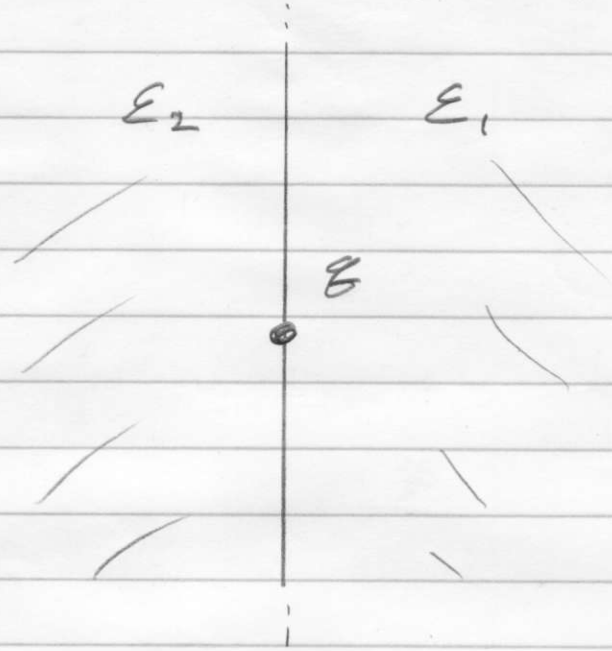
Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
November 10, 2020, 11am PST
On-line lecture

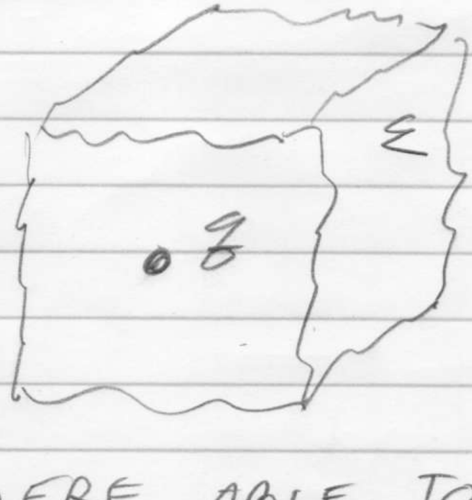
Administrative:
1. Homework 6 posted on
`faculty.washington.edu/ljrberg/AUT20_PHYS513`

Lecture: Multipoles, dielectrics. (Jackson chapter 4).
Section 4.4: Boundary-value problems involving dielectrics.

EXAMPLE: POINT CHARGE IN THE INTERFACE BETWEEN TWO (LINEAR) DIELECTRIC MEDIA ε_1 AND ε_2.

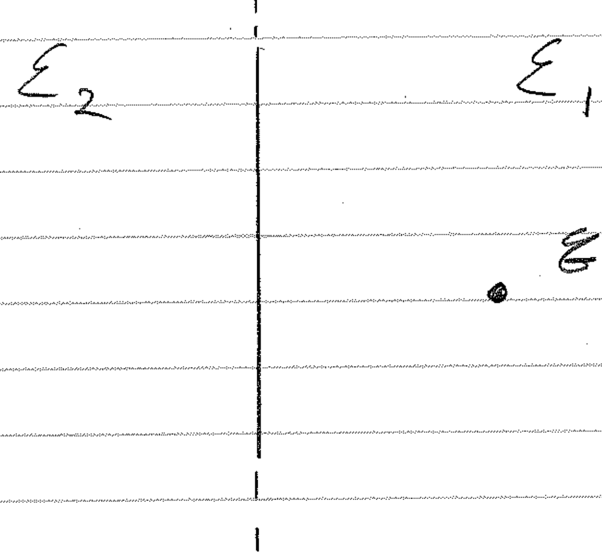


BACKGROUND. WE STUDIED A CHARGE EMBEDDED IN AN INFINITE DIELECTRIC.



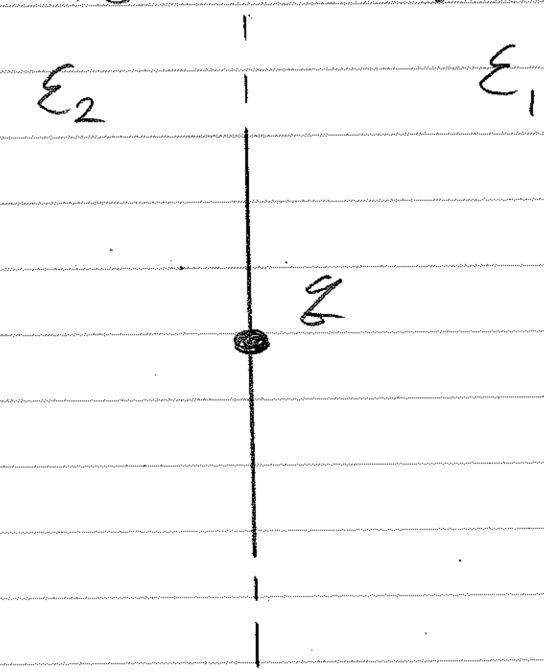
WE WERE ABLE TO DIRECTLY APPLY GAUSS'S LAW $\vec{\nabla}\cdot\vec{D} = \rho$, TO FIND $\vec{D}(\vec{r}) = \frac{1}{4\pi}\frac{q}{r^2}\hat{r}$ AND THEREFORE $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{r}$

WE ALSO STUDIED TO A MINOR EXTENT A CHARGE EMBEDDED IN A TWO-DIELECTRIC SYSTEM;



THE LACK OF SPHERICAL-SYMMETRY MEANS THAT, WHILE $\oint\!\!\oint \vec{D}\cdot\hat{n}\,dA$ HOLDS, IT CAN'T EASILY PROVIDE $\vec{D}$.

WHAT ABOUT THIS?



NOTICE THAT THERE'S NO LENGTH SCALE: IF FIELD LINES CURVE, THE RADIUS OF CURVATURE WOULD INTRODUCE A LENGTH SCALE. HENCE FIELD LINES ARE RADIAL:

$$\vec{D} = \frac{1}{4\pi} \frac{q}{r^2} \hat{r}$$

SO $\vec{E} = \frac{1}{4\pi\varepsilon_1} \frac{q}{r^2} \hat{r}$ (ε_1 SIDE)

OR $\vec{E} = \frac{1}{4\pi\varepsilon_2} \frac{q}{r^2} \hat{r}$ (ε_2 SIDE).

WE'LL LOOK AT THIS IN MORE DETAIL.

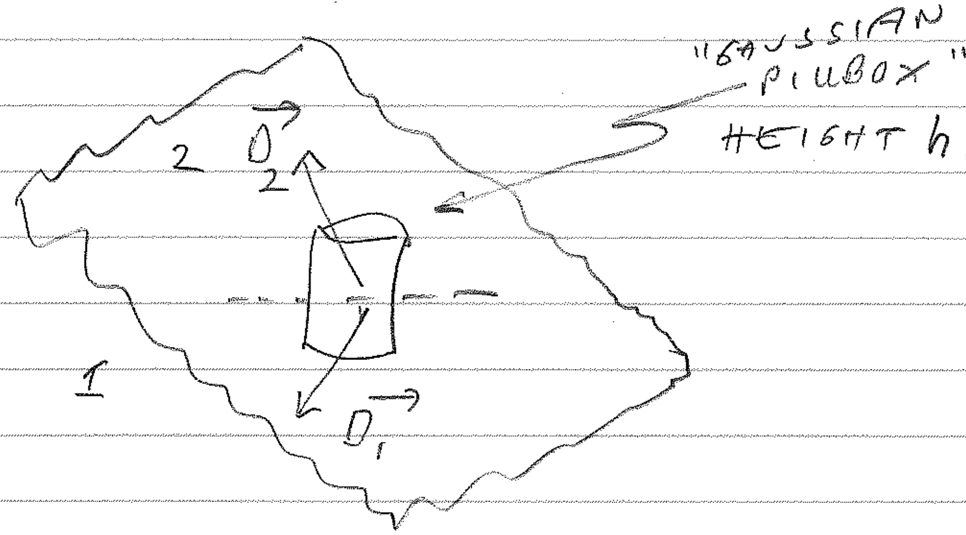
BOUNDARY-VALUE PROBLEMS WITH DIELECTRICS. JACKSON §4.4.

RECALL OUR INITIAL, EARLIER, DISCUSSION OF BOUNDARY-VALUE PROBLEMS. WHERE THE SYSTEM HAD CHARGE, WE DIVIDED THE SYSTEM CHARGE INTO CHARGE DENSITIES THAT VARY CONTINUOUSLY, AND DISCONTINUOUS CHARGE DENSITIES. WE DIDN'T WORRY ABOUT THE MEDIA: IT WAS EITHER CONDUCTOR OR VACUUM.

FOR THIS SECTION ON BOUNDARY-VALUE PROBLEMS WITH DIELECTRICS, WE LIKEWISE DIVIDE THE SYSTEM DIELECTRIC PROPERTIES INTO DIELECTRICS THAT VARY THE DIELECTRIC CONSTANT CONTINUOUSLY, AND DISCONTINUOUS VARIATIONS. BOUNDARY-VALUE PROBLEMS IN THIS SECTION MOSTLY SPEAK TO ABRUPT VARIATIONS. AT THE BOUNDARY BETWEEN TWO DIELECTRICS

(SEE JACKSON EQN. 4.40.)

BOUNDARY CONDITIONS FOR THE DIRECTION "NORMAL" TO THE SURFACE



$\vec{\nabla} \cdot \vec{D} = \rho_f$

$\oint \vec{D} \cdot \hat{n}\, dA = q_f$ WITH $q_f = \sigma_f A$

$\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = \sigma_f$

Q: DO I HAVE TO WORRY ABOUT $\vec{\nabla} \times \vec{D} \neq 0$?

A: NO. $\oint \vec{D} \cdot \hat{n}\, dA$ IS TRUE, IT'S JUST THAT IT DOESN'T UNIQUELY SPECIFY $\vec{D}$. BUT, GIVEN A $\vec{D}$, IT'S TRUE.

RECALL THE LINEAR DIELECTRIC $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$,

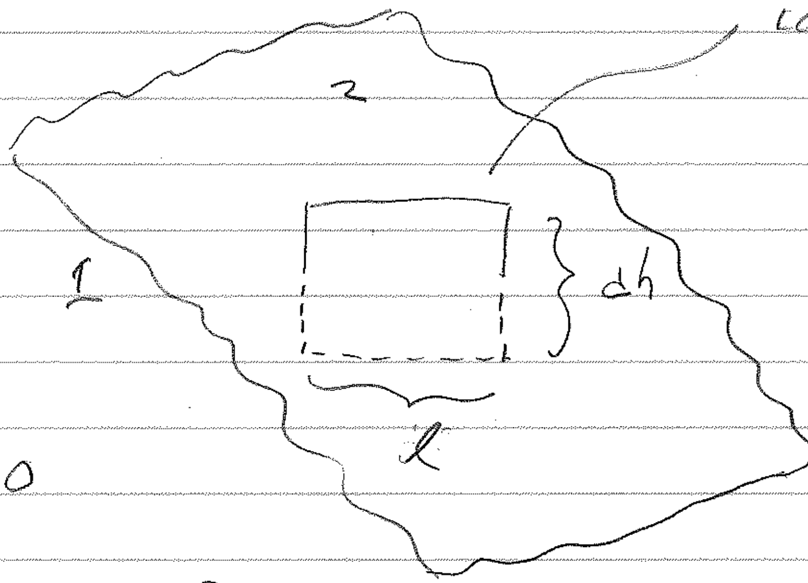
SO $\hat{n} \cdot (\varepsilon_{r_2} \vec{E}_2 - \varepsilon_{r_1} \vec{E}_1) = \sigma_f / \varepsilon_0$,

OR $\hat{n} \cdot (\varepsilon_{r_2} \vec{\nabla} \Phi_2 - \varepsilon_{r_1} \vec{\nabla} \Phi_1) = -\sigma_f / \varepsilon_0$,

OR $\varepsilon_{r_2} \frac{\partial}{\partial n} \Phi_2 - \varepsilon_{r_1} \frac{\partial}{\partial n} \Phi_1 = -\sigma_f / \varepsilon_0$.

BOUNDARY CONDITIONS FOR THE DIRECTIONS TANGENTIAL TO THE SURFACE (ALSO SEE JACKSON EQN. 4.40.)

THIS IS A LITTLE TRICKIER.



$\oint \vec{E} \cdot d\vec{l} = 0$

$\vec{E}_2 \cdot \vec{l} - \vec{E}_1 \cdot \vec{l}$ + TERMS LIKE $\vec{E} \cdot \vec{dh} = 0.$

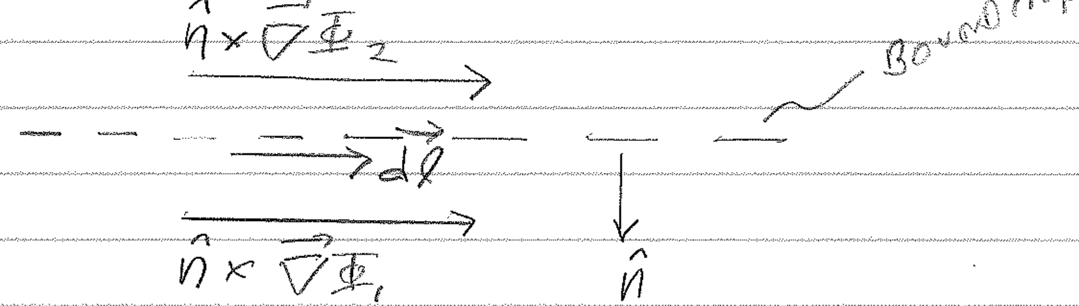
(ONE OF THE TRICKY POINTS: NOT THAT IT MATTERS HERE, BUT YOU CAN SHOW THE ABOVE IS VALID EVEN IF $\vec{\nabla} \times \vec{E} \neq 0$ SO LONG AS $\vec{\nabla} \times \vec{E}$ IS FINITE ON THE BOUNDARY.)

WITH $\vec{dh} \to 0$, $\vec{E}_2 - \vec{E}_1 = 0.$

IN TERMS OF POTENTIALS

$$\hat{n} \times (\vec{E}_2 - \vec{E}_1) = \hat{n} \times (\vec{\nabla}\Phi_2 - \vec{\nabla}\Phi_1) = 0.$$

NOW, ALONG THE BOUNDARY,
INTEGRATE ALONG THE DIRECTION
OF $\vec{\nabla}\Phi_2$ OR $\vec{\nabla}\Phi_1$



$$\hat{n} \times \int_a^b \left[\vec{\nabla}\Phi_2 - \vec{\nabla}\Phi_1\right] \cdot d\vec{\ell} = 0$$

WITH a, b TWO POINTS
ON THE BOUNDARY

$$\hat{n} \times \left[\Phi_2 - \Phi_1\right]_a^b = 0 \quad \text{FOR ALL } a, b.$$

SO $\Phi_2 = \Phi_1 +$ CONSTANT
(THE CONSTANT DISAPPEARS ON
TAKING THE GRADIENTS.)

IT'S CONVENIENT TO SET THE
CONSTANT TO ZERO.

WE SUMMARIZE THE BOUNDARY CONDITIONS FOR DIELECTRICS:

$$\hat{n}\cdot(\vec{D}_2-\vec{D}_1)=\sigma_F;$$

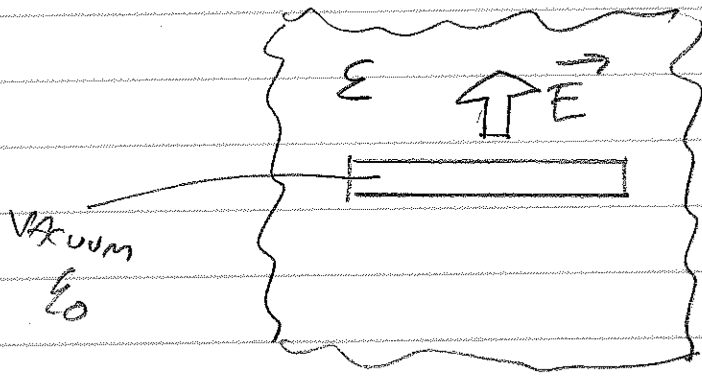
$$\epsilon_{r_2}\frac{\partial}{\partial n}\Phi_2-\epsilon_{r_1}\frac{\partial}{\partial n}\Phi_1=-\sigma_F/\epsilon_0;$$

$$\hat{n}\times(\vec{E}_2-\vec{E}_1)=0\quad\left\{\vec{E}_{t_2}-\vec{E}_{t_1}=0\right\}$$

$$\Phi_2=\Phi_1.$$

WE'LL CERTAINLY NEED THESE FOR BOUNDARY-VALUE PROBLEMS INVOLVING DIELECTRICS.

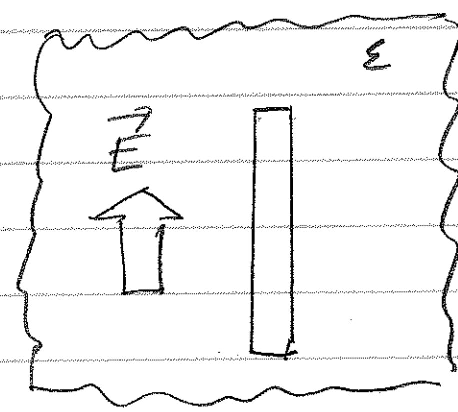
BOUNDARY-CONDITIONS; EXAMPLES, CAVITIES.



RECALL $\hat{n}\cdot(\varepsilon_{r_2}\vec{E}_2 - \varepsilon_{r_1}\vec{E}_1) = \sigma_f$

WITH $\sigma_f = 0$,

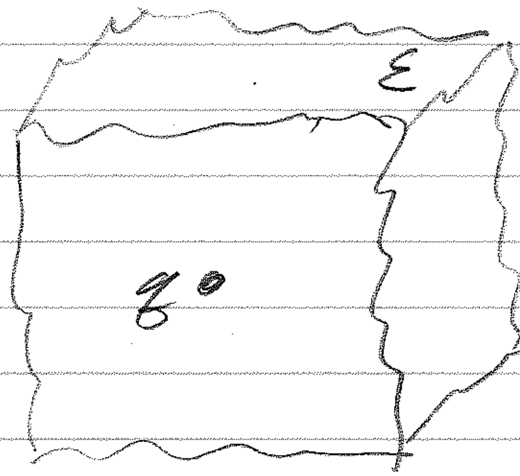
$$\vec{E}_{INSIDE} = \varepsilon\vec{E}$$



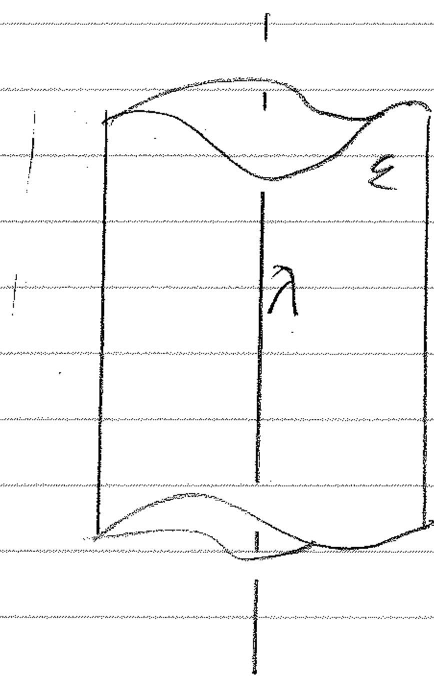
RECALL $\vec{E}_{t_2} - \vec{E}_{t_1} = 0$

SO $\vec{E}_{INSIDE} = \vec{E}$

THERE ARE SOME TRIVIAL PROBLEMS INVOLVING DIELECTRICS, e.g.,



INFINITE DIELECTRIC WITH POINT CHARGE

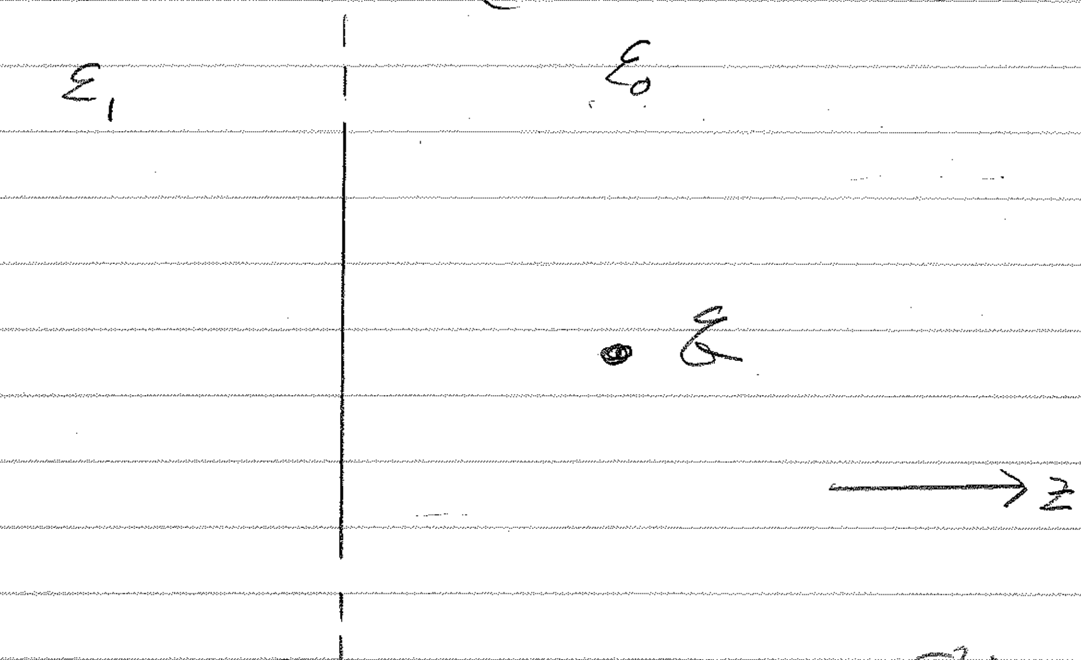


LINE CHARGE IN LONG DIELECTRIC CYLINDER

etc.

IN GENERAL, YOU'RE NOT THIS LUCKY.

EXAMPLE: POINT CHARGE OUTSIDE DIELECTRIC SLAB (JACKSON § 4.4).



Q: CAN $\vec{D}$ (AND THEREFORE $\vec{E}$) BE FOUND DIRECTLY FROM GAUSS'S LAW?

A: NO. NO OBVIOUS SYMMETRY.

WE'LL APPLY THE METHOD OF IMAGES.

$\vec{\nabla}\times\vec{E} = 0$ EVERYWHERE

$\vec{\nabla}\cdot\varepsilon_0\vec{E} = \rho_f$ $z > 0$ (ON SIDE WITH q)

$\vec{\nabla}\cdot\varepsilon_1\vec{E} = 0$ $z < 0$ (IN DIELECTRIC)

SUBJECT TO BOUNDARY CONDITIONS

$\hat{z}\cdot(\vec{D}_1 - \vec{D}_0) = 0$ NORMAL

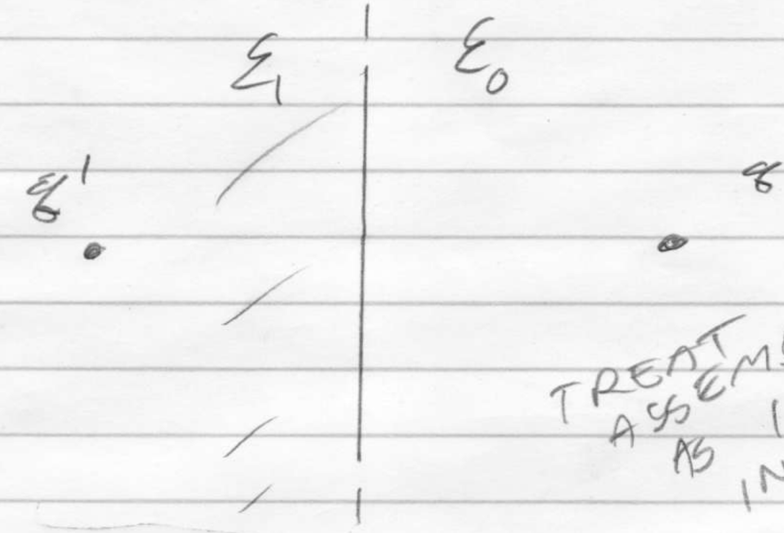
$\hat{x}\cdot(\vec{E}_1 - \vec{E}_0) = 0$ TANGENTIAL

$\hat{y}\cdot(\vec{E}_1 - \vec{E}_0) = 0$ TANGENTIAL.

SUBTLETY 1: WITHIN THE DIELECTRIC $\vec{E}_1 \neq 0$, SO THIS PROBLEM IS DIFFERENT THAN THE RELATED CONDUCTOR PROBLEM.

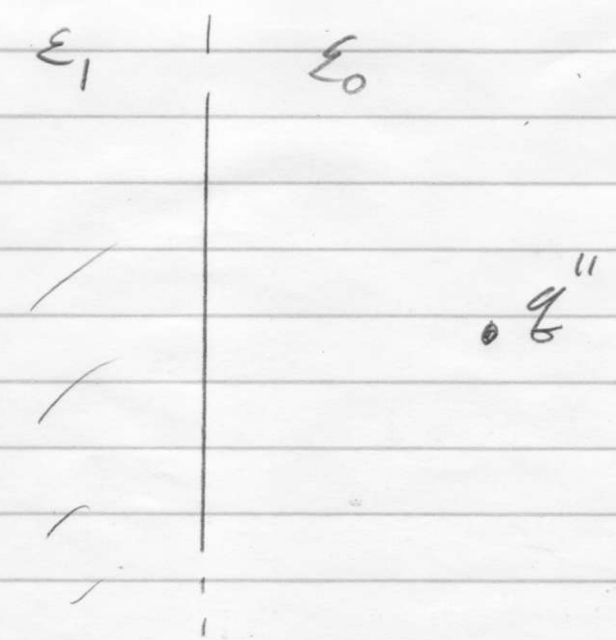
SUBLETY 2: FOR FIELD POINTS WITHIN THE DIELECTRIC, THE FIELD LOOKS LIKE THAT OF AN IMAGE "CHARGE q" ON THE VACUUM SIDE. THIS IS VERY DIFFERENT THAN THE RELATED CONDUCTOR PROBLEM: THIS IS TWO IMAGE CHARGE PROBLEMS, CALLED "CASE 1" AND "CASE 2".

CASE 1: THE REGION OF INTEREST (AND FIELD POINTS) ARE ON THE VACUUM SIDE:



TREAT THE ENTIRE ASSEMBLAGE OF CHARGES AS IF THEY WERE IN VACUUM.

CASE 2: THE REGION OF INTEREST (AND FIELD POINTS) ARE ON THE DIELECTRIC SIDE:



TREAT THE ENTIRE ASSEMBLAGE OF CHARGES AS IF THEY WERE IN DIELECTRIC

IT'S NOT OBVIOUS YET! WE'LL GUESS THE POSITIONS OF THE CHARGES, THEN FIND q' AND q'' FROM BOUNDARY CONDITIONS. WE'LL FIND IT IS A SOLUTION, THEREFOR A UNIQUE (ALMOST) SOLUTION.

WE HAVE THE FOLLOWING:

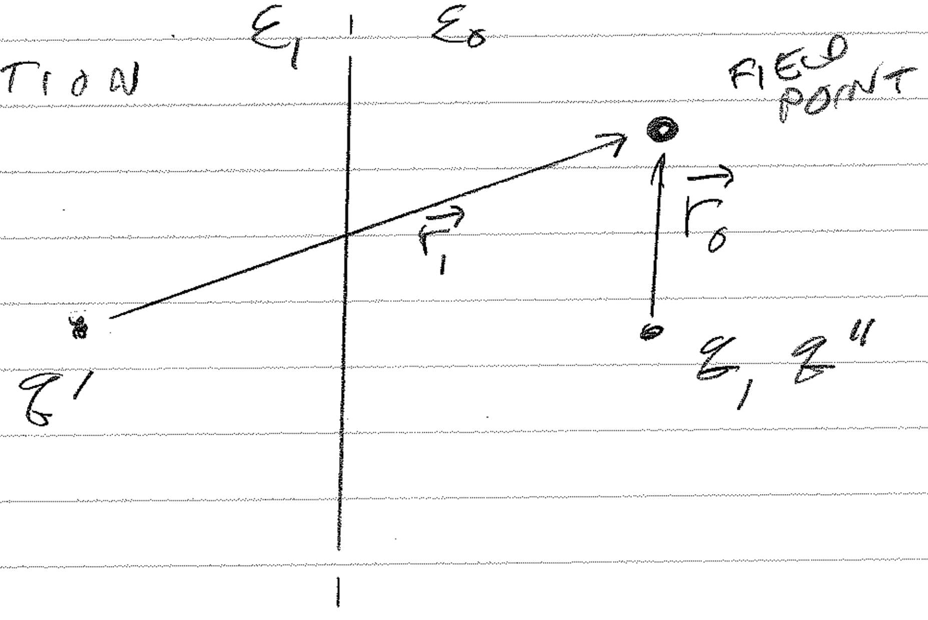
- ON THE VACUUM SIDE, THE POTENTIAL IS THAT OF q AND AN IMAGE CHARGE q'

- ON THE DIELECTRIC SIDE, THE POTENTIAL IS THAT OF THE IMAGE CHARGE q''.

THESE POTENTIALS SATISFY LAPLACE'S EQUATION (DIELECTRIC SIDE) OR POISSON'S EQUATION (VACUUM SIDE).

- TRY THE SIMPLIEST ASSUMPTION AND SEE IF THERE'S A SOLUTION: ALL CHARGES ARE AN EQUAL DISTANCE FROM THE PLANE.

- NOTATION



APPLY BOUNDARY CONDITION $\Phi_1\big|_S = \Phi_0\big|_S$:

$$\Phi_0 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_0} - \frac{1}{4\pi\varepsilon_0}\frac{q'}{r_1}$$

$$\Phi_1 = \frac{1}{4\pi\varepsilon_1}\frac{q''}{r_1}$$

NOTE, ON THE BOUNDARY $r_0 = r_1$.
HENCE $q - q' = q''\dfrac{\varepsilon_0}{\varepsilon_1}$

NOW APPLY BOUNDARY CONDITION

$$\frac{\varepsilon_1}{\varepsilon_0}\left[\frac{d}{dn}\Phi_1\right]_S - \left[\frac{d}{dn}\Phi_0\right]_S = 0.$$

APPLY $r_1 = r_2$ ON BOUNDARY;
APPLY $q - q' = q''\,\varepsilon_0/\varepsilon_1$;
EVALUATE $\frac{d}{dz}$ $\left(= \frac{d}{dn}\right)$:

$$q + q' = q''$$

WE CAN SOLVE FOR q' AND q'' IN TERMS OF q:

$$q' = q\,\frac{\varepsilon_1/\varepsilon_0 - 1}{\varepsilon_1/\varepsilon_0 + 1}\ ; \quad q'' = q\,\frac{2\varepsilon_1/\varepsilon_0}{\varepsilon_1/\varepsilon_0 + 1}.$$

SO, THIS IS A SOLUTION.

LINES OF $\vec{D}$ (AND $\vec{E}$):

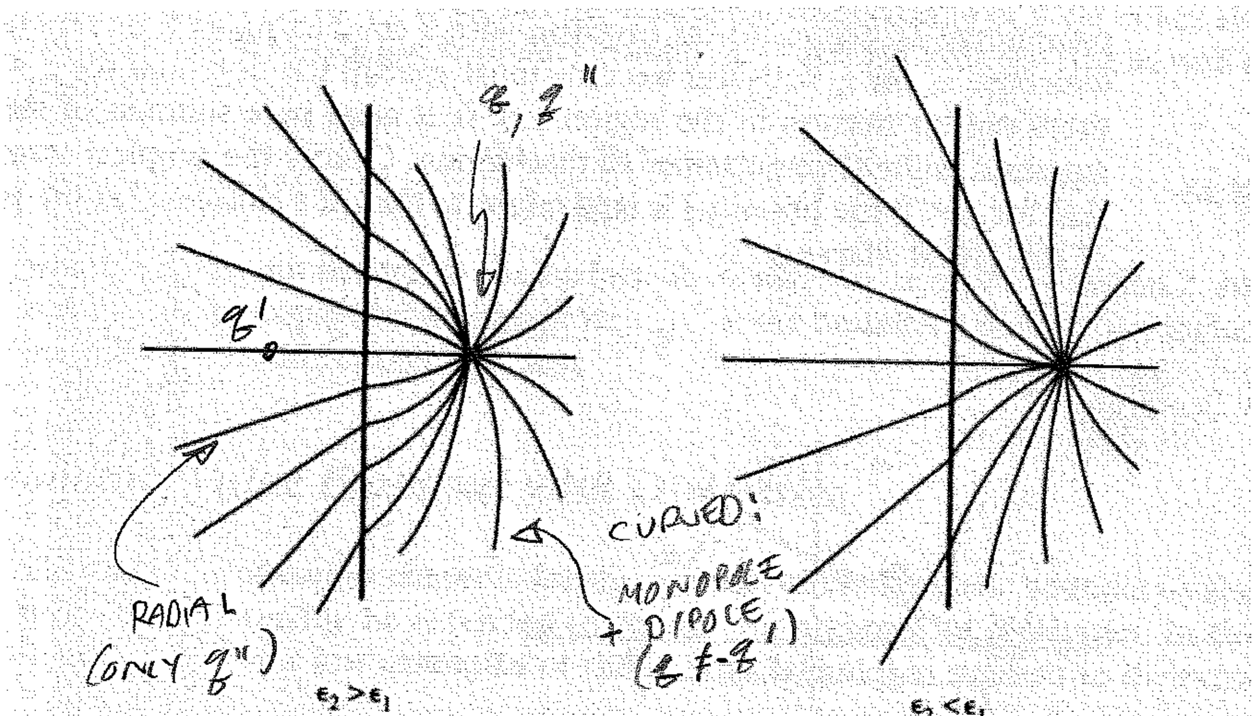


Figure 4.5 Lines of electric displacement for a point charge embedded in a dielectric ϵ_1 near a semi-infinite slab of dielectric ϵ_2.

CHECK: IF q IS IN THE INTERFACE, DO WE RECOVER OUR EARLIER RESULT?

NOTE THE LIMITING CASES:

- IF $\varepsilon_1 = \varepsilon_0$, $q' = 0$; $q'' = q$.

  Q: DOES THIS DESCRIBE A POINT CHARGE q IN VACUUM?

- IF $\varepsilon_1 \longrightarrow \infty$, $q' = q$, $q'' = 2q$.

  Q: IS THIS THE FIELD OF A POINT CHARGE q NEAR A CONDUCTING PLANE?

  A: IT SEEMS OK ON THE VACUUM SIDE: WE RECOVERED THE RESULT SEEN SEVERAL WEEKS AGO.

  ON THE DIELECTRIC SIDE, THE FIELD IS THAT OF A POINT CHARGE q'' IN INFINITE DIELECTRIC:

$$\vec{D} = \frac{1}{4\pi}\frac{q''}{r^2}\hat{r} \Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_1}\frac{q''}{r^2}\hat{r} \to 0.$$

A MEDIUM WITH $\varepsilon \to \infty$ IN MANY WAYS RESEMBLES A CONDUCTOR.

WHERE ARE POLARIZATION CHARGES ρ_P AND σ_P? (JACKSON PP. 156-7.)

RECALL $\rho_P = -\vec{\nabla} \cdot \vec{P}$; $\sigma_P = \vec{P} \cdot \hat{n}$.

RECALL $\vec{P} = \varepsilon_0 \chi_E \cdot \vec{E}$.

HENCE, ON THE VACUUM SIDE $\vec{\nabla} \cdot \vec{E} = 0 \rightarrow \vec{\nabla} \cdot \vec{P} = 0$ (EXCEPT RIGHT AT THE POSITION OF q).

SIMILARLY, ON THE DIELECTRIC SIDE $\vec{\nabla} \cdot \vec{E} = 0$ FOR $1/r$ POTENTIALS.

ON THE BOUNDARY

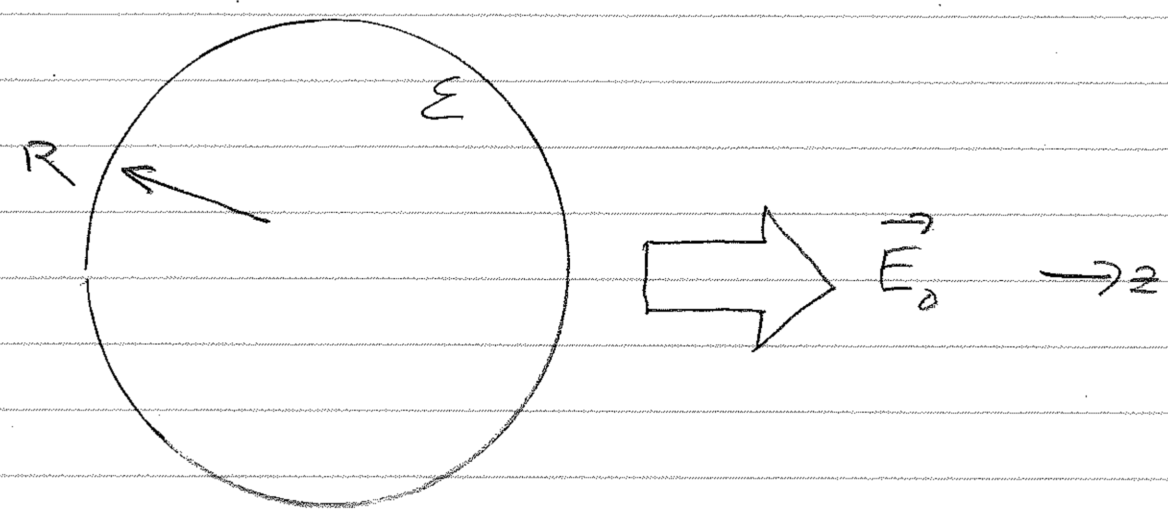
$\sigma_P = (\vec{P}_1 - \vec{P}_0) \cdot \hat{n}$ (IF q IS POSITIVE, σ_P IS NEGATIVE).

$$\sigma_P = (\varepsilon_1 - \varepsilon_0) \vec{E}_1 \cdot \hat{n} = (\varepsilon_1 - \varepsilon_0)\, \Delta\, \vec{\nabla} \Phi \cdot \hat{z}$$

$$= -q \frac{1}{2\pi} \frac{(\varepsilon_1 - \varepsilon_0)}{(\varepsilon_1 + \varepsilon_0)} \frac{d}{(\rho^2 + d^2)^{3/2}}$$

CHECK: DOES THE FACTOR $\dfrac{(\varepsilon_1 - \varepsilon_0)}{(\varepsilon_1 + \varepsilon_0)}$ REDUCE TO THE EARLIER RESULT?

EXAMPLE: DIELECTRIC SPHERE IN A UNIFORM APPLIED $\vec{E}$-FIELD.



WHAT DO WE EXPECT? IF $\varepsilon = \varepsilon_0$, THERE'S NO SURFACE POLARIZATION CHARGE. IF $\varepsilon \gg \varepsilon_0$, THERE'S A "$\cos\theta$" SURFACE POLARIZATION CHARGE. MAYBE AS ε INCREASES, THE "$\cos\theta$" SURFACE GROWS UNTIL IT APPROACHES THE $\varepsilon \gg \varepsilon_0$ CASE. WE'LL SEE.