Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
October 15, 2020, 11am
On-line lecture

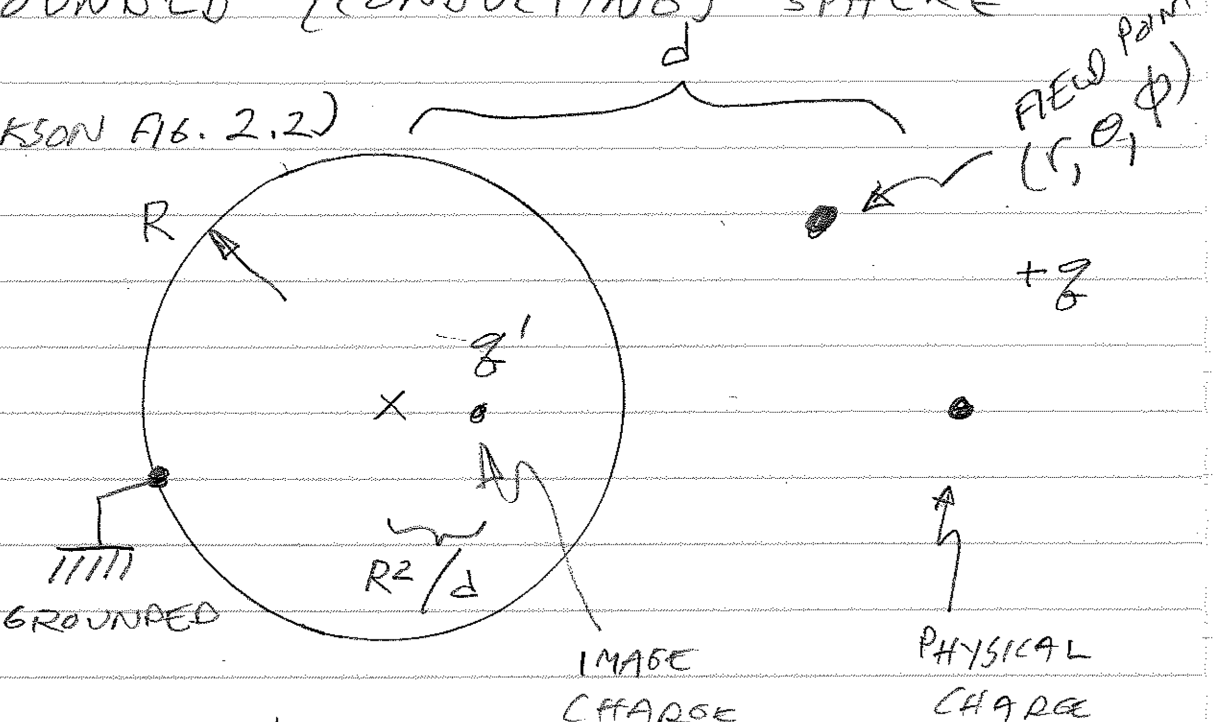
Administrative:

1. Homework 2 posted at
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
2. Draft of this lecture posted at
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
3. Office hours today after class at 12:30.

Lecture: Methods of finding potentials in boundary-value problems. (Jackson chapter 2).
Method of images: planar & spherical systems, continued.
Example of a "cos θ" field.
Green's function for the sphere and an example.

RECALL THE SPHERICAL IMAGE-CHARGE EXAMPLE: POINT CHARGE NEAR GROUNDED (CONDUCTING) SPHERE

(JACKSON FIG. 2.2)



$$q' = -q\,R/d$$

EXERCISE: SHOW THAT THE POTENTIAL AT $r = R$ (JACKSON EQN 2.3) IS ZERO.

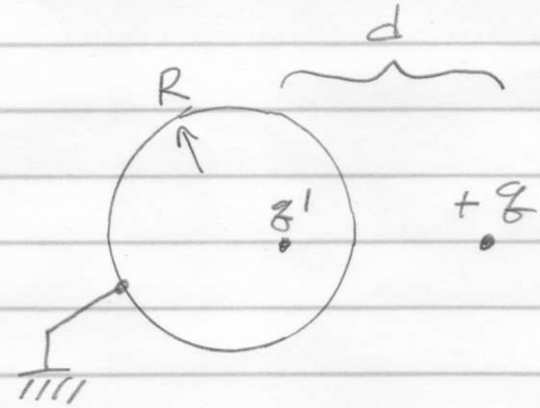
Q: HOW DOES THE PROBLEM CHANGE IF THE PHYSICAL CHARGE AND REGION OF INTEREST WERE AT $r < R$?

Q: DIFFERENT EXAMPLE: START WITH A NEUTRAL CONDUCTING SPHERE OF RADIUS R. THEN BRING CHARGE $+q$ TO A POSITION $r > R$. WHAT'S THE POTENTIAL OF THE SPHERE?

A: USE THE RESULT OF THE PREVIOUS PROBLEM, PLUS SUPERPOSITION

PROCEDURE:

① RESULT OF PREVIOUS PROBLEM.

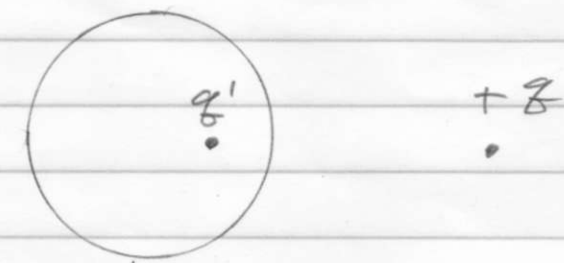


$q' = -q \frac{R}{d}$

Q: (TRIVIAL) WHAT'S THE POTENTIAL OF THE SPHERE?

Q: WHAT'S THE CHARGE ON THE GROUNDED SPHERE?

(2) REMOVE GROUND WIRE.



Q: DID ANYTHING CHANGE? (NO.)

(3) THIS IS NOT QUITE THE SOLUTION WE SEEK. WE'D LIKE THE CONDUCTING, ISOLATED, SPHERE TO HAVE ZERO CHARGE.

(4) Q: HOW MUCH CHARGE Q WOULD YOU NEED TO ADD TO MAKE THE SPHERE NEUTRAL?
A: $-q'$

(5) DUMP CHARGE $-q'$ ON THE SPHERE.
Q: CAN YOU DUMP IT ALL ON TOP OR DO YOU HAVE TO ADD IT UNIFORMLY.
A: FROM THE 2ND UNIQUENESS THEOREM, THE CHARGE $-q'$ IS ULTIMATELY DISTRIBUTED UNIFORMLY.

(6) THE POTENTIAL OF A UNIFORMLY CHARGED (WITH CHARGE $-q'$) SPHERE OF RADIUS R IS

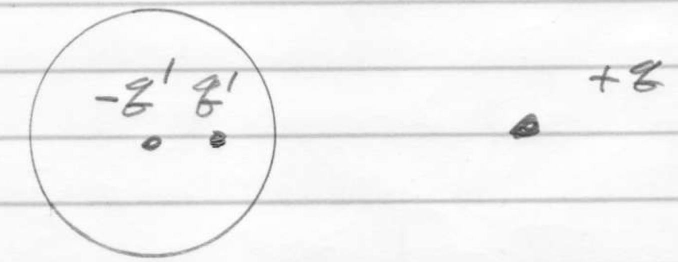
$$\frac{1}{4\pi\varepsilon_0}\frac{-q'}{R}$$

(7) SUPERIMPOSE THIS UNIFORMLY-CHARGED SYSTEM WITH THE POTENTIAL OF THE (NOT UNIFORMLY-CHARGED) GROUNDED-SPHERE EXAMPLE; THE LATER CONTRIBUTES ZERO.

(8) THE DESIRED POTENTIAL IS

$$\Phi(r=R) = \frac{1}{4\pi\varepsilon_0}\left(\frac{-qR}{d}\right)\frac{1}{R}$$

(9) NOTICE THE IMAGE-CHARGE PICTURE IS



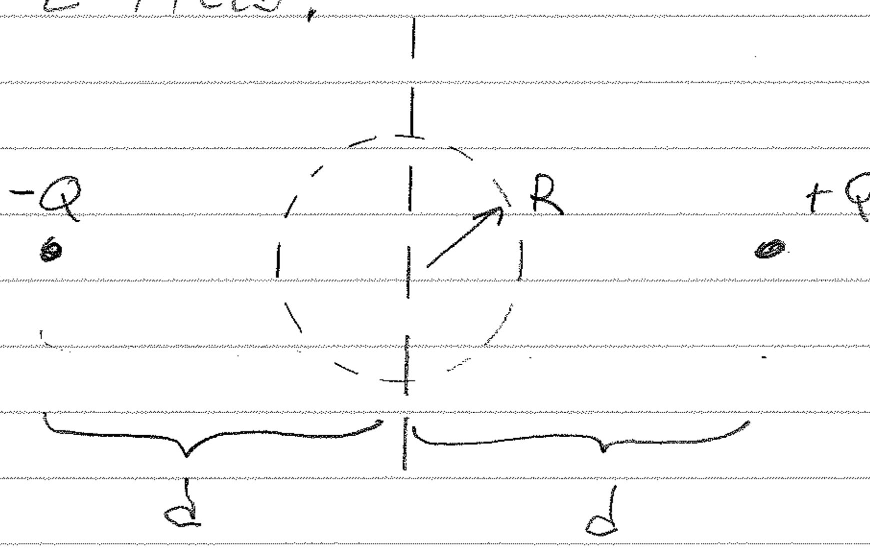
BY SUPERPOSITION OF POTENTIAL PLUS UNIQUENESS, THIS IS THE SOLUTION.

THERE ARE LOTS OF RELATED EXAMPLES, C.F., JACKSON §2.4: POINT CHARGE NEAR CONDUCTING SPHERE AT A SPECIFIED POTENTIAL

JACKSON §2.5

SOMETIMES THERE ARE SEVERAL SENSIBLE WAYS TO ADDRESS A PROBLEM. EXAMPLE: SOME EXTERNAL AGENT HAS ESTABLISHED A UNIFORM $\vec{E}$ FIELD. INTRODUCE A NEUTRAL, CONDUCTING SPHERE. WHAT'S $\Phi(r,\theta)$? THIS IS A CLASSIC PROBLEM FOR SEPARATION-OF-VARIABLES AND LEGENDRE POLYNOMIALS. WE'LL FOR NOW ADDRESS IT WITH METHOD-OF-IMAGES.

- CONSTRUCT A UNIFORM $\vec{E}$ FIELD.



WITH $d \gg R$, THE FIELD $\vec{E_0}$ NEAR THE SPHERE IS UNIFORM. (FROM DIMENSIONALITY, ALL GRADIENTS IN THIS TWO-CHARGE SYSTEM $\sim 1/d$.)

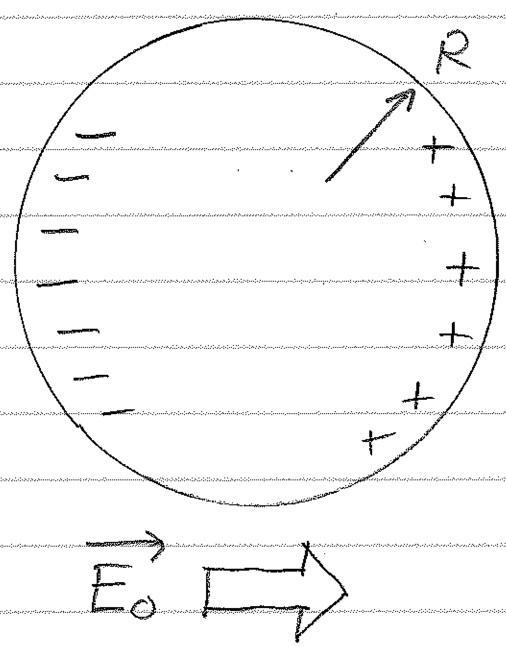
$$|\vec{E_0}| \approx 2 \frac{1}{4\pi\epsilon_0} \frac{Q}{d^2}.$$

THE POTENTIAL FOR A UNIFORM $\vec{E}$ FIELD $\vec{E_0}$ IS

$$\Phi(r,\theta) = -E_0 r \cos\theta$$

Q: WHICH DIRECTION IS $\theta = 0$?

• INTRODUCE A NEUTRAL CONDUCTING SPHERE AT THE ORIGIN.

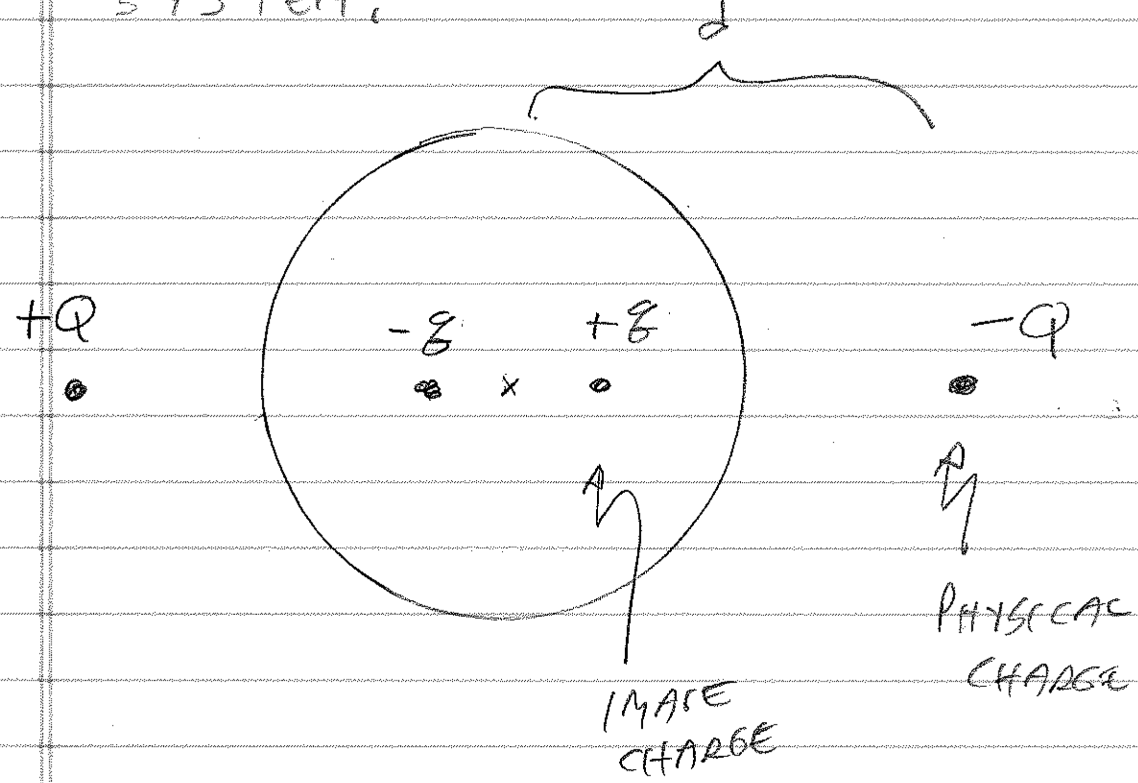


(5)

THE CONDUCTING SPHERE BECOMES POLARIZED. THE LEADING (AND ONLY, AS IT HAPPENS) MOMENT IS A DIPOLE WITH

$$\Phi_{DIPOLE} \sim 1/r^2.$$

- RETURN TO THE IMAGE SYSTEM.



• FIND $\Phi(r,\theta)$

1. AT THE FIELD POINT, SUPER-IMPOSE THE FOUR POTENTIALS.

2. TAKE THE LIMIT $d \gg r$ (MORE EASILY DONE BY FACTORING OUT 'd^2') IN THE TWO TERMS WITH CHARGES $\pm Q$).

3. TAKE THE LIMIT $r \gg R^2/d$ (MORE EASILY DONE BY FACTORING OUT r^2) IN THE TWO TERMS CONTAINING CHARGES $\pm \tilde{z}$.

4. EXPAND SQUARE-ROOTS IN THE DENOMINATORS; KEEP LEADING TERM.

IN DETAIL

$$\Phi(r,\theta) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{(r^2+d^2+2rd\cos\theta)^{1/2}}$$

$$-\frac{Q}{4\pi\varepsilon_0}\frac{1}{(r^2+d^2-2rd\cos\theta)^{1/2}}$$

$$-\frac{Q\,R/d}{4\pi\varepsilon_0}\frac{1}{\left(r^2+\frac{R^4}{d^2}+2\frac{R^2}{d}r\cos\theta\right)^{1/2}}$$

$$+\frac{Q\,R/d}{4\pi\varepsilon_0}\frac{1}{\left(r^2+\frac{R^4}{d^2}-2\frac{R^2}{d}r\cos\theta\right)^{1/2}}$$

FACTOR OUT d^2 (FIRST 2), r^2 (LATER 2)

$$\Phi(r,\theta) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{d}\frac{1}{\left(\frac{r^2}{d^2}+1+\frac{2r}{d}\cos\theta\right)^{1/2}}$$

$$-\frac{Q}{4\pi\varepsilon_0}\frac{1}{d}\frac{1}{\left(\frac{r^2}{d^2}+1-\frac{2r}{d}\cos\theta\right)^{1/2}}$$

$$-\frac{Q\,R/d}{4\pi\varepsilon_0}\frac{1}{r}\frac{1}{\left(1+\frac{R^4}{d^2r^2}+2\frac{R^2}{dr}\cos\theta\right)^{1/2}}$$

$$+\frac{Q\,R/d}{4\pi\varepsilon_0}\frac{1}{r}\frac{1}{\left(1+\frac{R^4}{d^2r^2}-2\frac{R^2}{dr}\cos\theta\right)^{1/2}}$$

EXPAND SQUARE ROOTS.

$$\Phi(r,\theta) \approx \frac{Q}{4\pi\epsilon_0}\frac{1}{d}\left\{1 - \frac{1}{2}\left(\frac{r^2}{d^2} + \frac{2r}{d}\cos\theta\right)\right\}$$

$$-\frac{Q}{4\pi\epsilon_0}\frac{1}{d}\left\{1 - \frac{1}{2}\left(\frac{r^2}{d^2} - \frac{2r}{d}\cos\theta\right)\right\}$$

$$-\frac{Q\,R/d}{4\pi\epsilon_0}\frac{1}{r}\left\{1 - \frac{1}{2}\left(\frac{R^4}{d^2r^2} + 2\frac{R^2}{dr}\cos\theta\right)\right\}$$

$$+\frac{Q\,R/d}{4\pi\epsilon_0}\frac{1}{r}\left\{1 - \frac{1}{2}\left(\frac{R^4}{d^2r^2} - 2\frac{R^2}{dr}\cos\theta\right)\right\}$$

TAKE LIMITS $d \gg r$ & $r \gg R^2/d$.

$$\Phi(r,\theta) \approx \underbrace{\frac{2Q}{4\pi\epsilon_0}\frac{1}{d^2}\, r\cos\theta}_{\text{"CONSTANT } \vec{E}\text{" TERM}}$$

$$\underbrace{-\frac{2Q}{4\pi\epsilon_0}\frac{1}{d^2}\frac{R^3}{r^2}\cos\theta}_{\text{"DIPOLE" TERM}}$$

RECALL THE POTENTIAL FROM A POINT DIPOLE IS $\Phi_{DIPOLE}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$.

SO $\Phi(r,\theta)$ CONTAINS A POINT-DIPOLE POTENTIAL WITH

$$|\vec{p}| = -2Q\frac{R^3}{d^2}$$

THE OTHER TERM IS THAT OF A UNIFORM $\vec{E}$ FIELD WITH

$$|\vec{E}_0| = \frac{2Q}{4\pi\varepsilon_0}\frac{1}{d^2}.$$

SO $\Phi(r,\theta) = E_0 r\cos\theta - E_0\frac{R^3}{r^2}\cos\theta$

OF COURSE, INSIDE $\Phi(r<R) = 0$.

WHAT'S THE SURFACE CHARGE?

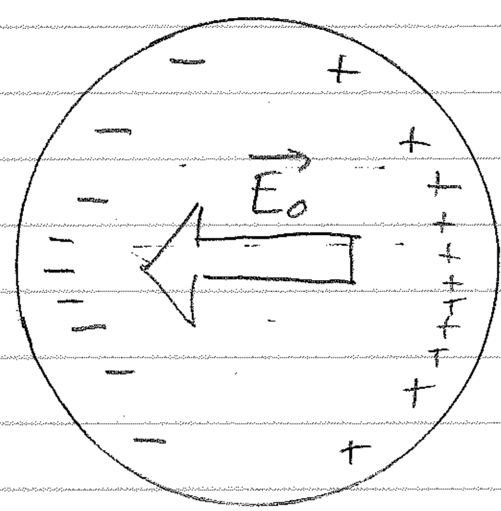
AGAIN $\sigma = -\varepsilon_0 \left.\frac{d\Phi}{dr}\right|_{r=R}$

$$= 3\varepsilon_0 \frac{2Q}{4\pi\varepsilon_0}\frac{1}{d^2}\cos\theta$$

IT'S INTERESTING TO LOOK IN MORE DETAIL AT THE INTERIOR FIELD.

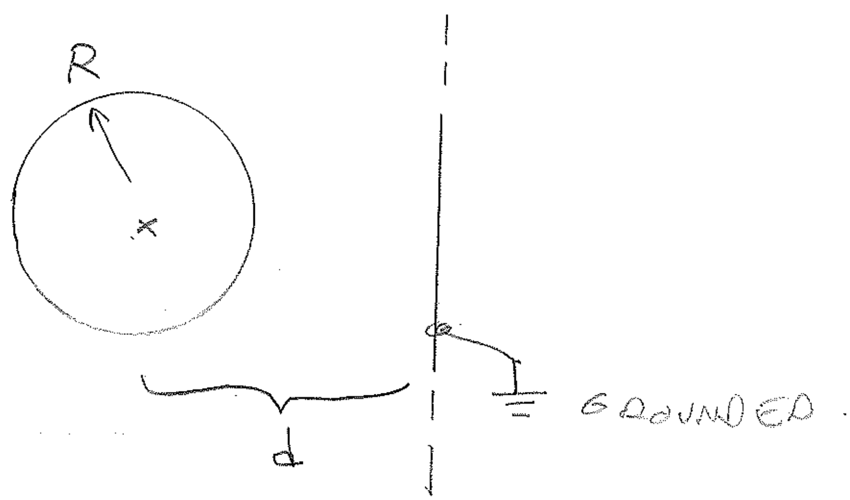
THE INTERIOR FIELD IS ZERO.
Q: WHY?
THIS INTERIOR FIELD IS A SUPERPOSITION OF THE UNIFORM EXTERNAL FIELD $\vec{E_0}$ PLUS THE FIELD FROM THE SURFACE CHARGE.

SINCE $\vec{E_0}$ IS UNIFORM, THE FIELD FROM THE SURFACE CHARGE IS LIKEWISE UNIFORM. IT FOLLOWS THAT A "COS θ" DISTRIBUTION OF SURFACE CHARGE PRODUCES A UNIFORM $\vec{E}$ FIELD.

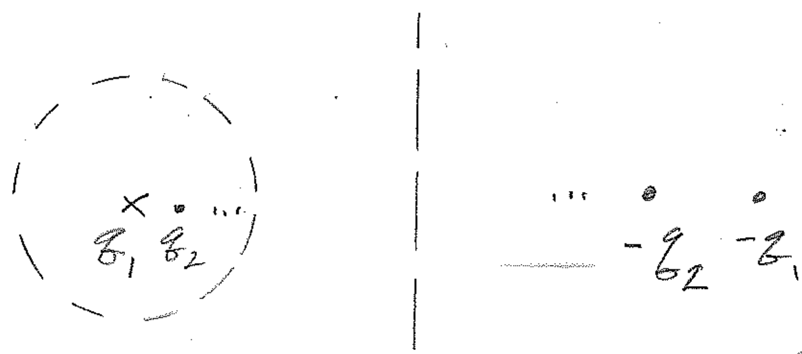


<u>EXAMPLE</u> CAPACITANCE OF A CONDUCTING SPHERE AND A GROUND PLANE.

RECALL THE CAPACITANCE OF A SPHERE (AND THE "GROUNDED" SURFACE AT ∞) IS $4\pi\varepsilon_0 R$. WHAT'S THE EFFECT OF THE GROUND PLANE?



WE'LL USE METHOD OF IMAGES.



q_1 MAKES THE SPHERE AN EQUIPOTENTIAL (BUT NOT PLANE)
$-q_1$ MAKES PLANE EQUIPOTENTIAL (BUT NOT SPHERE)
q_2 MAKES SPHERE EQUIPOTENTIAL (BUT NOT PLANE)
$-q_2$ MAKES PLANE EQUIPOTENTIAL (BUT NOT SPHERE)

AD INFINITUM

NOW, $-q_1$ HAS THE SAME MAGNITUDE AS q_1. $q_2 = +q_1 \frac{R}{2d}$.

THE TOTAL CHARGE ON THE SPHERE IS

$$q_1 + q_2 + \cdots = q_1 \left(1 + R/2d + \cdots\right)$$

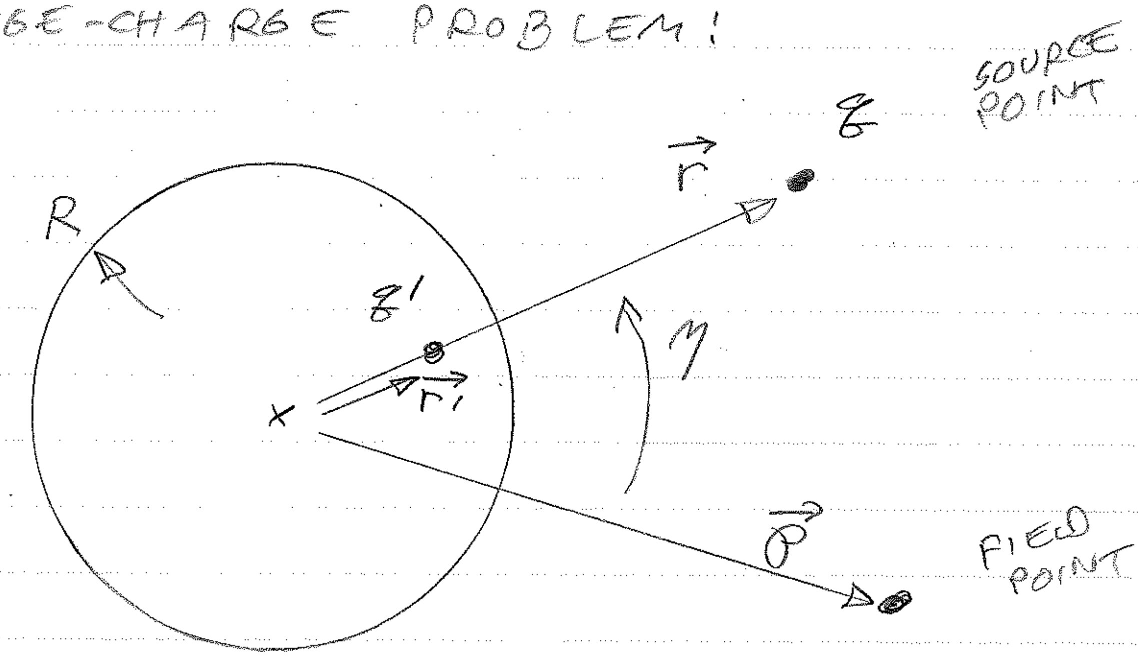
NOW TO FIND THE CAPACITANCE. THIS IS THE CHARGE ON A CONDUCTOR DIVIDED BY THE POTENTIAL DIFFERENCE. BUT NOTICE IT'S ONLY CHARGE q_1 THAT ESTABLISHES THE POTENTIAL OF THE SPHERE. ALL THE OTHER CHARGES ONLY SERVE TO MAKE THE POTENTIAL OF THE SPHERE ZERO. (PLUS THE POTENTIAL DUE TO q_1).

$$\text{HENCE } C = \frac{Q}{V} = \frac{q_1\left(1 + R/2d + \cdots\right)}{q_1/4\pi\epsilon_0 R}$$

$$= 4\pi\epsilon_0 R\left(1 + R/2d + \cdots\right)$$

MOVING THE GROUND PLANE NEAR THE SPHERE INCREASES THE CAPACITANCE (AS EXPECTED).

GREEN'S FUNCTION FOR THE SPHERE.
RECALL THE INITIAL SPHERICAL IMAGE-CHARGE PROBLEM!



$$\Phi(\vec{\rho}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{\rho}|} + \frac{1}{4\pi\varepsilon_0}\frac{q'}{|\vec{r'}-\vec{\rho}|}$$

WITH $q' = -q\,R/r$

AND $r' = R^2/r$

THIS Φ IS VERY CLOSE TO G.
WE WANT $\nabla^2 G = -4\pi\,\delta(\vec{r}-\vec{r'})$, SO

$$G(\vec{r},\vec{r'}) = \frac{1}{|\vec{r}-\vec{r'}|} + \cdots$$

FOR UNIT CHARGE $q \rightarrow 4\pi\varepsilon_0$,

$$G(\vec{r},\vec{\rho}) = \frac{1}{|\vec{r}-\vec{\rho}|} - \frac{R}{r}\frac{1}{\left|\frac{R^2}{r^2}\vec{r}-\vec{\rho}\right|}$$

THIS IS THE SPHERICAL GREEN'S FUNCTION.

WE'LL TURN THIS INTO SPHERICAL COORDINATES (r, θ, ϕ):

$$G(\vec{r}, \vec{\rho}) = \frac{1}{(r^2 + \rho^2 - 2r\rho\cos\eta)^{1/2}}$$

$$- \frac{}{\left(R^2 + \left(\frac{r}{R}\rho\right)^2 - 2r\rho\cos\eta\right)^{1/2}}$$

WITH η THE ANGLE BETWEEN $\vec{r}$ AND $\vec{\rho}$.

FOR DIRICHLET BOUNDARY CONDITIONS, WE'LL NEED

$$\Phi(\vec{\rho}) = \frac{1}{4\pi\varepsilon_0} \iiint \rho(\vec{r})\, G(\vec{\rho}, \vec{r})\, d\tau$$

$$- \frac{1}{4\pi} \oiint \Phi(\vec{r})\Big|_s \frac{\partial}{\partial n} G(\vec{\rho}, \vec{r})\Big|_s \, dA .$$

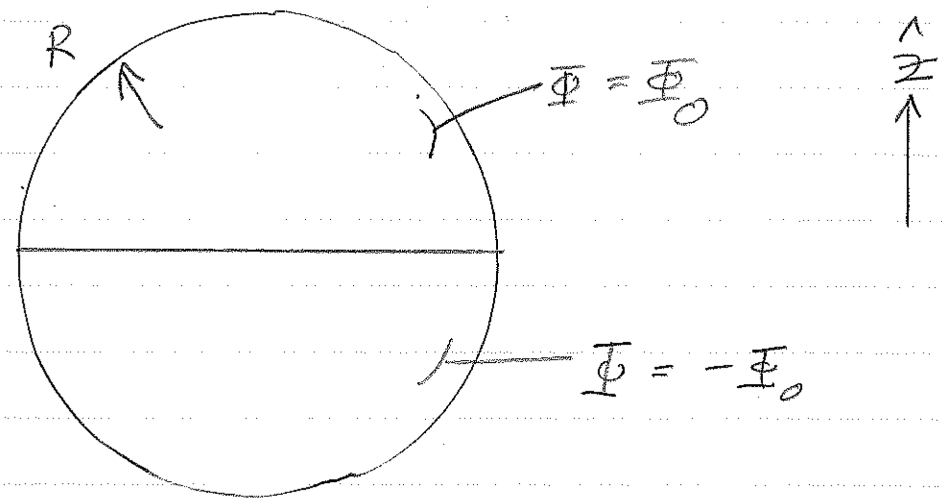
SO, WE'LL NEED $\frac{\partial}{\partial n} G\Big|_{s\ (r=R)}$:

$$\frac{\partial G}{\partial n}\Big|_{r=R} = - \frac{\rho^2 - R^2}{R\{R^2 + \rho^2 - 2R\rho\cos\eta\}^{3/2}}$$

WE COULD HAVE GUESSED THIS. IT'S THE SURFACE-CHARGE FOR THE IMAGE-CHARGE PROBLEM FOR UNIT CHARGE.

JACKSON §2.7

EXAMPLE; TWO CONDUCTING HEMISPHERES AT DIFFERENT POTENTIALS. (N.B., WE'LL DO THIS LATER VIA SEPARATION-OF-VARIABLES AND LEGENDRE POLYNOMIALS.)



NO BULK CHARGE ρ, ONLY BOUNDARY TERMS.

$$\Phi(r,\theta,\phi) = \frac{\Phi_0}{4\pi} \underbrace{(2\pi)}_{\text{FROM } \int d\phi} \underbrace{(R^2)}_{\text{FROM } d\Omega \text{ IN } dA}$$

$$\times \left[\int_0^{+1} d\cos\theta' \frac{r^2 - R^2}{R\{R^2 + r^2 - 2Rr\cos\eta\}^{3/2}} - \int_{-1}^{0} d\cos\theta' \frac{r^2 - R^2}{R\{R^2 + r^2 - 2Rr\cos\eta\}^{3/2}} \right]$$

THIS IS A MESS. BUT THERE'S A COUPLE TRICKS.

COMBINE THE TWO INTEGRALS BY A SUBSTITUTION IN $\int_{-1}^{0} d\cos\theta'$ VIA $\theta' \longrightarrow \pi - \theta'$. HENCE

$$\Phi(r,\theta,\phi) = \frac{\Phi_0}{4\pi} 2\pi R^2$$

$$\times \int_0^1 d\cos\theta' \left\{ \frac{r^2 - R^2}{R(r^2+R^2-2rR\cos\eta)^{3/2}} - \frac{r^2 - R^2}{R(r^2+R^2+2rR\cos\eta)^{3/2}} \right\}$$

IT'S STILL A MESS.
BUT, THERE'S A SPECIAL CASE FOR THE VALUE OF Φ ON THE $+z$-AXIS:

$$\Phi(r,\theta=0,\phi) = \Phi_0 \left\{ 1 - \frac{z^2 - R^2}{z} \frac{1}{\sqrt{z^2+R^2}} \right\}$$

NOTICE $\Phi \to \Phi_0$ AT $z \to R$, AS EXPECTED.

AT LARGE DISTANCES $z \gg R$, WE SEE A DIPOLE (NOT A PURE DIPOLE)

$$\Phi \longrightarrow \Phi_0 \frac{R^2}{z^2}.$$

NOTICE I SKIPPED THE GREEN'S FUNCTION FOR A CYLINDER. THIS IS BECAUSE THERE'S NO SIMPLE IMAGE-CHARGE SOLUTION; BUT THERE IS A COMPLICATED IMAGE-CHARGE SOLUTION. [SEE, e.g., SMYTHE §5.298],

WE'LL SHORTLY MOVE INTO SEPARATION-OF-VARIABLES, THIS WILL CONSIDER THE POINT CHARGE NEAR A CONDUCTING SPHERE, AND FROM THAT THE CORRESPONDING GREEN'S FUNCTION.