Lecture: Methods of finding potentials in boundary-value problems. (Jackson chapter 2).
Method of images: planar & spherical systems, continued.
Example of a “cos θ” field.
Green’s function for the sphere and an example.
RECALL THE SPHERICAL IMAGE-CHARGE EXAMPLE: POINT CHARGE NEAR GROUNDED (CONDUCTING) SPHERE

\[ q' = -\frac{q}{d} \]

EXERCISE: SHOW THAT THE POTENTIAL AT \( r = R \) (JACKSON E\&N 2.3) IS ZERO.

Q: HOW DOES THE PROBLEM CHANGE IF THE PHYSICAL CHARGE AND REGION OF INTEREST WERE AT \( r < R \)?
Q: Different example: start with a neutral conducting sphere of radius $R$. Then bring charge $+z$ to a position $r > R$. What's the potential of the sphere?

A: Use the result of the previous problem, plus superposition procedure:

1. Result of previous problem.

$$\phi = \frac{-z}{r}$$

$R$

Q: (trivial) What's the potential of the sphere?

Q: What's the charge on the grounded sphere?
2) REMOVE GROUND WIRE.

\[ q_1 + q_2 \]

Q: DID ANYTHING CHANGE? (NO)

3) THIS IS NOT QUITE THE SOLUTION WE SEEK. WE'D LIKE THE CONDUCTING, ISOLATED, SPHERE TO HAVE ZERO CHARGE.

4) Q: HOW MUCH CHARGE \( q_2 \) WOULD YOU NEED TO ADD TO MAKE THE SPHERE NEUTRAL?
   A: \(-q_1\)

5) DUMP CHARGE \(-q_1\) ON THE SPHERE.
   Q: CAN YOU DUMP IT ALL ON TOP OR DO YOU HAVE TO ADD IT UNIFORMLY.
   A: FROM THE 2ND UNIQUENESS THEOREM, THE CHARGE \(-q_1\) IS ULTIMATELY DISTRIBUTED UNIFORMLY.
6. **The potential of a uniformly charged (with charge \(-q\')) sphere of radius \(R\) is**

\[
\frac{1}{4\pi\varepsilon_0} \frac{-q'}{R}
\]

7. **Superimpose this uniformly charged system with the potential of the (not uniformly charged) grounded sphere example; the later contributes zero.**

8. **The desired potential is**

\[
\Phi(r=R) = \frac{1}{4\pi\varepsilon_0} \left( \frac{-q \, R}{d} \right) \frac{1}{R}
\]

9. **Notice the image-charge picture is**

<table>
<thead>
<tr>
<th>-q'</th>
<th>q'</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
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By **superposition of potential plus uniqueness, this is the solution.**
There are lots of related examples, C.F. Jackson §2.4: Point Charge near a Conducting Sphere at a Specified Potential

Jackson §2.5

Sometimes there are several sensible ways to address a problem. Example: Some external agent has established a uniform $E$ field. Introduce a neutral conducting sphere, what is $E(r, \theta)$? This is a classic problem for Separation-of-Variables and Legendre polynomials. We'll for now address it with Method-of-Images.

Construct a uniform $E$ field.

\[ \begin{matrix}
-Q & \rightarrow & +Q \\
\end{matrix} \]
With \( d \gg R \), the field \( \vec{E}_0 \) near the sphere is uniform. (From dimensionality, all gradients in this two-charge system \( \sim 1/d \).)

\[ \left| \vec{E}_0 \right| \approx 2 \frac{1}{4\pi\varepsilon_0} \frac{Q}{d^2}. \]

The potential for a uniform \( \vec{E} \) field \( \vec{E}_0 \) is

\[ \Phi(r, \theta) = -E_0 r \cos \theta. \]

Q: Which direction is \( \theta = 0 \)?

*Introduce a neutral conducting sphere at the origin.*
The conducting sphere becomes polarized. The leading (and only, as it happens) moment is a dipole with
\[ \vec{D} \sim \frac{1}{r^2}. \]

Return to the image system.
Find $\Phi(r, \theta)$.

1. At the field point, superimpose the four potentials.

2. Take the limit $d \gg r$ (more easily done by factoring out $d^2$) in the two terms with charges $\pm Q$.

3. Take the limit $r \gg R^2/d$ (more easily done by factoring out $r^2$) in the two terms containing charges $\pm Q$.

4. Expand square-roots in the denominators, keep leading term.
IN DETAIL

\[ \Phi (r, \theta) = \frac{Q}{4 \pi \varepsilon_0} \frac{1}{(r^2 + d^2 + 2rd \cos \theta)^{1/2}} \]

\[ - \frac{Q}{4 \pi \varepsilon_0} \frac{1}{(r^2 + d^2 - 2rd \cos \theta)^{1/2}} \]

\[ - \frac{Q}{4 \pi \varepsilon_0} \frac{r(R)}{(r^2 + R^2 - 2R^2 \cos \theta)^{1/2}} \]

\[ + \frac{Q}{4 \pi \varepsilon_0} \frac{R/R}{(r^2 + R^2 - 2R^2 \cos \theta)^{1/2}} \]

Factoring out \( d^2 \) (first 2), \( r^2 \) (inter 2)

\[ \Phi (r, \theta) = \frac{Q}{4 \pi \varepsilon_0} \frac{1}{d} \frac{1}{(r^2 + \frac{R^2}{d^2} + 1 + \frac{2r}{d} \cos \theta)^{1/2}} \]

\[ - \frac{Q}{4 \pi \varepsilon_0} \frac{1}{d} \frac{1}{(r^2 + \frac{R^2}{d^2} + 1 - \frac{2r}{d} \cos \theta)^{1/2}} \]

\[ - \frac{Q}{4 \pi \varepsilon_0} \frac{r(R)}{r} \frac{1}{(1 + \frac{R^2}{d^2r^2} + 2 \frac{R^2}{d^2r} \cos \theta)^{1/2}} \]

\[ + \frac{Q}{4 \pi \varepsilon_0} \frac{R/R}{r} \frac{1}{(1 + \frac{R^2}{d^2r^2} - 2 \frac{R^2}{d^2r} \cos \theta)^{1/2}} \]
EXPAND SQUARE ROOTS.

\[ \Phi(r, \theta) \approx \frac{Q}{4\pi \varepsilon_0} \left\{ 1 - \frac{1}{2} \left( \frac{r^2}{d^2} + \frac{2r}{d} \cos \theta \right) \right\} \]

\[ - \frac{Q}{4\pi \varepsilon_0} \frac{1}{d} \left\{ 1 - \frac{1}{4} \left( \frac{r^2}{d^2} - \frac{2r}{d} \cos \theta \right) \right\} \]

\[ - \frac{Q}{4\pi \varepsilon_0} \frac{R^2}{r} \left\{ 1 - \frac{1}{2} \left( \frac{R^4}{d^4 r^4} + 2 \frac{R^2}{d^2} \cos \theta \right) \right\} \]

\[ + \frac{Q}{4\pi \varepsilon_0} \frac{R^2}{r} \left\{ 1 - \frac{1}{2} \left( \frac{R^4}{d^4 r^4} - 2 \frac{R^2}{d^2} \cos \theta \right) \right\} \]

TAKE LIMITS \(d \to 0\) & \(r \to R r^2 / d\),

\[ \Phi(r, \theta) \approx \frac{2Q}{4\pi \varepsilon_0} \frac{1}{d^2} \cos \theta \] "CONSTANT TERM"

\[ - \frac{2Q}{4\pi \varepsilon_0} \frac{1}{d^2} \frac{R^3}{r^2} \cos \theta \] "DIPOLE TERM"

RECALL THE POTENTIAL FROM A POINT DIPOLE IS

\[ \Phi_{\text{Dipole}} (r) = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \] so \( \Phi(r, \theta) \) contains a point dipole potential with

\[ 10^{-3} \frac{1}{r^2} = -2\pi \frac{Q}{d^2} \frac{R^3}{d^2} \]
The other team is that of a uniform electric field with
\[ |E_0| = \frac{2\Phi}{\pi \varepsilon_0 d^2}. \]

So \( \Phi(r, \theta) = E_0 r \cos \theta = E_0 \frac{R^3}{r^2} \cos \theta \)

Of course, inside \( \Phi(r < R) = 0 \).

What's the surface charge?
Again \( \sigma = -\varepsilon_0 \frac{d\Phi}{dr} |_{r=R} \)

\[ = 3\varepsilon_0 \frac{2\Phi}{4\pi \varepsilon_0 d^2} \cos \theta \]

It's interesting to look in more detail at the interior field.
The interior field is zero.
Q: Why?
This interior field is a superposition of the uniform external field $\vec{E}_0$ plus the field from the surface charge.

Since $\vec{E}_0$ is uniform, the field from the surface charge is likewise uniform. It follows that a "\( \cos \theta \)" distribution of surface charge produces a uniform $\vec{E}$ field.
EXAMPLE CAPACITANCE OF A CONDUCTING SPHERE AND A GROUND PLANE.

RECALL THE CAPACITANCE OF A SPHERE (AND THE "GROWN-0EO" SURFACE AT 0) IS \( \frac{q}{2\pi \varepsilon_0 R} \). WHAT'S THE EFFECT OF THE GROUND PLANE?

WE'LL USE METHOD OF IMAGES,

\[ \begin{array}{c}
\frac{q}{2} \\
(\frac{q}{2}, \frac{q}{2}) \\
\vdots
\end{array} \]

\(-\frac{q}{2}, -\frac{q}{2}, -\frac{q}{2}, \ldots\)

\( z_1 \) makes the sphere an equipotential (but not plane)
\(-z_1\) makes plane equipotential (but not sphere)
\( z_2 \) makes sphere equipotential (but not plane)
\(-z_2\) makes plane equipotential (but not sphere)

\( \ddots \)

AD INFINITUM
Now, \(-\varepsilon_1\) has the same magnitude as \(\varepsilon_1\). \(\varepsilon_2 = +\varepsilon_1 \frac{R}{2d}\).

The total charge on the sphere is 
\[\varepsilon_1 + \varepsilon_2 + \cdots = \varepsilon_1 (1 + \frac{R}{2d} + \cdots)\]

Now to find the capacitance. This is the charge on a conductor divided by the potential difference. But notice its only charge \(\varepsilon_1\) that establishes the potential of the sphere. All the other charges only serve to make the potential of the sphere zero, (plus the potential due to \(\varepsilon_1\)).

Hence \[C = \frac{Q}{V} = \frac{\varepsilon_1 (1 + \frac{R}{2d} + \cdots)}{\varepsilon_1/4\pi\varepsilon_0 R} = 4\pi\varepsilon_0 R (1 + \frac{R}{2d} + \cdots)\]

Moving the ground plane near the sphere increases the capacitance (as expected).
GREEN'S FUNCTION FOR THE SPHERE.
RECALL THE INITIAL SPHERICAL IMAGE-CHARGE PROBLEM!

\[ \Phi (\vec{r}) = \frac{1}{4\pi \varepsilon_0} \frac{\varepsilon}{|\vec{r} - \vec{r}'|} + \frac{1}{4\pi \varepsilon_0} \frac{\varepsilon'}{|\vec{r}' - \vec{r}|} \]

WITH \( \varepsilon' = -\varepsilon \frac{R}{r} \)
AND \( r' = \frac{R^2}{r} \)

This \( \Phi \) is very close to \( G \).
We want \( \nabla^2 G = -4\pi \varepsilon \delta (\vec{r} - \vec{r}') \), so
\[ G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} + \cdots \]

For unit charge \( \varepsilon \rightarrow 4\pi \varepsilon_0 \),
\[ G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{R}{r} \frac{1}{|\vec{r}' - \vec{r}|} \]

This is the spherical Green's function.
We'll turn this into spherical coordinates \((r, \theta, \phi)\):

\[
G(r, \theta, \phi) = \frac{1}{(r^2 + \theta^2 - 2r \theta \cos \gamma)^{1/2}}
\]

\[
= \frac{1}{(R^2 + (\frac{r}{R} \theta)^2 - 2r \theta \cos \gamma)^{1/2}}
\]

with \(\gamma\) the angle between \(\vec{r}\) and \(\vec{\theta}\).

For Dirichlet boundary conditions, we'll need

\[
\Phi(\theta) = \frac{1}{4\pi \varepsilon_0} \oint_{S} \Phi(r) G(\theta, \theta) \, dr - \frac{1}{4\pi} \oint_{S} \partial_{\theta} G(\theta, \theta) \frac{\partial \Phi(\theta)}{\partial \theta} \, dA.
\]

So, we'll need \(\frac{\partial}{\partial \theta} G\) at \(r = R\):

\[
\left. \frac{\partial G}{\partial \theta} \right|_{r=R} = -\frac{\theta^2 - R^2}{R^2 + \theta^2 - 2R \theta \cos \gamma}^{3/2}
\]

We could have guessed this, it's the surface-charge for the image-charge problem for unit charge.
Example: Two conducting hemispheres at different potentials. (N.B., we'll do this later via separation of variables and Legendre polynomials.)

\[ \Phi = \Phi_0 \]

\[ \Phi = -\Phi_0 \]

No bulk charge, only boundary terms.

\[ \Phi (r, \theta, \phi) = \frac{\Phi_0}{4\pi} \left[ \begin{array}{c} 2\pi \int \frac{r^2 - R^2}{R \left[ R^2 + r^2 - 2Rr \cos \theta \right]^{3/2}} \, dr \\ + \int_0^1 \int_{\theta_1}^{\theta_2} \frac{r^2 - R^2}{R \left[ R^2 + r^2 - 2Rr \cos \theta \right]^{3/2}} \, d\cos \theta \, d\phi \\ - \int_0^1 \int_{\theta_1}^{\theta_2} \frac{r^2 - R^2}{R \left[ R^2 + r^2 - 2Rr \cos \theta \right]^{3/2}} \, d\cos \theta \, d\phi \end{array} \right] \]

This is a mess. But there's a couple tricks.
Combine the two integrals by a substitution in \( \int d \cos \theta' \) via 
\[ \theta' \rightarrow \pi - \theta', \] hence

\[
\Phi(r, \theta, \phi) = \frac{\Phi_0}{4\pi} 2\pi R^2 
\times \int_0^1 d \cos \theta' \left\{ \frac{r^2 - R^2}{R (r^2 + R^2 - 2rR \cos \theta')^{3/2}} - \frac{r^2 - R^2}{R (r^2 + R^2 + 2rR \cos \theta')^{3/2}} \right\}.
\]

It's still a mess.

But, there's a special case for the value of \( \Phi \) on the +z-axis:

\[
\Phi(r; \theta = 0; \phi) = \Phi_0 \left[ 1 - \frac{z^2 - R^2}{z \sqrt{z^2 + R^2}} \right]^{2/3}.
\]

Notice \( \Phi \rightarrow \Phi_0 \) at \( z \rightarrow R \), as expected.

At large distance \( z \gg R \), we see a dipole (not a pure dipole)

\[ \Phi \rightarrow \Phi_0 \frac{R^2}{z^2}. \]
Notice I skipped the Green's function for a cylinder. This is because there's no simple image-charge solution, but there is a complicated image-charge solution. (See, e.g., Smythe §§ 298).

We'll shortly move into separation-of-variables, this will consider the point charge near a conducting sphere, and from that the corresponding Green's function.