Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
December 10, 2020, 11am
On-line lecture

Administrative:

1. The last lecture is today.
2. Midterm 2 will be posted today at 4pm Pacific time.
3. The draft of this lecture is posted at `faculty.washington.edu/ljrberg/AUT20_PHYS513`
4. Office hours are today after class at 12:30.

Lecture:

Magnetostatics, Faraday's Law. Quasi-Static Fields. (Jackson chapter 5).

Section 5.16-18 Energy in the magnetic field. Mutual- and self-inductance. "Quasi-static" magnetic fields in conductors (preview of wave propagation in conductors).

Maxwell equations, Macroscopic Electromagnetism, Conservation Laws. (Jackson chapter 6).

Section 6.3 Gauge transformations. Lorentz and Coulomb gauges. The wave equations in Lorentz gauge. Historical roots of gauge invariance.

WE FOUND FROM THE LAST LECTURE

$$U_M = \frac{1}{2}\iiint \vec{A}\cdot\vec{J}\,dv$$

(RELATED TO $U_M = \frac{1}{2}\iiint \vec{B}\cdot\vec{H}\,dv$ FOR LINEAR PERMEABLE MEDIA).

FOR "LINE" CIRCUITS (WIRES, ETC.)

$$U_M \rightarrow \frac{1}{2}\sum_K I_K \oint \vec{A}\cdot d\vec{l}_K$$

APPLYING STOKES THEOREM AND $\vec{B} = \vec{\nabla}\times\vec{A}$,

$$U_M = \frac{1}{2}\sum_K I_K \iint \vec{B}\cdot\hat{n}\,da = \frac{1}{2}\sum_K I_K \Phi_{M_K}.$$

INDUCTANCE: WE SEEK TO EXPRESS U_M IN TERMS OF A FACTOR CONTAINING CURRENTS TIMES A FACTOR CONTAINING GEOMETRY (SIMILAR TO $U_E = \frac{1}{2} C \Phi^2$).

RECALL $\Phi_{M_K} = \iint \vec{B}\cdot\hat{n}\,da_K = \iint \vec{\nabla}\times\vec{A}\cdot\hat{n}\,da_K = \oint \vec{A}\cdot d\vec{l}_K$.

ALSO RECALL FROM $\vec{A} = \frac{\mu_0}{4\pi}\iiint \frac{J(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dv'$;

$$\vec{A}(\vec{r}_K) = \frac{\mu_0}{4\pi}\sum_K \oint \frac{I_K\,d\vec{l}_j}{|\vec{r}_K - \vec{r}_j|}$$

WE'LL PUT THIS FORM OF $\vec{A}$ INTO Φ_{M_K}:

- $\Phi_{M_k} = \sum_{k \neq j} M_{jk} I_j$

WITH $M_{jk} = \frac{\mu_0}{4\pi} \oint\oint \frac{d\vec{l}_j \cdot d\vec{l}_k}{|\vec{r}_k - \vec{r}_j|} = M_{kj}$

M_{jk} IS THE "MUTUAL INDUCTANCE" OF CIRCUIT K DUE TO CIRCUIT J.

(Q: HOW WOULD YOU PUT JACKSON EQN 5.155 INTO THIS FORM?)

- HENCE $U_M = \frac{1}{2} \sum_{k \neq j} M_{jk} I_j I_k$.

"SELF INDUCTANCE" L.
WE DEFINE L FROM $\Phi_{M_k} = L_k I_k$.
THIS IS THE MAGNETIC FLUX FROM CIRCUIT K THROUGH ITSELF.
FROM THE ABOVE FORM FOR M_{jk}, (THAT IS, FOR M_{kk}),

$$L_k = \frac{\mu_0}{4\pi} \oint_1\oint_2 \frac{d\vec{l}_{k_1} \cdot d\vec{l}_{k_2}}{|\vec{r}_{k_1} - \vec{r}_{k_2}|}$$

(Q: HOW WOULD YOU PUT JACKSON EQN. 5.154 INTO THIS FORM?)

SOME THINGS TO REMEMBER:

$$\Phi_{M_k} = \sum_{k \neq j} M_{jk} I_j$$

WITH $M_{jk} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_k \cdot d\vec{l}_j}{|\vec{r}_k - \vec{r}_j|} = M_{kj}$

FOR TWO CIRCUITS $\mathcal{E}_2 = -M \frac{d}{dt} I_1$

(Q: SHOW THIS).

$\Phi_{M_k} = L_k I_k \qquad (L_k = M_{kk}).$

$$U_M = \frac{1}{2} \sum_k L_k I_k^2$$

FOR ONE CIRCUIT $\mathcal{E} = -L \frac{d}{dt} I$

(Q: SHOW THIS).

QUASI-STATIC FIELDS IN CONDUCTORS
JACKSON § 5.18

SOME IMPLICATIONS OF "QUASI-STATIC".
TIME-DEPENDENCE IS ALLOWED, BUT ...
FIELDS ASSUMED TO PROPAGATE INSTANTANEOUSLY;
SYSTEM IS SMALL COMPARED TO THE WAVELENGTH;
NEGLIGIBLE CONTRIBUTION OF THE DISPLACEMENT CURRENT TO CURRENTS.

OUR QUASI-STATIC EQUATIONS ARE
$\vec{\nabla}\times\vec{H}=\vec{J}$, $\vec{\nabla}\cdot\vec{B}=0$, $\vec{\nabla}\times\vec{E}+\frac{\partial}{\partial t}\vec{B}=0$.

THERE IS NO ACCUMULATION OF CHARGE SO $\vec{\nabla}\cdot\vec{E}=0$.

THE MEDIUM OBEYS OHM'S LAW: $\vec{J}=\sigma\vec{E}$.

ALSO, WITH NO CHARGE ACCUMULATION $\Phi=0$.

WE HAVE $\vec{E}=-\frac{\partial}{\partial t}\vec{A}$.

{HERE, WE BORROWED
$\vec{E}=-\vec{\nabla}\Phi-\frac{\partial}{\partial t}\vec{A}$ FROM
JACKSON CHAPTER 6.}

FROM $\vec{\nabla}\times\vec{H} = J$, $\vec{\nabla}\times\vec{B} = \mu\vec{J} = \mu\sigma\vec{E}$.

WITH $\vec{B} = \vec{\nabla}\times\vec{A}$.

$$\nabla\times\vec{\nabla}\times\vec{A} = -\mu\sigma\frac{d}{dt}A$$

$$-\nabla^2 A + \vec{\nabla}(\vec{\nabla}\cdot A) = -\mu\sigma\frac{d}{dt}\vec{A}$$

WITH $\vec{\nabla}\cdot\vec{A} = 0$,

$$\nabla^2\vec{A} - \mu\sigma\frac{d}{dt}\vec{A} = 0.$$

Q: DOES THIS HOLD FOR $\vec{E}$?
Q: DOES THIS HOLD FOR $\vec{B}$?

THIS IS A DIFFUSION EQUATION.

SUPPOSE THERE ARE NON-EQUILIBRIUM FIELDS OF LENGTH SCALE l: HOW LONG (τ) FOR THEM TO DIFFUSE AWAY?

THE SCALINGS ARE $\nabla^2\vec{A} \sim \mathcal{O}(\vec{A}/l^2)$; $\frac{d}{dt}\vec{A} \sim \mathcal{O}(\vec{A}/\tau)$: THESE COMBINE TO $\tau \sim \mu\sigma l^2$.

ALTERNATIVELY, THE LENGTH SCALE FIELDS ARE SET UP IN A CONDUCTOR FOR APPLIED FIELDS WITH FREQUENCY $\nu = 1/\tau$ IS

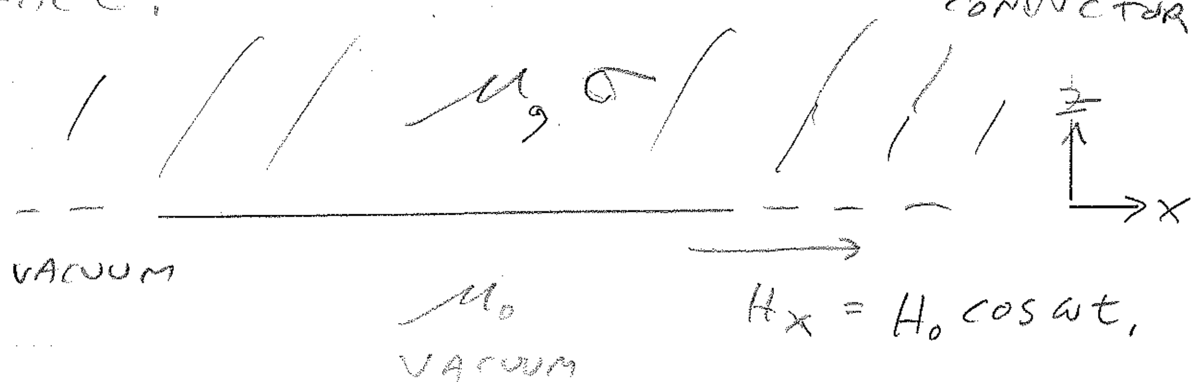
$$l \sim \frac{1}{\sqrt{\mu\sigma\nu}}.$$

JACKSON QUOTES A FUN EXAMPLE. THERE ARE CURRENTS (AND FIELDS) IN THE EARTH'S CORE. SINCE THE CORE IS SO LARGE (ℓ), THE DECAY TIME $\tau \sim \mu \sigma \ell^2$ IS LONG, AROUND 10^5 YEARS. THIS IS INTRIGUINGLY CLOSE TO THE FIELD-FLIPPING INTERVAL OF THE EARTH'S FIELD.

WE'LL COME BACK TO THIS FOR WAVE PROPAGATION IN CONDUCTORS. WHAT'S NOTABLE HERE IS THE LENGTH SCALE ℓ IS THAT OF FULL ELECTRODYNAMICS IN A GOOD CONDUCTOR. IN THIS VERY SIMPLE QUASI-STATIC ASSUMPTION WE ANTICIPATE A "SKIN DEPTH" IN GOOD CONDUCTORS.

EXAMPLE:

SKIN DEPTH. SEMI-INFINITE CONDUCTOR WITH AN $\vec{H}$ FIELD APPLIED ALONG ITS FACE:



THE BOUNDARY CONDITIONS GIVE THE CHARACTER OF FIELDS IN THE MEDIUM: PARALLEL COMPONENT OF $\vec{H}$ CONTINUOUS AT $z=0$ AND SYMMETRY REQUIRES $\vec{H} \sim \hat{x}$ INSIDE. THAT IS INSIDE $\vec{H} = H(z,t)\hat{x}$.

THE SYSTEM IS LINEAR, THE TIME DEPENDENCE EVERYWHERE FACTORS OUT AS

$$\vec{H}(z,t) = H(z)\, e^{-i\omega t}\, \hat{x}$$

WHERE WE HAVE TO REMEMBER TO ALWAYS AT THE END OF THE CALCULATION TO TAKE THE REAL PART.

THE DIFFUSION EQUATION

$$\left(\nabla^2 - \mu\sigma \frac{d}{dt}\right)\vec{H} = 0$$ BECOMES

$$\left(\frac{d^2}{dz^2} + i\mu\sigma\omega\right) H_x(z) = 0.$$

THIS IS A VERY WELL-KNOWN EQUATION... YOU'LL SEE IT AGAIN NEXT QUARTER. IT HAS SOLUTION

$$H_x(z) = e^{ikz}$$ WITH

$$k^2 = i\mu\sigma\omega$$ OR

$$k = \pm(1+i)\sqrt{\mu\sigma\omega/2}$$

Q: SHOW THIS IS THE SOLUTION.

THE TERM IN THE SQUARE ROOT IS AN INVERSE LENGTH, THE "SKIN DEPTH" δ

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}.$$

THE MAGNETIC (AND AS WE'LL SEE, ELECTRIC) FIELD FALLS EXPONENTIALLY INTO THE MEDIUM WITH $1/e$ LENGTH δ. IT HAPPENS TO BE THE SAME AS THE "GOOD CONDUCTOR" δ JACKSON CHAPT. 7.

SUMMARIZE THIS QUARTER.

$$\vec{\nabla}\cdot\vec{D} = \rho; \qquad \vec{\nabla}\cdot\vec{B} = 0;$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}\vec{B} = 0; \qquad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial}{\partial t}\vec{D}.$$

WE ALSO REQUIRE THE CONSTITUTIVE RELATIONS

$$\vec{B}(\vec{H}, t, \cdots), \quad \vec{D}(\vec{E}, t, \cdots), \quad \vec{J}(\vec{E}, t, \cdots).$$

WE ALSO REQUIRE BOUNDARY CONDITIONS.

WE INTRODUCED POTENTIALS; THE FIELDS ARE DERIVATIVES OF THE POTENTIALS. THE POTENTIALS SATISFY 2$^{\text{ND}}$ ORDER DIFFERENTIAL EQUATIONS. THE FORM OF THE POTENTIALS ENSURE THE FIELDS SATISFY TWO MAXWELL EQUATIONS BY CONSTRUCTION

THAT IS,

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{FOR} \quad \vec{B} = \vec{\nabla} \times \vec{A};$$

FOR $\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t} \vec{B} = 0$, IT'S A LITTLE TRICKIER. REWRITE $\vec{\nabla} \times \vec{E}$

$$\vec{\nabla} \times \left\{ \vec{E} + \frac{\partial}{\partial t} \vec{A} \right\} = 0.$$

THIS IS SATISFIED BY CONSTRUCTION FOR $\vec{E} + \frac{\partial}{\partial t} \vec{A} = -\vec{\nabla} \Phi$. ($\Phi$: WHY?)

REWRITING: $\vec{E} = -\vec{\nabla} \Phi - \frac{\partial}{\partial t} \vec{A}$.

ON TRANSITIONING TO DYNAMICS, THE POTENTIALS ENSURE THESE TWO MAXWELL EQUATIONS TO BE SATISFIED.

IN MORE DETAIL, SINCE THE EQUATIONS $\vec{\nabla} \cdot \vec{B}$ AND $\vec{\nabla} \times \vec{E}$ ARE SATISFIED BY THE FORM OF THE POTENTIALS, WHAT ARE THE DYNAMICAL EQUATIONS FOR THE FIELDS? HOW DO THE FIELDS EVOLVE? THROUGH $\vec{\nabla} \cdot \vec{D}$ AND $\vec{\nabla} \times \vec{H}$. (THAT IS, $\vec{\nabla} \cdot \vec{B}$ AND $\vec{\nabla} \times \vec{E}$ MAXWELL EQUATIONS CONTAIN NO NEW INFORMATION.)

WE'D LIKE TO FIND Φ AND $\vec{A}$ IN A DYNAMICAL (NON-STATIC) SYSTEM FROM THE SOURCES. (FOR SOME REASON, JACKSON RESTRICTS THIS DISCUSSION TO VACUUM.) EVALUATE THE 2 DYNAMICAL EQUATIONS:

- $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left\{-\vec{\nabla}\Phi - \frac{\partial}{\partial t}\vec{A}\right\} = \rho/\varepsilon_0.$

THAT IS $\nabla^2\Phi + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\varepsilon_0.$

- $\vec{\nabla}\times\vec{B} - \varepsilon_0\mu_0\frac{\partial}{\partial t}\vec{E} = \mu_0\vec{J}.$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - \varepsilon_0\mu_0\frac{\partial}{\partial t}\left(-\vec{\nabla}\Phi - \frac{\partial}{\partial t}\vec{A}\right) = \mu_0\vec{J}.$$

$$-\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) + \varepsilon_0\mu_0\vec{\nabla}\frac{\partial}{\partial t}\Phi + \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\vec{A} = \mu_0\vec{J}.$$

THAT IS $\nabla^2\vec{A} - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\vec{A} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \varepsilon_0\mu_0\frac{\partial}{\partial t}\Phi\right) = 0.$

COMMENTS:

- WE'LL SEE $\varepsilon_0\mu_0 = 1/c^2$;
- RECOGNIZE THE LORENTZ CONDITION.
- WE'LL SEE A CHOICE OF GAUGE CONSISTS OF FIXING $\vec{\nabla}\cdot\vec{A}$.

WE HAVE TWO COMPLICATED 2^{ND} ORDER DIFFERENTIAL EQUATIONS

$$\nabla^2 \Phi + \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = -\rho/\varepsilon_0 ;$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi\right) = -\mu_0 \vec{J}$$

FURTHER, THESE EQUATIONS ARE COUPLED, MAKING THEM YET HARDER TO SOLVE.

FORTUNATELY, WE CAN EXPLOIT "GAUGE FREEDOM" TO UNCOUPLE THEM. (FOR A CONTRAIRIAN VIEW OF THE CONCEPT OF THE GAUGE TRANSFORMATION, SEE PENROSE "THE ROAD TO REALITY"). THAT IS, DEFINE NEW POTENTIALS

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\Lambda ;$$

$$\Phi \longrightarrow \Phi' = \Phi - \frac{\partial}{\partial t}\Lambda .$$

WITH Λ ANY "REASONABLE" SCALAR FIELD.

EXERCISE: SHOW $\vec{A}$, $\vec{A}'$ AND Φ, Φ' DESCRIBE THE SAME FIELDS $\vec{E}$ AND $\vec{B}$.

NOW TO DELVE DEEPER INTO GAUGES.

<u>LORENTZ GAUGE.</u> FIX $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial}{\partial t}\Phi$ (THE LORENTZ CONDITION).

EXERCISE: SHOW IT'S ALWAYS POSSIBLE TO FIX $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial}{\partial t}\Phi$.

SUPPOSE IT'S NOT THE CASE $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi = 0$, THAT IS, SUPPOSE

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi = \eta \quad (\neq 0).$$

LET Φ' AND $\vec{A}'$ SATISFY THIS GAUGE CONDITION (LORENTZ CONDITION), THAT IS

$$\vec{\nabla}\cdot\vec{A}' + \frac{\partial}{\partial t}\Phi' = 0.$$

FURTHER SUPPOSE Φ, $\vec{A}$ AND Φ', $\vec{A}'$ ARE RELATED BY THE GAUGE FUNCTION Λ.

THAT IS,

$$\vec{\nabla}\cdot\vec{A}' + \frac{\partial}{\partial t}\Phi' = 0 \quad \text{BECOMES}$$

$$\vec{\nabla}\cdot\vec{A} + \nabla^2\Lambda + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda = 0.$$

RECALLING $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi = \eta$,

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda + \eta = 0.$$

THIS IS, IN PRINCIPLE, A SOLVABLE WAVE EQUATION IN Λ. NEXT QUARTER YOU'LL FIND THE EXPLICIT FORM OF THE SOLUTION Λ.

WITH THIS LORENTZ CONDITION, THE EQUATIONS DECOUPLE

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = -\rho/\varepsilon_0;$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t}\vec{A} = -\mu_0\vec{J}.$$

EXERCISE! SHOW THIS IS THE CASE.

EXERCISE. SUPPOSE Φ AND $\vec{A}$ SATISFY THE LORENTZ CONDITION, THEN A GAUGE TRANSFORMATION

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda, \quad \Phi' = \Phi - \frac{d}{dt}\Lambda$$

WITH $\nabla^2\Lambda - \frac{1}{c^2}\frac{d^2}{dt^2}\Lambda = 0$

ALSO SATISFIES THE LORENTZ CONDITION.

THE GAUGE AREN'T UNIQUE IN THE SENSE, e.g., THERE IS ONLY ONE GAUGE SATISFYING A PARTICULAR GAUGE CONDITION.

COULOMB GAUGE. SOMETIMES CALLED "RADIATION" OR "TRANSVERSE" GAUGE. (VERY UNFORTUNATELY SOME ENGINEERS CALL THE LORENTZ GAUGE THE "RADIATION GAUGE"!)

THE "GAUGE-FIXING" CONDITION IS

$$\vec{\nabla}\cdot\vec{A} = 0.$$

EXERCISE: SHOW IT'S ALWAYS POSSIBLE TO FIX $\vec{\nabla}\cdot\vec{A} = 0$. THE ARGUMENT IS SIMILAR TO THAT LEADING TO $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial}{\partial t}\Phi$.

SUPPOSE $\vec{\nabla}\cdot\vec{A} \neq 0$. WE INTRODUCE NEW POTENTIALS THROUGH A GAUGE TRANSFORMATION

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda;$$

$$\longrightarrow \vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\Lambda.$$

WE'D HAVE $\vec{\nabla}\cdot\vec{A}' = 0$ IF

$$\nabla^2\Lambda = -\vec{\nabla}\cdot\vec{A}.$$

WE KNOW THE SOLUTION

$$\Lambda(\vec{r}) = \frac{1}{4\pi}\iiint \frac{\vec{\nabla}\cdot\vec{A}}{|\vec{r}-\vec{r}'|}\, dv'.$$

THIS IS PECULIAR... TIME DOES NOT ENTER. CHANGING $\vec{\nabla}\cdot\vec{A}$ SOMEWHERE INSTANTANEOUSLY CHANGES Λ FAR AWAY. BUT OK... Λ IS NOT FIELDS ... .

RECALL WE STARTED WITH EQUATIONS

$$\nabla^2\Phi + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\varepsilon_0 \, ;$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi\right) = \mu_0\vec{J}$$

HENCE IN COULOMB GAUGE WE HAVE POISSON'S EQUATION

$$\Phi(r,t) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\,dv' ,$$

THIS IS THE USUAL STATIC COULOMB SOLUTION. NOTICE ITS INSTANTANEOUS, A CHANGE IN ρ AT $\vec{r}',t$ CAUSES AN INSTANTANEOUS CHANGE IN Φ AT $\vec{r},t$. IT'S NOT THE CASE THOUGH THAT COULOMB GAUGE IS ACAUSAL... BUT THE QUESTION REQUIRES SOME THOUGHT.

THIS MAY BE ADDRESSED NEXT QUARTER. (SEE FOOTNOTE JACKSON P242.)

THE VECTOR POTENTIAL SATISFIES

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{d^2}{dt^2}\vec{A} - \frac{d}{dt}\vec{\nabla}\Phi = -\mu_0 \vec{J},$$

THIS IS A MESS, SO COULOMB GAUGE IS VERY USEFUL FOR SYSTEMS OF NO CHARGE. IT ALSO HAS UTILITY IN FIELD THEORY. (THE TERM "TRANSVERSE GAUGE" SUGGESTS THE TRANSVERSE PHOTON POLARIZATION... ..).

CLASSIC PROBLEM! DECOMPOSE $\vec{J}$ INTO LONGITUDINAL AND TRANSVERSE CURRENTS $\vec{J} = \vec{J}_L + \vec{J}_T$.
(JACKSON EQN 6.30). IN THIS CASE $\nabla^2 \vec{A} - \frac{1}{c^2}\frac{d^2}{dt^2}\vec{A} = -\mu_0 \vec{J}_T$.

CLASSIC PROBLEM. SUPPOSE THE SYSTEM HAS NO CHARGE ($\rho = 0$):

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{d^2}{dt^2}\vec{A} = -\mu_0 \vec{J}.$$

IT'S WORTH LOOKING AT "HISTORICAL ROOTS OF GAUGE INVARIANCE", J.D. JACKSON & L.B. OKUN (2001).

HISTORICAL ROOTS OF GAUGE INVARIANCE (GI).

GI GOES BACK TO AT LEAST 1820's WHEN E&M WAS BEING DEVELOPED AND THE FIRST E&M THEORIES WERE BEING DEVELOPED.

THE GAUGE CONDITION $\vec{\nabla}\cdot\vec{A}=0$ WAS ADVANCED AS "NATURAL" BY MAXWELL;

THE GAUGE $\vec{\nabla}\cdot\vec{A}=-\frac{1}{c^2}\frac{\partial}{\partial t}\Phi$ WAS ADVANCED BY LORENTZ CIRCA 1860.

THE TERM "GAUGE TRANSFORMATION" APPEARED IN 3 CONTEXTS:

I. WYLE CIRCA 1918 SUGGESTED A GENERALIZATION OF GENERAL RELATIVITY WHEREBY "LENGTH" IS PATH-DEPENDENT. THE METRIC $g_{\mu\nu}$ IS SET UP TO A CONSTANT OF PROPORTIONALITY: $g_{\mu\nu}\to\lambda g_{\mu\nu}$.
WEYL THEN WENT ON TO MODIFY MAXWELL'S EQUATIONS APPROPRIATELY.

EINSTEIN POINTED OUT PROBLEMS WITH THIS IDEA.

II. IN RELATIVISTIC QUANTUM MECHANICS CIRCA 1926 (SCHRÖDINGER, KLEIN & FOCK) WAS A FORMULATION FOR SPINLESS CHARGED PARTICLES. THE EVOLUTION EQUATION (DYNAMICAL EQUATION) IS INVARIANT UNDER SCALING THE PARTICLE WAVE FUNCTION BY $e^{ie\Lambda/\hbar c}$, SO LONG AS

$$\Phi' \rightarrow \Phi - \frac{1}{c}\frac{\partial}{\partial t}\Lambda \; ; \; \vec{A}' \rightarrow \vec{A} + \vec{\nabla}\Lambda$$

THIS IS THE USUAL QUANTUM FORMULATION OF GAUGE INVARIANCE. IT'S SOMETIMES STATED "E&M IS A U(1) SYMMETRY".

(CIRCA 1929 WEYL CALLED THIS "EICHINVARIANZ" (GAUGE INVARIANCE).

III THE PRESENT ERA OF "NON-ABELAN GAUGE THEORIES" STARTED CIRCA 1954 WITH A PAPER BY YANG & MILLS.

SOME OTHER GAUGES (USING THE FORM $A^\mu = (\Phi/c^2, \vec{A})$.

$n_\mu A^\mu = 0$ (PLUS $n_\mu n^\mu = 0$) "LIGHT CONE GAUGE".

$r_\mu A^\mu$ ($r_\mu r^\mu \neq 0$) "FOCK - SCHWINGER GAUGE".

$\vec{n} \cdot \vec{A} = 0$ "POINCARÉ GAUGE".

$\Phi = 0$ "HAMILTONIAN (OR TEMPORAL, OR WEYL" GAUGE.

$\hat{n} \cdot \vec{A} = 0$ "AXIAL GAUGE" (VERY COMMON IN QFT).

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THE GENERAL IDEA IS THAT AN APPROPRIATE GAUGE CHOICE SIMPLIFIES CALCULATIONS.

COULOMB GAUGE HAS HISTORICALLY BEEN APPLIED TO THE QUANTUM MECHANICS ON NON-RELATIVISTIC CHARGED PARTICLES INTERACTING WITH RADIATION. THIS IS DUE TO ONLY HAVING TO QUANTIZE THE TRANSVERSE VECTOR POTENTIAL OF THE PHOTONS, LEAVING THE INSTANTANEOUS SCALAR POTENTIAL DESCRIBING STATIC INTERACTIONS UNQUANTIZED.

THIS ENDS THE QUARTER.

I WON'T BE THE INSTRUCTOR IN 514 AND 515.

NEXT QUARTER, YOU'LL FIND THE FULL DYNAMICAL SOLUTIONS TO (LORENTZ GAUGE)

$$\nabla^2 \begin{Bmatrix} \Phi \\ \vec{A} \end{Bmatrix} - \frac{1}{c^2} \begin{Bmatrix} \Phi \\ \vec{A} \end{Bmatrix} = - \begin{Bmatrix} \rho/\varepsilon_0 \\ \mu_0 \vec{J} \end{Bmatrix};$$

THIS BRINGS IN FULL DYNAMICS, ALLOWING FOR, e.g., DESCRIPTIONS OF RADIATION AND WAVE PROPAGATION.