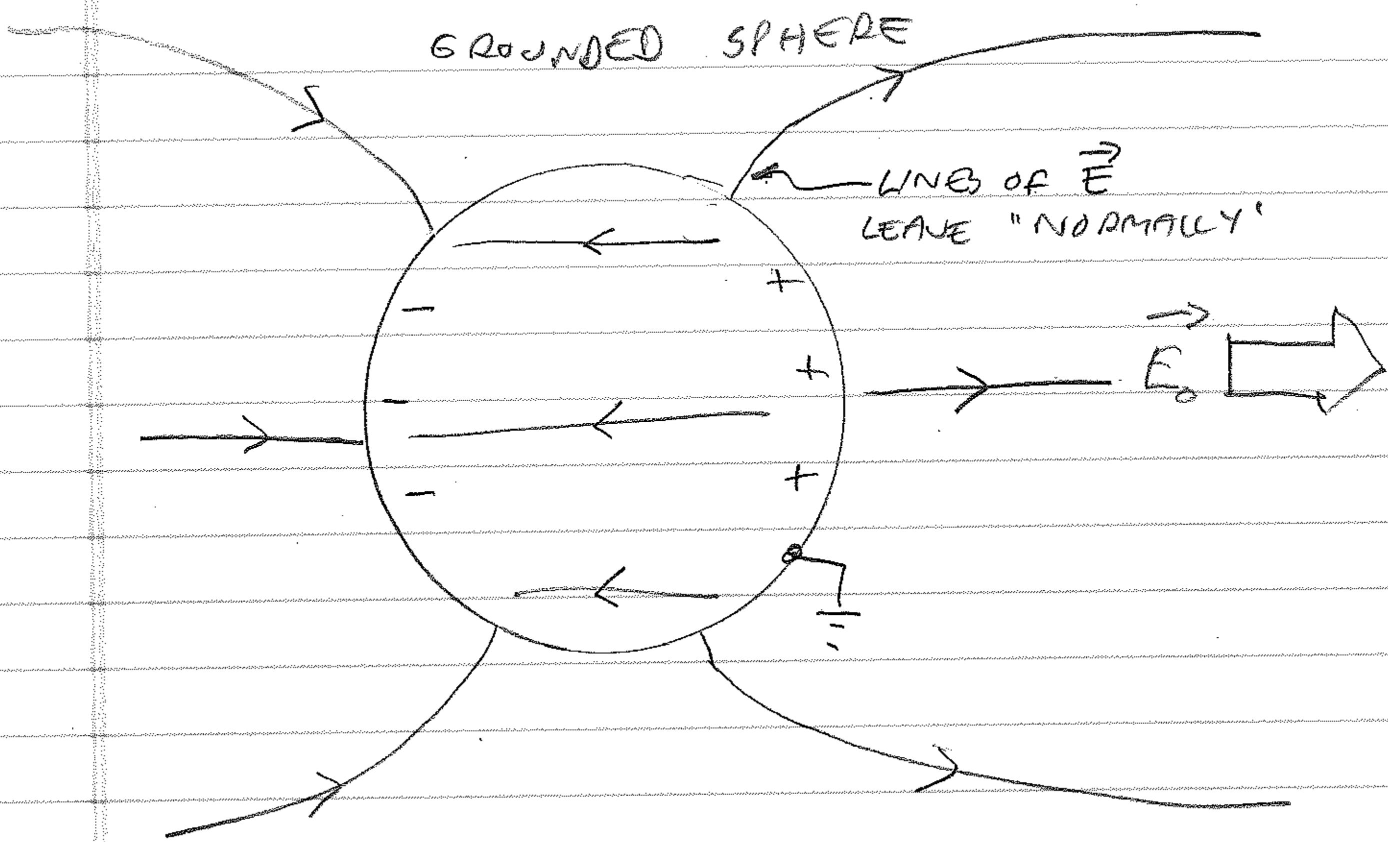
RECALL THE PURE DIPOLE POTENTIAL

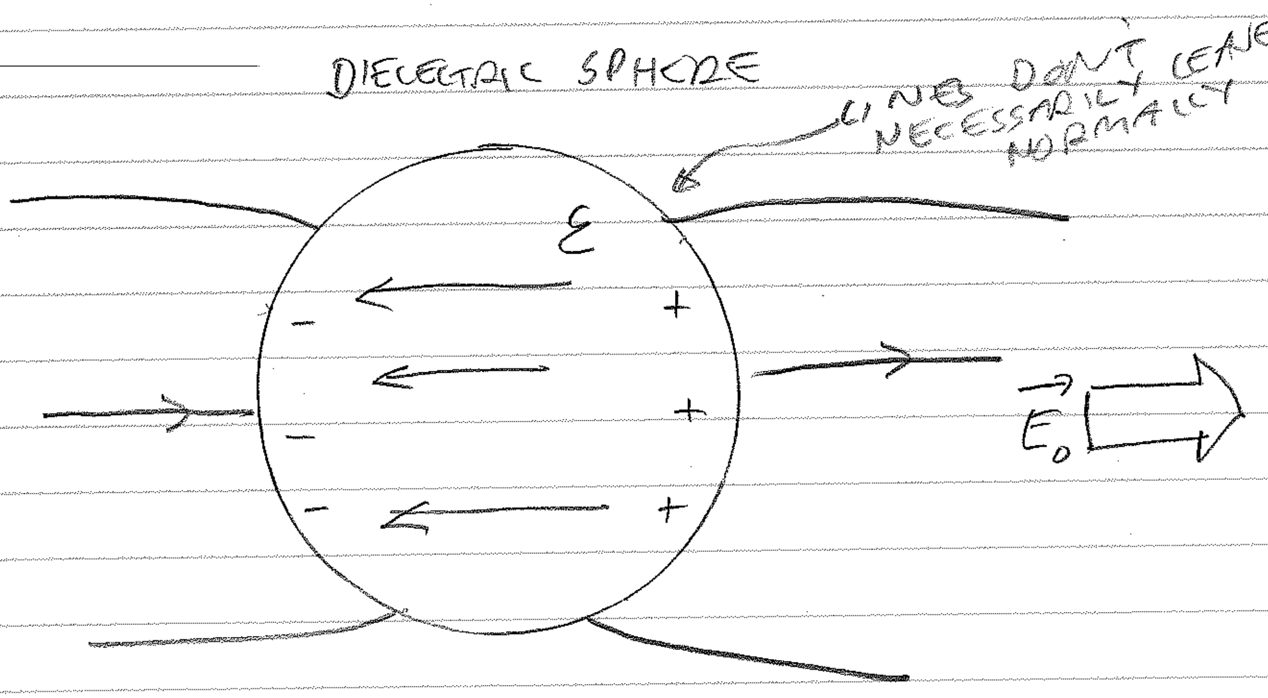
$$\Phi^{(2)}(r,\theta) = \frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^2},$$ p THE DIPOLE MOMENT.

IN $\Phi_{r>R}$, THE $-E_0 r\cos\theta$ TERM IS THAT OF THE APPLIED FIELD $E_0\hat{z}$.

THE OTHER TERM IS THAT OF A PURE DIPOLE. Q: WHY? YOU CAN READ OFF:

$$p = 4\pi\varepsilon_0 \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} E_0 R^3.$$


GROUNDED SPHERE
LINES OF $\vec{E}$
LEAVE "NORMALLY"
$\vec{E}_0$


DIELECTRIC SPHERE
LINES DON'T NECESSARILY LEAVE NORMALLY
ε
$\vec{E}_0$

ELECTROSTATIC ENERGY IN DIELECTRIC MEDIA (JACKSON §4.7).

WE START BY POSING A QUESTION: CAN ALL THE MECHANICAL PROPERTIES OF AN ELECTRICALLY-INTERACTING SYSTEM BE DESCRIBED EITHER IN TERMS OF THE INTERACTING SOURCES ON IN TERMS OF THE FIELDS WHICH ARE PRODUCED BY THE SOURCES?

FOR THE LATER (FIELD THEORY APPROACH) TO BE SENSIBLE, THE INTERACTIONS MUST BE FORMULATED IN TERMS OF THE FIELDS THEMSELVES, AND NOT EXPLICITLY ON THE CONFIGURATION OF SOURCES THAT PRODUCE THE FIELD.

WE'VE SEEN THIS DISCUSSION BEFORE: IT SHOULD BE POSSIBLE TO HAVE A FIELD THEORY IN WHICH WE CAN DESCRIBE THE MECHANICAL PROPERTIES (SUCH AS ENERGY OR FORCE) EQUIVALENTLY IN TERMS OF THE SOURCES OF THE FIELD OR IN TERMS OF THE FIELDS OF THE SOURCES.

SUMMARY OF THIS DISCUSSION EARLIER THIS QUARTER (VACUUM, CHARGES AND CONDUCTORS):

- WORK AN EXTERNAL AGENT NEEDS TO APPLY TO ASSEMBLE A CHARGED SYSTEM (IGNORING SELF-ENERGY)

$$W = \frac{\varepsilon_0}{2} \sum_i \varepsilon_i \Phi_i^{0}$$

THE "0" REMINDS US TO NOT INCLUDE THE SELF ENERGY.

THIS IS A FREE ENERGY.

- $$U = \frac{\varepsilon_0}{2} \iiint E^2 \, dv$$

THIS IS A TOTAL ENERGY.

ENERGY & DIELECTRICS:

BOTTOM LINE: UNDER CERTAIN CIRCUMSTANCES

$$U = \frac{\varepsilon_0}{2} \iiint \vec{E} \cdot \vec{E} \, dv$$

CAN BE REPLACED WITH

$$U = \frac{1}{2} \iiint \vec{E} \cdot \vec{D} \, dv;$$

WE'LL EXPLORE THIS.

WHERE DOES $\frac{1}{2} \iiint \vec{E} \cdot \vec{D}$ COME FROM?

- START WITH A SYSTEM THAT INCLUDES DIELECTRICS.
- ADD A SMALL AMOUNT OF "FREE" CHARGE $\delta \rho$ TO THE SYSTEM. ASSUME "RIGID" CONSTRAINTS SO NO MECHANICAL WORK IS DONE ON ADDING CHARGE.
- TO EVADE ISSUES OF SELF ENERGY, WE'LL ASSUME CONTINUOUS CHARGE DISTRIBUTIONS.

- ON ADDING $\delta \rho$, THE INCREMENT OF WORK DONE δW IS

$$\delta W = \iiint \Phi \, \delta \rho \, dv \quad (\text{"}qV\text{" WORK}).$$

$$= \iiint \Phi \, \delta(\vec{\nabla} \cdot \vec{D}) \, dv$$

$$= \iiint \Phi \, \vec{\nabla} \cdot \delta \vec{D} \, dv.$$

INTERATE BY PARTS: THE IDENTITY

$$\vec{\nabla} \cdot (\vec{A} \, \Psi) = \Psi \, \vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{\nabla} \Psi;$$

$$\delta W = \oiint \Phi \, \delta \vec{D} \cdot \hat{n} \, dv$$

$$- \iiint \delta \vec{D} \cdot \vec{\nabla} \Phi \, dv$$

WITH THE SOURCES LOCALIZED, THE SURFACE TERM AT ∞ VANISHES, SO

$$\delta W = \iiint \vec{E} \cdot \delta \vec{D} \, dv.$$

AND HERE, WE'RE STUCK.

$$\delta W = \iiint \vec{E} \cdot \delta\vec{D} \, d\omega.$$

WE'D LIKE TO KEEP ADDING INCREMENTS OF CHARGE $\delta\rho$ TO BRING $\delta\vec{D}$ UP TO A FINAL $\vec{D}$. BUT WE DON'T A PRIORI KNOW THE CONSTITUTIVE RELATION $\vec{E}(\vec{D})$, SO WE DON'T KNOW HOW TO DO THAT INTEGRATION.

NOW! A SPECIAL CASE, SUPPOSE IT'S A LINEAR DIELECTRIC

$$\vec{D} = \varepsilon\vec{E}, \quad \delta\vec{D} = \varepsilon\delta\vec{E}.$$

HENCE

$$U = \int_0^D \delta W = \int_0^D \iiint \vec{E} \cdot \delta\vec{D} \, dv$$

$$= \iiint \int_0^E \varepsilon\vec{E} \cdot \delta\vec{E} \, dv$$

$$= \iiint \int_0^E \tfrac{1}{2}\varepsilon\,\delta(E^2) \, dv$$

$$= \frac{1}{2} \iiint \tfrac{1}{2}\varepsilon E^2 \, dv$$

$$= \frac{1}{2} \iiint \vec{E} \cdot \vec{D} \, dv$$

SO, $U = \frac{1}{2} \iiint \vec{E} \cdot \vec{D} \, dV$

APPLIES TO LINEAR DIELECTRICS,

IT'S NOT A FUNDAMENTAL EQUATION

OF ELECTROSTATICS.

Q: FOR THE ELECTRET, IS

$U = \frac{1}{2} \iiint \vec{E} \cdot \vec{D} \, dV$?

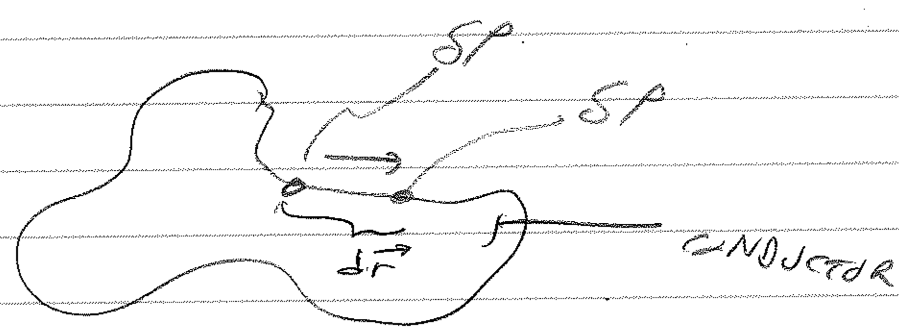
IF NOT, HOW WOULD YOU

FIND U?

APPLICATION OF THIS PROCEDURE:
THOMSON'S THEOREM.

"CHARGES REDISTRIBUTE THEMSELVES ON A CONDUCTOR SO AS TO MINIMIZE THE (FREE) ENERGY."

THERE COULD PERHAPS BE A NUMBER OF CONDUCTORS. SUPPOSE THE CHARGES ARE STATIC (IN EQUILIBRIUM). THEN MAKE A VIRTUAL DISPLACEMENT OF THE CHARGES ALONG ONE OF THE CONSTANT-POTENTIAL SURFACES OF A CONDUCTOR:



(SUBTLETIES ABOUND HERE; FOR INSTANCE, ONE CONVENTION IN ELECTRODYNAMICS IS TO CALL THE FREE ENERGY U INSTEAD OF F. ALSO, THE TEMPERATURE NEEDS TO BE CONSTANT... THERE'S A HEAT BATH ... SO AS TO BE ABLE TO EQUATE CHANGES IN FREE ENERGY TO MECHANICAL WORK. ETC.)

RECALL OUR PRIOR RESULT: FOR LOCALIZED SOURCES

$$\delta U = \iiint \vec{E} \cdot \delta \vec{D} \, dv$$

REDUCES TO

$$\delta U = \iiint \Phi \, \delta \rho \, dv.$$

THERE COULD BE A NUMBER OF CONDUCTORS, EACH WITH THEIR OWN POTENTIAL Φ_i. SO

$$\delta U = \sum_{i=1}^{N} \iiint_i \Phi_i \, \delta \rho \, dv_i$$

NOTE Φ_i IS CONSTANT OVER CONDUCTOR i AND $\delta\rho$ IS NON-ZERO ONLY ON THE CONDUCTOR, SO:

$$\delta U_i' = \Phi_i \iiint \delta\rho \, dV = \Phi_i \, \delta q_i' = 0$$

SINCE CHARGE WAS MOVED, BUT THE TOTAL CHARGE ON EACH CONDUCTOR IS UNCHANGED.

WITH $\delta U_i'$, EACH CONDUCTOR IS AT A STATIONARY POINT IN U.

THUS, THE EQUILIBRIUM CONDITION THAT EACH CONDUCTOR BE AN EQUIPOTENTIAL IS EQUIVALENT TO MAKING THE FREE ENERGY A MINIMUM IN THOMSON'S THEOREM.

JACKSON LEAVES IT HERE. THERE'S A FEW DIRECTIONS FROM HERE:

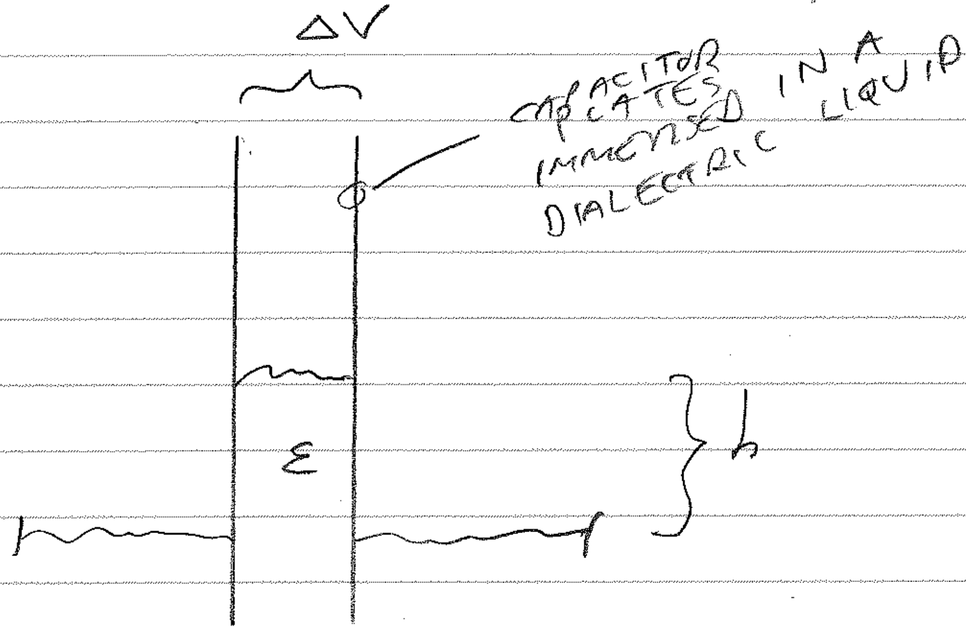
- THERMODYNAMICS OF U. E.g., SUPPOSE AN ELECTRIC FIELD THREADS A DIELECTRIC, HOW MUCH HEAT IS GIVEN OFF?

- SPECIFIC FORMS OF THE DIELECTRIC CONSTANT. E.g., THE CLAUSIUS - MOSSITTI RELATION

$$\varepsilon_r = 1 + C/T; \quad T \text{ THE TEMPERATURE.}$$

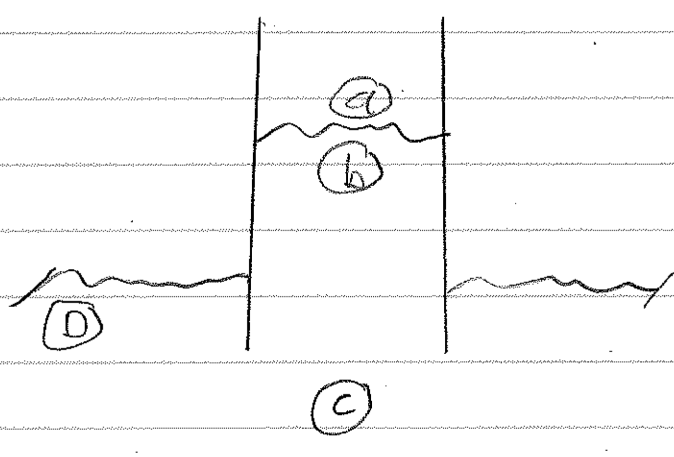
- DIELECTRIC LIQUIDS. WE'LL LOOK AT THIS.

DIELECTRIC LIQUIDS.
THE GRIFFITHS-TYPE DISCUSSION IS ALSO DONE IN JACKSON:



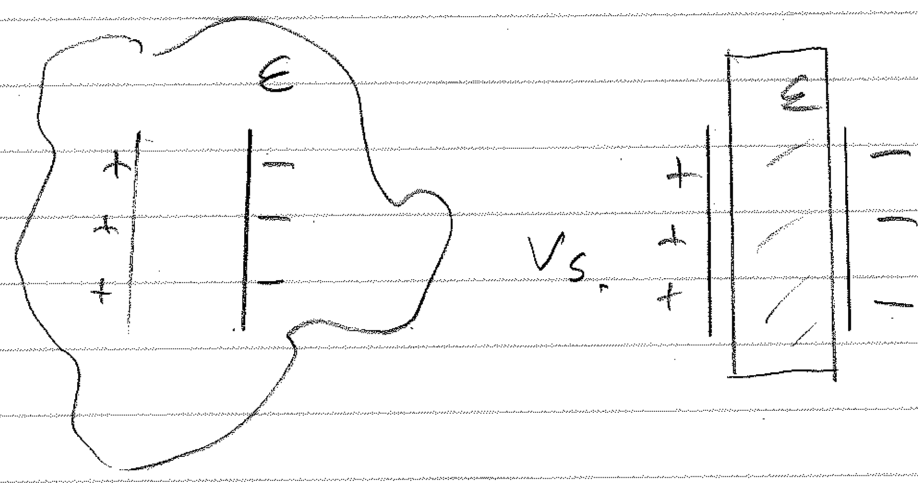
THE COLUMN HEIGHT h IS DEDUCED FROM BALANCING ELECTROSTATIC ENERGY WITH WITH THE GRAVITATIONAL POTENTIAL.

THIS IS FINE. BUT WHERE DOES THE UPWARDS FORCE COME FROM? HERE'S SOME POSSIBILITIES:



UNDERSTANDING THE ANSWER TO THIS ALLOWS YOU TO ANSWER ANOTHER QUESTION:

CONSIDER PARALLEL CAPACITOR PLATES IN TWO CONFIGURATIONS. CASE 1 HAS PLATES COMPLETELY SUBMERGED IN A DIELECTRIC LIQUID ε, CASE 2 HAS A SOLID DIELECTRIC SLAB ε BETWEEN BUT NOT TOUCHING THE PLATES.



Q: IS THE ATTRACTIVE FORCE BETWEEN THE PLATES IN THE TWO CASES THE SAME? FIELD THEORY OF ELECTRODYNAMICS WOULD SUGGEST SO... BUT WHAT'S THE ANSWER?