**Physics 513, Electrodynamics I**
**Department of Physics, University of Washington**
**Autumn quarter 2020**
**October 13, 2020, 11am**
**On-line lecture**

***Administrative:***

**1. Homework 2 posted on**
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
**2. Homework 1 solutions posted on**
`faculty.washington.edu/ljrberg/AUT20_PHYS513`

***Lecture: Methods of finding potentials in boundary-value problems. (Jackson chapter 2).***

**Energy relations in the electrostatic field II: self-energy.**
**2D geometry via conformal mapping (not in Jackson).**
**Method of images: planar & spherical systems.**

# ENERGY, IN VARIOUS FORMS II

ELECTROSTATICS IS BASED ENTIRELY ON A SINGLE PHYSICAL LAW: COULOMB'S LAW. THIS LAW DESCRIBES ACTION-AT-A-DISTANCE FORCE BETWEEN CHARGES. WE INTRODUCED THE ELECTRIC FIELD $\vec{E}$ AS AN INTERMEDIARY AGENT TO SIMPLIFY THE DESCRIPTION OF THE INTERACTION BETWEEN CHARGES.

JUST AS FOR MECHANICS, IT HAPPENS THAT ALL MECHANICAL PROPERTIES OF AN ELECTRICALLY-INTERACTING SYSTEM CAN BE DESCRIBED EITHER IN TERMS OF THE SOURCES ($\rho$) OR THE FIELDS DUE TO THE SOURCES ($\vec{E}$); THIS LATER IS "FIELD THEORY".

FOR DEEP AND FUNDAMENTAL REASONS, THESE TWO APPROACHES PRODUCE (NEARLY) EQUIVALENT RESULTS.

IN THE LAST LECTURE, WE FOUND THE MECHANICAL WORK REQUIRED TO ASSEMBLE CHARGES $\{q_i\}$ WHERE THERE ARE POINT CHARGES AT $\infty$:

$$W = \frac{1}{2}\sum_i q_i \Phi_i' \Rightarrow \frac{1}{2}\iiint \rho \Phi \, dv$$

$\Phi'$ IS THE POTENTIAL AT THE LOCATION OF $q_i$ DUE TO ALL THE OTHER CHARGES, NOT INCLUDING $q_i$.

IN ASSEMBLING THESE CHARGES, WE CHANGED THE ENERGY OF THE SYSTEM. SINCE ELECTROSTATIC FORCES ARE CONSERVATIVE, THE ASSEMBLY WORK $W$ IS THE SYSTEM ENERGY. IT STANDS TO REASON THIS ENERGY MUST BE STORED SOMEWHERE.

BUT, JUST AS FOR MECHANICS, THE PLACE OF STORAGE DEPENDS ON YOUR POINT OF VIEW. (RECALL LAST LECTURE'S EXAMPLE OF MASSES AND SPRINGS.)

FOR DEEP REASONS, THEREFORE, THERE MUST BE AN EXPRESSION THAT MAKES IT APPEAR AS IF THE SYSTEM ENERGY IS IN THE "ELASTIC" PROPERTY OF THE FIELD (THE MECHANICS EQUIVALENT IS SAYING THE ENERGY OF THE MASS-SPRING SYSTEM IS IN THE SPRING).

MAXWELL (AND WE, IN THE LAST LECTURE) FOUND THIS FIELD ENERGY

$$U = \frac{\varepsilon_0}{2} \iiint E^2 \, dv.$$

THE PROOF OF THIS IS SUBTLE. AT A FIELD POINT, DUE TO AN ASSEMBLAGE OF POINT CHARGES, THE FIELD IS

$$\vec{E} = \sum_i \vec{E_i}. \quad \text{HENCE}$$

$$E^2 = \sum_i E_i^2 + \sum_i \sum_{j \neq i} \vec{E_i} \cdot \vec{E_j}$$

MAXWELL NOTICED THE TERM $\sum_i E_i^2$ DIVERGES AT THE CHARGES, BUT THE VOLUME INTEGRAL OF THIS TERM IS INDEPENDENT OF THE RELATIVE POSITIONS OF THE CHARGES. THEREFORE

$$U_s = \frac{\varepsilon_0}{2} \iiint \sum_i E_i^2 \, dv$$

REPRESENTS THE WORK TO ASSEMBLE THE SET OF POINT CHARGES. MAXWELL CALLED $U_s$ THE "SELF ENERGY" OF THE SYSTEM.

YOU COULD, e.g., TRY TO "CONTROL" OR REGULATE THIS BY INTRODUCING A FINITE POINT-CHARGE RADIUS (MORE ON THIS LATER... THIS APPROACH HAS FUNDAMENTAL DIFFICULTIES).

HENCE

$$U = U_s + \frac{\varepsilon_0}{2} \iiint \sum_i \sum_{j \neq i} \vec{E}_i \cdot \vec{E}_j \, dv$$

$$= U_s + \frac{\varepsilon_0}{2} \sum_i \iiint \vec{E}_i \cdot \left( -\nabla \sum_{j \neq i} \Phi_j \right) dv$$

NOW INTEGRATE BY PARTS VIA THE IDENTITY

$$\vec{\nabla}\cdot(\vec{A}\Phi) = \Phi\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}\Phi,$$

$$U = U_s - \frac{\varepsilon_0}{2}\sum_i \iiint \vec{\nabla}\cdot\left\{\vec{E}_i \sum_{j\neq i}\Phi_j\right\} dv$$

$$+ \frac{\varepsilon_0}{2}\sum_i \iiint \sum_{j\neq i}\Phi_j \, \vec{\nabla}\cdot\vec{E}_i \, dv$$

NOTICE $\vec{\nabla}\cdot\vec{E}_i$ VANISHES EXCEPT AT THE POSITION OF $q_i$

$$\vec{\nabla}\cdot\vec{E}_i = \frac{q_i}{\varepsilon_0}\,\delta(\vec{r}_i), \quad \text{SO}$$

$$U = U_s - \frac{\varepsilon_0}{2}\sum_i \oiint \vec{E}_i \sum_{j\neq i}\Phi_j \cdot \hat{n}\, dA$$

$$+ \frac{1}{2}\sum_i \Phi_i' \, q_i \quad \text{(NOTICE THE "$'$" ON $\Phi_i'$)}$$

WITH SPATIALLY-BOUNDED CHARGES $\{q_i\}$, THE SURFACE TERM VANISHES, (Q: WHY), LEAVING

$$U = U_s + \frac{1}{2}\sum_i \Phi_i' \, q_i$$

THIS IS THE MECHANICAL WORK PLUS THE WORK TO ASSEMBLE POINT CHARGES:
"FREE-" VS. "TOTAL-" ENERGY.

IF THERE WERE DIELECTRICS IN THE SYSTEM, WE'D NEED GREATER CARE. BUT, FORTUNATELY, FOR ELECTROSTATICS WITHOUT DIELECTRICS, $\vec{E} \to 0$ INSIDE THE CONDUCTORS, AND THERMODYNAMICALLY, THIS IS A GREAT SIMPLIFICATION.

# BOUNDARY-VALUE PROBLEMS IN ELECTROSTATICS I.

IN SOME VOLUME, $\rho$ IS SPECIFIED. ON SURFACES BOUNDING THE VOLUME (INCLUDING, PERHAPS, THE SURFACE AT $\infty$), THERE ARE DIRICHLET OR NEUMANN BOUNDARY CONDITIONS.

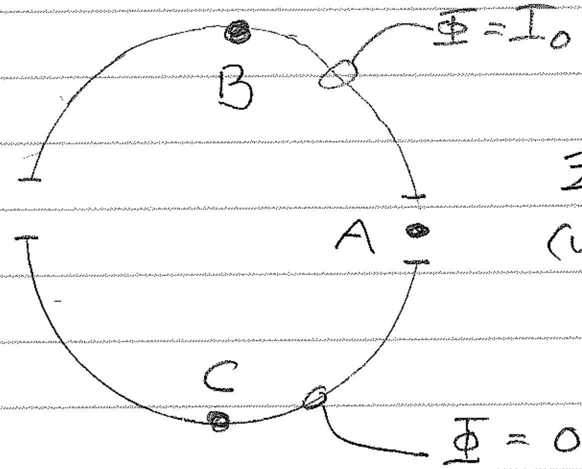
WE DESIRE A SOLUTION TO

$$\nabla^2 \Phi = -\rho/\varepsilon_0 \quad \left\{ \text{OR } \nabla^2 \Phi = 0 \right\}.$$

POISSON'S EQUATION IS FOUND EVERYWHERE IN S.T.E.M. FIELDS, SO THERE HAVE DEVELOPED MANY WAYS OF FINDING $\Phi$. e.g., GREEN'S FUNCTION (FROM LAST LECTURE), OR "INVERSION" (NOT COVERED IN JACKSON), OR "SCHWARTZ TRANSFORMATION" (IN 2D, NOT COVERED IN JACKSON.

EXAMPLE OF CONFORMAL MAPPING ("SCHWARTZ TRANSFORMATION"): "SPLIT" CYLINDER.

THIS IS A TOPIC NOT COVERED IN JACKSON AND YOU'RE NOT EXPECTED TO KNOW IT, BUT IT'S INTERESTING.

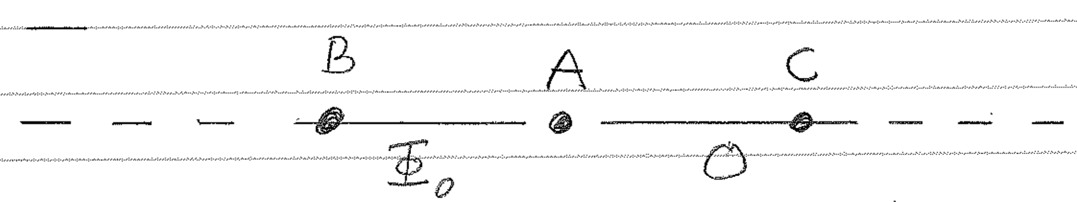



$z = x + iy$ (UNPRIMED)

APPLY A CONFORMAL TRANSFORMATION TO THE $z$-PLANE:

$$z \to z' = \frac{1}{i}\,\frac{z-1}{z+1} = x' + i y'$$

THE CYLINDER TRANSFORMS INTO



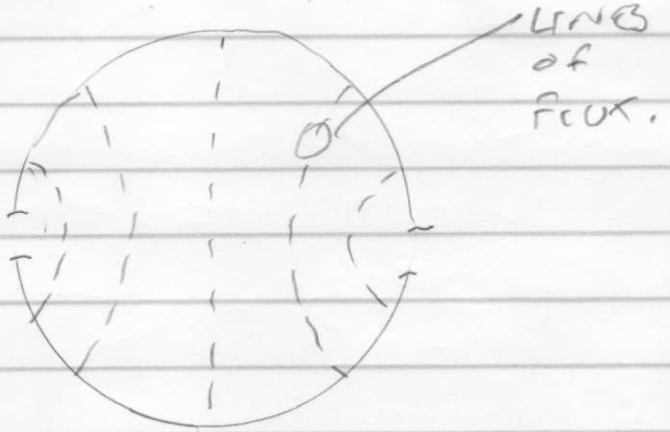
IN THE UPPER-HALF PLANE, THIS HAS SOLUTION

$$\Phi'(x', y') = \frac{\Phi_0}{\pi} \tan^{-1} \frac{y'}{x'}$$

- NOTICE IT "WORKS" FOR $y' = 0$; BY UNIQUENESS IT IS THE SOLUTION.

TRANSFORMING BACK

$$\Phi(x, y) = \Phi_0 \tan^{-1} \frac{1 - x^2 + y^2}{2y}$$

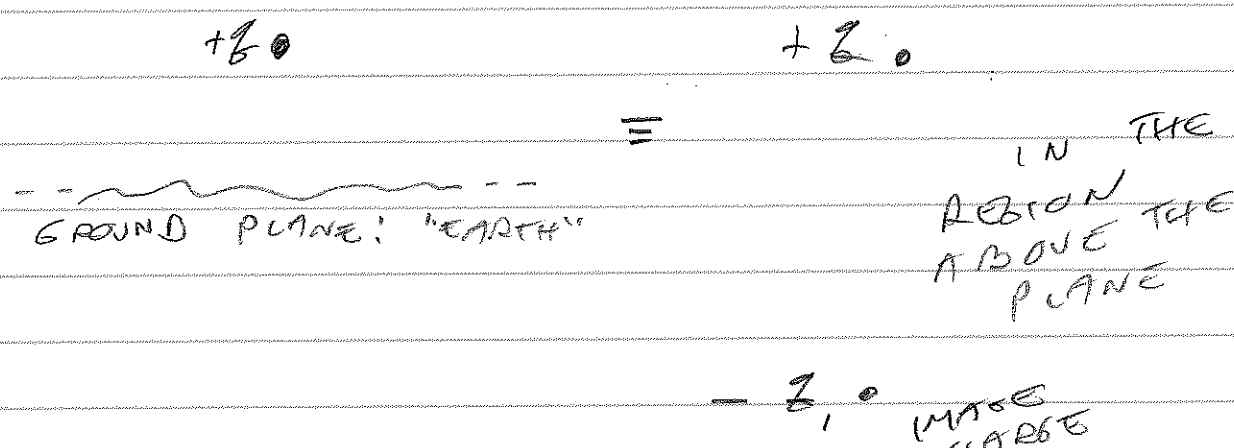


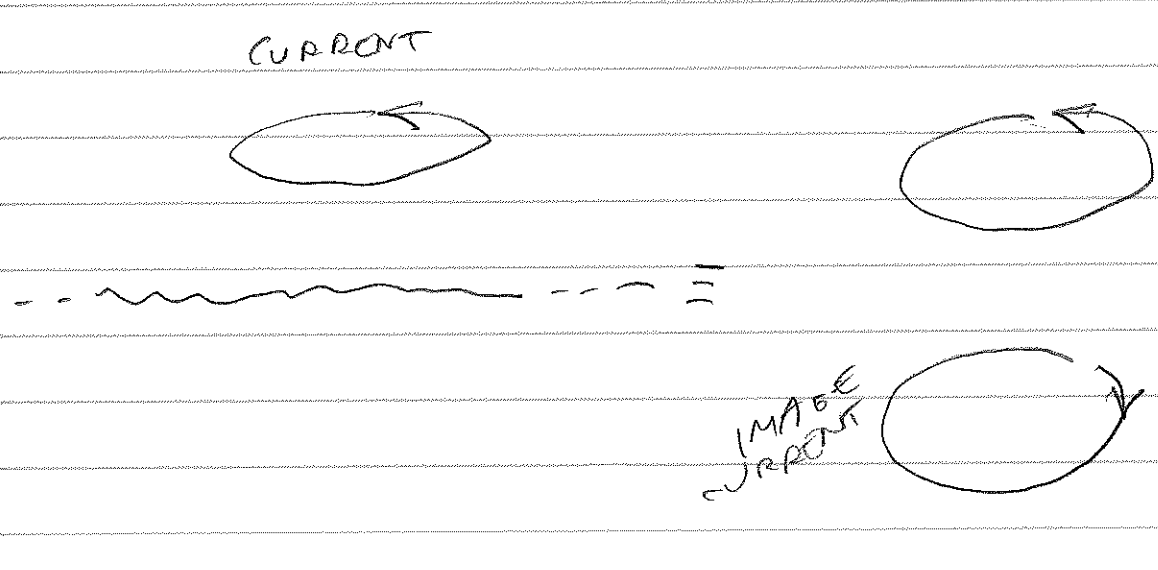

THERE ARE MANY OTHER EXAMPLES OF 2D SOLUTIONS VIA CONFORMAL MAPPING.

METHOD OF IMAGES — MOST DEFINITELY COVERED IN JACKSON.

THE USUAL FORMULATION IS ONE OR MORE POINT CHARGES (OR CHARGED CONDUCTING SPHERES; SEE SMYTHE) PLUS BOUNDARY SURFACES. THIS METHOD IS ALSO USEFUL FOR, e.g., IMAGE DIPOLES AND IMAGE CURRENTS.

e.g.,

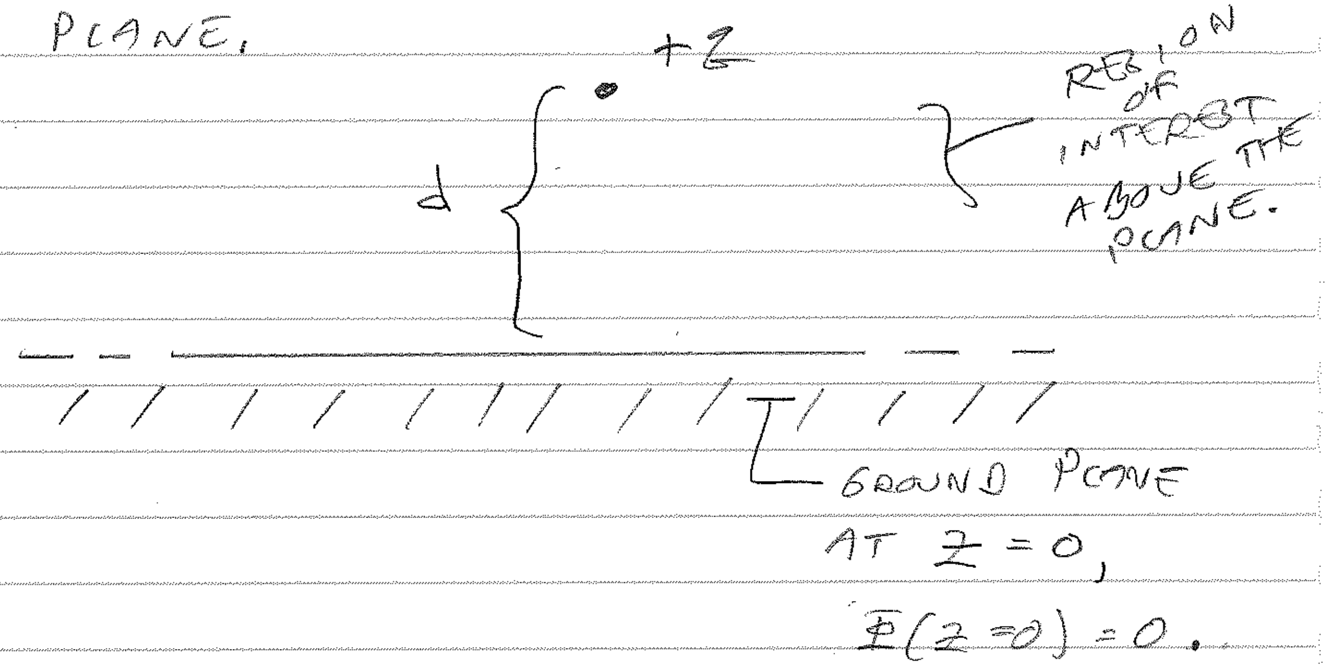


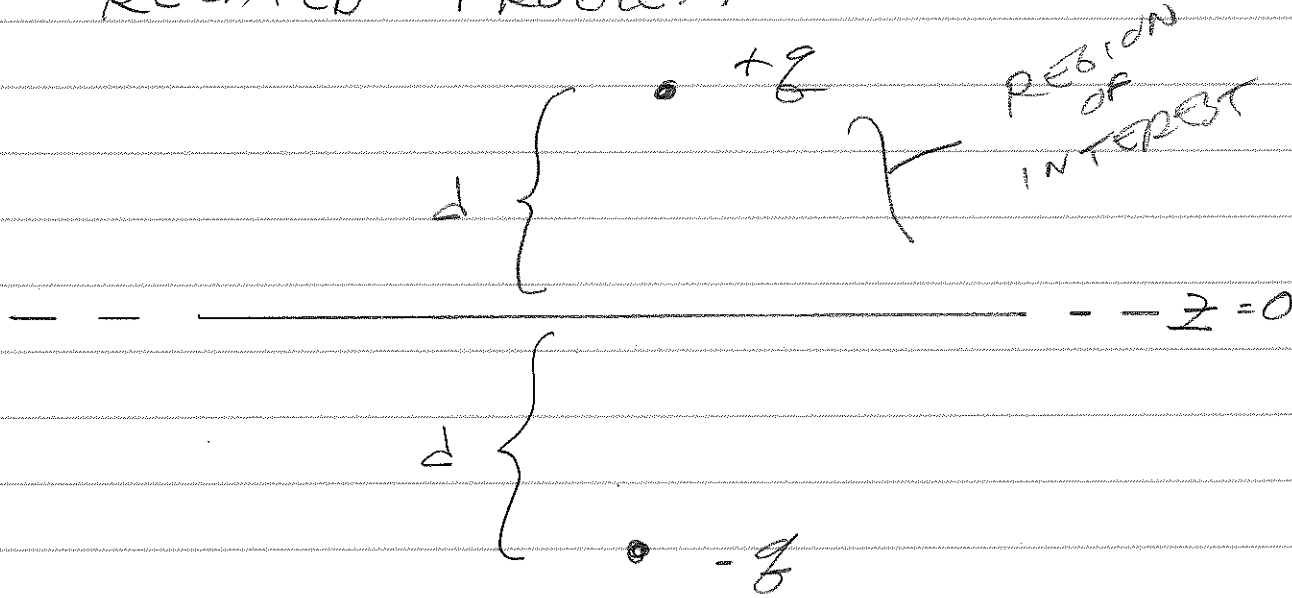
FOR POINT SOURCES WITHIN THE VOLUME OF INTEREST, IT MAY BE POSSIBLE TO INFER THAT A COUNTABLE (MAYBE INFINITE) NUMBER OF IMAGE CHARGES OUTSIDE THE REGION OF INTEREST PROVIDES THE CORRECT BOUNDARY CONDITIONS.

THIS METHOD IS BASED ON THE UNIQUENESS OF SOLUTIONS TO $\nabla^2 \Phi = \rho/\varepsilon_0$.

CLASSIC (SIMPLE) IMAGE-CHARGE PROBLEM: POINT CHARGE ABOVE GROUND PLANE.




THE IMAGE-CHARGE SOLUTION IS A SOLUTION TO A DIFFERENT, BUT RELATED PROBLEM




NOTICE, BY SYMMETRY $\Phi(z=0)=0$ AND BOUNDARY CONDITIONS ARE SATISFIED. WHAT ABOUT AT $\infty$?

IN THE REGION OF INTEREST, THE SUPERPOSITION OF $+q$ AND $-q$ IS THE SOLUTION TO THE ORIGINAL PROBLEM.

Q: IS THIS A SOLUTION TO THE POTENTIAL IN THE LOWER 1/2 PLANE? WHY NOT? GIVE AN EXAMPLE OF WHY NOT.

Q: CAN THERE BE IMAGE CHARGES IN THE REGION OF INTEREST?

IN CYLINDRICAL COORDINATES $(\rho, \phi, z)$, THE POTENTIAL IS

$$\Phi(\rho, z) = \frac{1}{4\pi\varepsilon_0} \frac{+q}{\sqrt{\rho^2 + (z-d)^2}}$$

$$+ \frac{1}{4\pi\varepsilon_0} \frac{-q}{\sqrt{\rho^2 + (z+d)^2}}$$

Q: HOW WOULD YOU USE THIS RESULT TO FIND THE GREENS FUNCTION FOR A PLANE?

FROM GAUSS'S LAW, THE SURFACE CHARGE ON THE PLANE IS

$$\sigma = -\varepsilon_0 \frac{\partial \Phi}{\partial z}\Big|_{z=0} = \frac{1}{2} \frac{-q d}{(\rho^2 + d^2)^{3/2}}$$

Q: SHOW THIS.

Q: WHAT IS THE FORCE ON $+q$.
A: FIELD THEORY SAYS THE FORCE IS THE FIELD $\vec{E}$ NEAR $+q$ TIMES $+q$! IT DOESN'T MATTER WHAT MADE THE $\vec{E}$: BOTH THE ORIGINAL PROBLEM AND THE IMAGE PROBLEM HAVE THE SAME $\vec{E}$ AND HENCE THE SAME FORCE ON $+q$.

Q: HOW MUCH MECHANICAL WORK IS REQUIRED TO ASSEMBLE THE SYSTEM?

A: FOR THE (TWO-CHARGE) IMAGE PROBLEM $W = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}$, IT'S EASY.

A: FOR THE ORIGINAL (GROUND PLANE) PROBLEM, EVALUATE

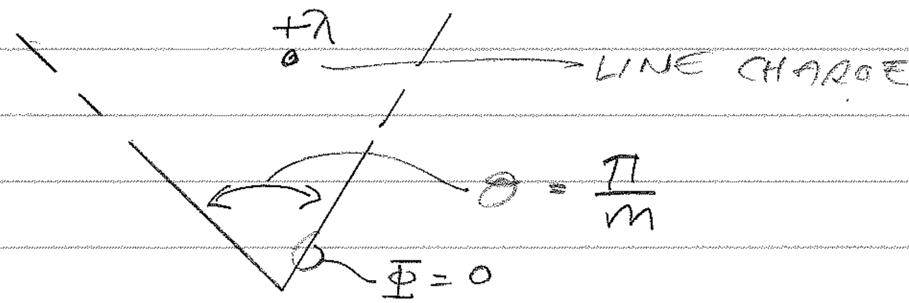
$$W = \int_{\infty}^{d} \vec{F}\cdot d\vec{l} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d},$$

THIS IS 1/2 THAT OF THE 2-CHARGE IMAGE SYSTEM.

Q: WHY IS THIS SO?

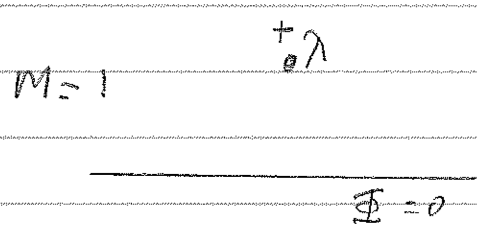
SOME PROBLEMS HAVE A LARGE NUMBER OF IMAGE CHARGES.

e.g., A HARD PROBLEM (SEE SMYTHE §4.06).



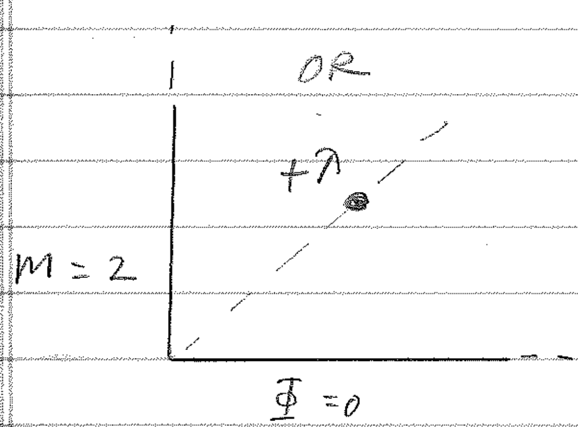

IN GENERAL THIS HAS AN INCREASING NUMBER OF IMAGE LINE CHARGES AS m GROWS

FOR EXAMPLE



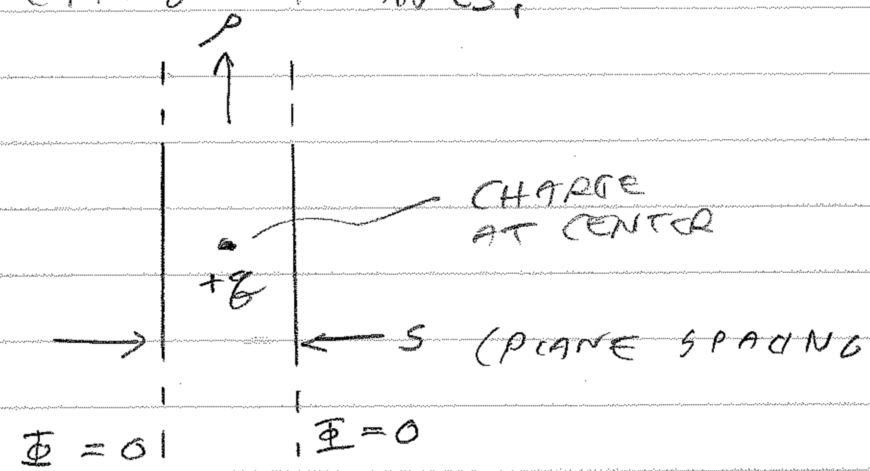

Q! WHERE'S THE IMAGE CHARGE?

OR

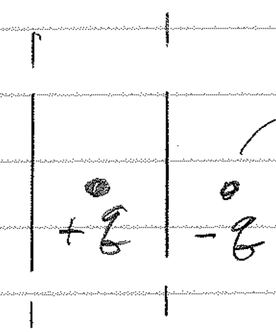



Q! WHERE'S THE IMAGE CHARGES?

ANOTHER IMAGE-CHARGE EXAMPLE:
POINT CHARGE BETWEEN TWO GROUNDED CONDUCTING PLANES.



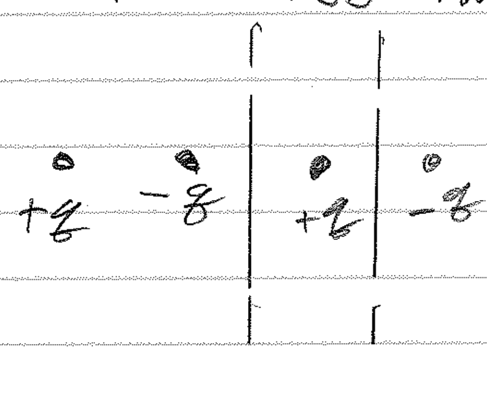

STEP 1. ADD AN IMAGE CHARGE.




Q: WHERE IS THIS IMAGE CHARGED PLACED?

BUT NOW THE LEFT SURFACE HAS NON-ZERO POTENTIAL

STEP 2. ADD TWO MORE IMAGE CHARGES




Q: WHERE ARE THE TWO NEW CHARGES PLACED.

NOW, THE LEFT SURFACE IS AT ZERO POTENTIAL, BUT NOT THE RIGHT. SO, KEEP ADDING PAIRS OF CHARGED AD INFINITEM.

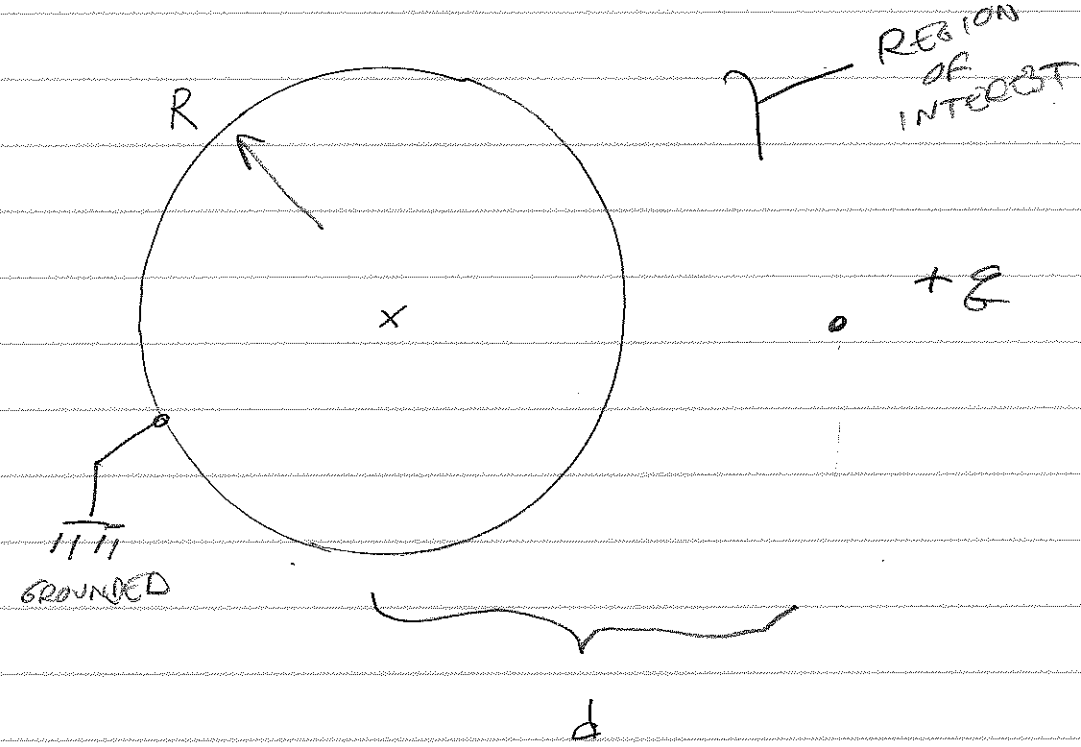
IT HAPPENS THIS SERIES IS SUMMABLE (SEE SMYTHE §4.)

$$\Phi(\rho, z) \sim e^{-\rho/s/2} \sin \pi z/s$$

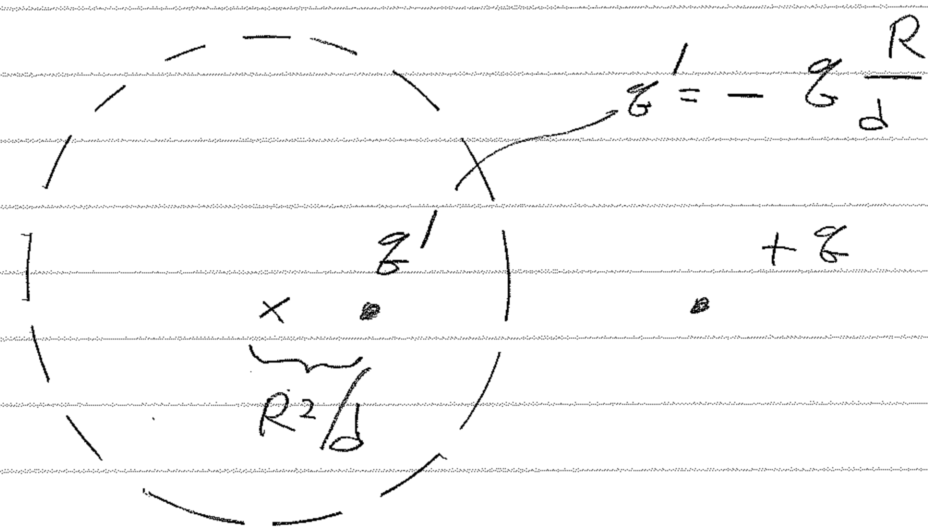
Q: SHOW THIS SATISFIES THE BOUNDARY CONDITIONS

COMMENT, THIS PROBLEM IS ALSO DONE VIA "SEPARATION OF VARIABLES" (LATER).

ANOTHER IMAGE-CHARGE PROBLEM: POINT CHARGE INSIDE (OR OUTSIDE) A GROUNDED SPHERE.




THE IMAGE CHARGE SYSTEM IS




Q: SHOW THAT $\Phi(r=R)=0$.
(SEE JACKSON EQN. 2.3 AND P. 59)

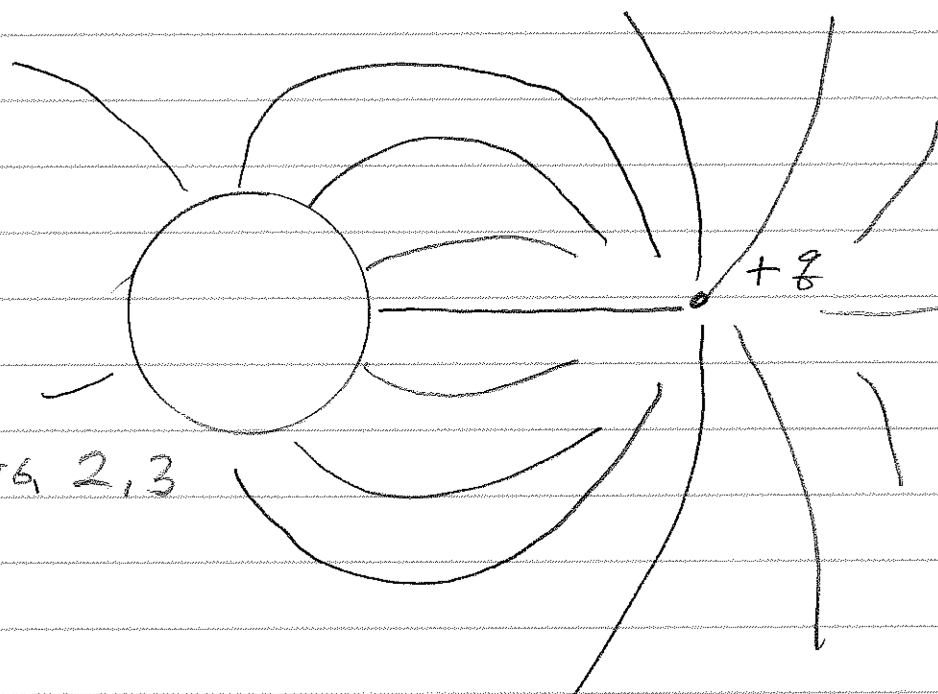
THE FORCE ON $+q$ CAN COME FROM COULOMB'S LAW (WITH 2 CHARGES).

THE SURFACE CHARGE IS

$$\sigma = -\varepsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=R}$$

= A MESS (SEE JACKSON EQN. 2.5)

THE LINES OF $\vec{E}$ LOOK LIKE




SEE, e.g.,
JACKSON FIG. 2.3

Q: IS THE TOTAL CHARGE ON THE SPHERE POSITIVE, NEGATIVE OR ZERO?

Q: FAR FROM THE SYSTEM, IS THERE A SIMPLE DESCRIPTION OF THE FIELD?