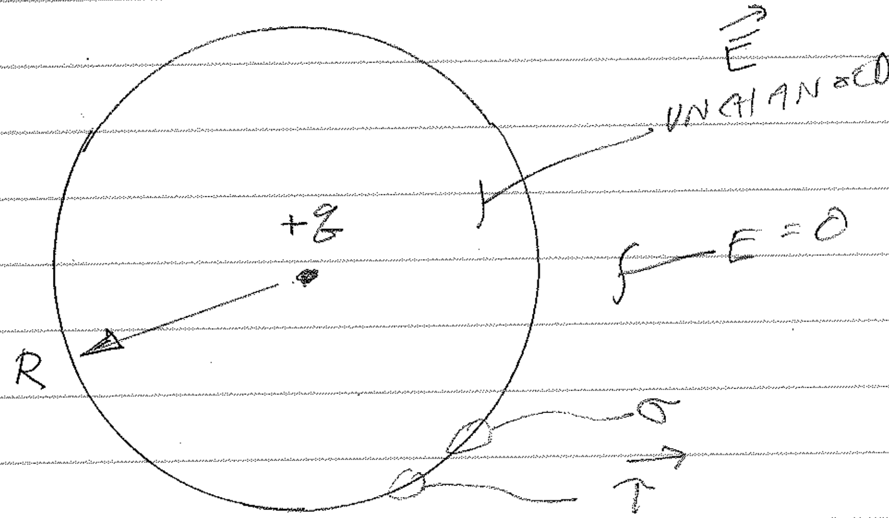
Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
October 8, 2020, 11am
On-line lecture

Administrative:
1. Homework 1 posted at
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
2. Draft of this lecture posted at
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
3. Office hours today after class at 12:30.

Lecture: Jackson Chapter 1. Tools to calculate potential.
Surface singularities.
Uniqueness of solutions to Poisson's equation.
Greens function solutions I.
Energy relations in the electrostatic field I.

"SURFACE SINGULARITIES" J. §1, 6

SIMPLE EXAMPLE: CONSIDER A SPHERICAL SURFACE WITH A CHARGE q AT THE CENTER. WHAT SURFACE-CHARGE σ AND SURFACE-DIPOLE LAYER $\vec{\tau}$ RESULT IN THE FIELD OUTSIDE VANISHING AND THE FIELD INSIDE UNCHANGED.



PLACE SURFACE CHARGE σ: $\sigma = \frac{1}{4\pi}\frac{-q}{R^2}$

THE POTENTIAL Φ_σ DUE TO THIS SURFACE CHARGE IS

$$\Phi_\sigma(r) = -\frac{1}{4\pi\varepsilon_0}\frac{q}{R} \quad \text{(INSIDE)}$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{q}{r} \quad \text{(OUTSIDE)}$$

Q: WHY IS THE POTENTIAL Φ_σ CONSTANT INSIDE?

NOW PLACE SURFACE DIPOLE LAYER $\vec{\tau}$ (WITH CONVENTIONS OF PREVIOUS LECTURE):

$$\vec{\tau} = \frac{+1}{4\pi} \frac{q}{R} \hat{r}$$

THE POTENTIAL Φ_τ DUE TO THIS SURFACE DIPOLE LAYER IS

$$\Phi_\tau(r) = +\frac{1}{4\pi\epsilon_0} \frac{q}{R} \quad \text{(INSIDE)}$$

$$= 0 \quad \text{(OUTSIDE)}$$

Q: WHY IS Φ_τ CONSTANT INSIDE, WHY IS IT CONSTANT OUTSIDE, WHY IS IT ZERO OUTSIDE?

THE POTENTIAL Φ_q DUE TO THE POINT CHARGE q IS

$$\Phi_q(r) = +\frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad \text{(INSIDE \& OUTSIDE)}.$$

NOW, SUM THEM UP:

$$\Phi(r) = \Phi_q + \Phi_\sigma + \Phi_\tau$$

$$= \frac{+1}{4\pi\epsilon_0} \frac{q}{r} \quad \text{INSIDE}$$

$$= 0 \quad \text{OUTSIDE.}$$

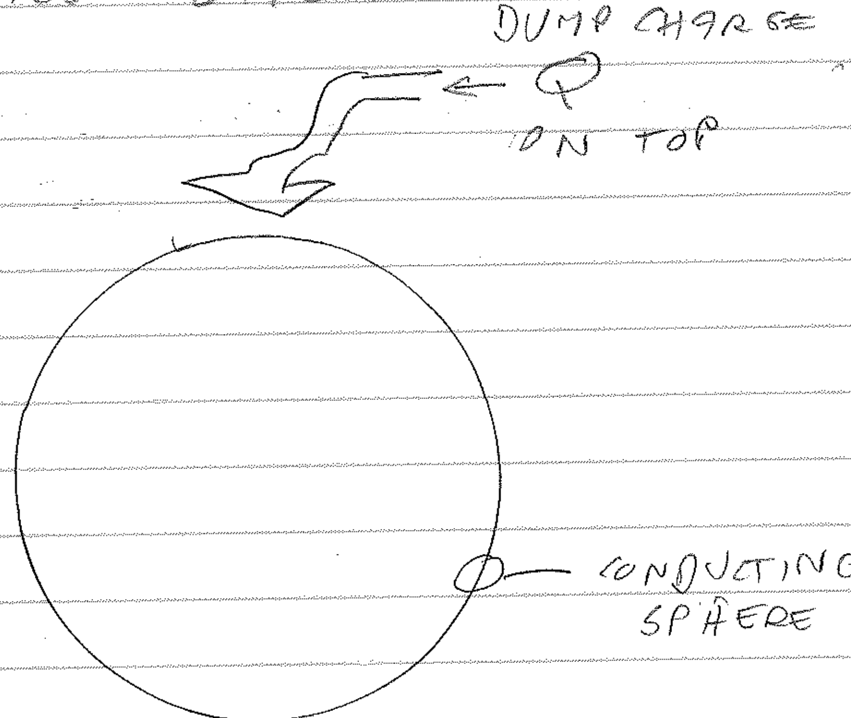
BACK TO POISSON'S EQUATION $\nabla^2 \Phi = -\rho/\varepsilon_0$.
ARE THE SOLUTIONS Φ UNIQUE?
YES (ALMOST): SEE JACKSON §1.9.

1. $\Phi(\vec{r})$ WITHIN THE VOLUME IS UNIQUE IF Φ_S (DIRICHLET) OR $d\Phi/dn|_S$ (NEUMANN) IS SPECIFIED. YOU DON'T HAVE SUFFICIENT DEGREES-OF-FREEDOM TO INDEPENDENTLY SPECIFY Φ_S AND $d\Phi/dn|_S$ ON THE SAME SURFACE ELEMENTS.

2. IT FOLLOWS THAT: SUPPOSE YOU HAVE A VOLUME BOUNDED BY ONE OR MORE CONDUCTING SURFACES S_i WITH SURFACE CHARGE DENSITIES σ_i. THE RESULTING POTENTIAL $\Phi(\vec{r})$ IS (ALMOST) UNIQUE IF THE TOTAL CHARGES ON EACH CONDUCTOR Q_i IS GIVEN.

THIS IS SOMETIMES CALLED THE "SECOND UNIQUENESS THEOREM". IT CAN SAVE A LOT OF WORK.

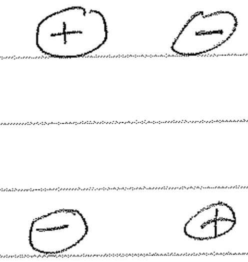
SIMPLE EXAMPLE: YOU HAVE A CONDUCTING SPHERE. DUMP CHARGE Q ON TOP... SHOW THE CHARGE IS UNIFORMLY DISTRIBUTED AS A SURFACE LAYER.



METHOD 1 (HARD, LANDAU & LIFSHITZ). START WITH A UNIFORM SURFACE CHARGE σ. SHOW AND PERTURBATION $\delta\sigma$ SOMEWHERE INCREASES THE ELECTROSTATIC SOLUTION.

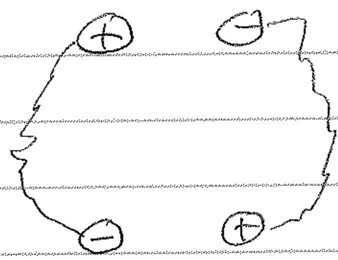
METHOD 2. Q/r^2 IS A STATIONARY SOLUTION. BY UNIQUENESS THE SURFACE CHARGE IS THEREFORE UNIFORMLY DISTRIBUTED.

ANOTHER EXAMPLE. PURCELL'S QUANDRY.
START WITH 4 CHARGES:



4 CHARGES

JOIN THEM WITH WIRES, THUS:



HOW IS THE CHARGE DISTRIBUTED?

THIS IS TRIVIAL WITH UNIQUENESS.

YOU SHOULD ALSO LOOK AT THOMSON'S THEOREM, JACKSON PROBLEM 1.15. ... THIS A VARIANT OF THE SECOND UNIQUENESS THEOREM.

GREEN'S FUNCTION I. JACKSON §1.10.
ESSENTIALLY, A STATEMENT OF SUPERPOSITION.

RECALL GREEN'S THEOREM FROM VECTOR CALCULUS. JACKSON EQN 1.35

$$\iiint (\phi \nabla^2 \psi - \psi \nabla^2 \phi)$$
$$= \oiint (\phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi) \cdot \hat{n} \, dA$$

FOR "REASONABLE" SCALAR FUNCTIONS ϕ AND ψ.

WE'D LIKE TO FIND THE POTENTIAL $\Phi(\vec{r})$ FOR A BOUNDARY-VALUE PROBLEM; THAT IS, THERE ARE CHARGE $\rho(\vec{r})$ AND POTENTIAL Φ_S (OR $\partial\Phi/\partial n|_S$).

WE'LL INTRODUCE THE CONCEPT OF THE GREEN'S FUNCTION G FOR THIS PARTICULAR GEOMETRY CONSISTING OF A PARTICULAR VOLUME AND A PARTICULAR BOUNDARY.

A GREEN'S FUNCTION G IS A SOLUTION TO A DIFFERENT BUT RELATED PROBLEM: THIS RELATED PROBLEM PLACES A UNIT POINT CHARGE $1/4\pi\epsilon_0$ AT SOURCE-POSITION $\vec{r}\,'$ WITHIN THE VOLUME, AND THE SURFACES ARE AT UNIFORM ZERO POTENTIAL $\Phi_S = 0$ (THAT IS, THE SURFACE IS AN EQUIPOTENTIAL, A CONDUCTOR). THERE IS NO OTHER CHARGE WITHIN THE VOLUME, BUT OF COURSE THERE IS INDUCED SURFACE CHARGE. THE CONDITION $\Phi_S = 0$ IS FOR DIRICHLET BOUNDARY CONDITIONS, THE MORE COMMON CASE.

TRIVIAL EXAMPLE. THE GREEN'S FUNCTION G FOR FREE SPACE.

HERE, THE "CONDUCTING BOUNDARY" IS AT $r \to \infty$. SO THE SOLUTION TO POISSON'S EQUATION

$$\nabla' G(\vec{r}, \vec{r}\,') = -4\pi\,\delta(\vec{r} - \vec{r}\,')$$

IS $$G(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|}.$$

$\frac{1}{|\vec{r}-\vec{r}\,'|}$ IS THE POTENTIAL IN FREE SPACE, DUE TO A UNIT CHARGE AT $\vec{r}\,'$: $\frac{1}{4\pi\epsilon_0}\delta(\vec{r}-\vec{r}\,')$. NOTICE THE SURFACE $r \to \infty$ IS AN EQUIPOTENTIAL $\Phi_s = 0$.

IN GENERAL, G WILL BE MORE COMPLICATED SINCE THE BOUNDARY IN GENERAL MAY NOT BE AT $r \to \infty$; IN THIS CASE $G(\vec{r}, \vec{r}\,')$ PICKS UP CONTRIBUTIONS FROM CHARGES ON THE BOUNDARY.

I SHOULD MENTION GREEN'S RECIPROCATION THEOREM JACKSON P. 40. (ALSO SEE JACKSON PROBLEM 1.12).

$$G(\vec{r}, \vec{r}\,') = G(\vec{r}\,', \vec{r}).$$

THERE ARE MANY SUCH RECIPROCATION THEOREMS IN THE PHYSICAL SCIENCES.

Q! WHAT'S ANOTHER ONE?

N.B. THE CASE OF NEUMANN BOUNDARY CONDITIONS IS TRICKER: SEE FOOTNOTE JACKSON P. 40.

TO APPLY THE GREEN'S FUNCTION, WE'LL GO BACK TO GREEN'S THEOREM, WITH $\phi \to \Phi$ AND $\psi \to G(\vec{r}, \vec{r}\,')$:

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iiint \rho(\vec{r}\,') G(\vec{r}, \vec{r}\,') \, d\tau'$$

$$+ \frac{1}{4\pi} \oint\!\!\oint G(\vec{r}, \vec{r}\,') \frac{\partial \Phi(\vec{r}\,')}{\partial n'} \, dA'$$

$$- \frac{1}{4\pi} \oint\!\!\oint \Phi_S(\vec{r}\,') \frac{\partial G(\vec{r}, \vec{r}\,')}{\partial n'} \, dA'$$

IN GENERAL G HAS FORM

$$G(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|} + F(\vec{r}, \vec{r}\,')$$

WHERE F IS THE CONTRIBUTION FROM CHARGES ON THE BOUNDARY.

IF WE HAVE DIRICHLET BOUNDARY CONDITIONS WE NEED $G_S(\vec{r}, \vec{r}\,') = 0$.

AS WE SAID, THE VALUE OF $\partial G / \partial n |_S$ IS TRICKIER. YOU MIGHT SUPPOSE YOU WANT

$$\left.\frac{\partial G}{\partial n}\right|_S = 0$$

BUT RECALL G HAS TO SATISFY

$$\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$$

J, EQN. 1.39.

APPLY DIVERGENCE THEOREM

$$\oiint \vec{\nabla}' G \cdot \hat{n}' dA' = \iiint \left(-4\pi\delta(\vec{r}-\vec{r}')\right) dV'$$

FOR A SMALL SPHERE AROUND THE SOURCE AT $\vec{r}'$!

$$\left.\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\right|_S \times \text{AREA} = -4\pi$$

SO, WE DON'T HAVE THE FREEDOM TO SET $\frac{\partial G}{\partial n} = 0$.

THE SIMPLIST THING WE CAN DO IS SET $\left.\partial G / \partial n\right|_S = -4\pi / \text{AREA}$.

(SEE COMMENT BOTTOM OF JACKSON P. 39).

IT HAPPENS THE EFFECT OF $-4\pi/\text{AREA}$ IS TO ADD A CONSTANT TO Φ (SEE JACKSON EQN. 1.46).

(I HAVEN'T MUCH USED GREEN'S FUNCTION SOLUTIONS FOR NEUMANN BOUNDARY CONDITIONS.)

NOTE: THIS GREEN'S FUNCTION PROCEDURE REPLACES WHAT ARE POSSIBLY VERY COMPLICATED BOUNDARY CONDITIONS WITH THE SIMPLER GREEN'S-FUNCTION BOUNDARY CONDITIONS.

TRIVIAL EXAMPLE: DIRICHLET BOUNDARY CONDITIONS WITH $\Phi_S = 0$. ONLY 1 TERM IN JACKSON EQN 1.42 REMAINS.

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint G \rho \, dv'$$

THIS IS OBVIOUS. IT'S JUST THE STATEMENT OF SUPERPOSITION OF POTENTIALS FROM CHARGES WITHIN THE VOLUME.

ANOTHER EXAMPLE. THERE'S NO ρ WITHIN THE VOLUME, BUT Φ_S IS SPECIFIED. AGAIN, ONLY ONE TERM IN JACKSON EQN 1.42 REMAINS

$$\Phi(\vec{r}) = -\epsilon_0 \iint \Phi_s(\vec{r}\,') \frac{\partial G}{\partial n'} dA'$$

BUT RECALL FOR A SURFACE CHARGE σ $\sigma \sim \partial\Phi/\partial n|_S$.

SO $\partial G/\partial n|_S$ IS THE INDUCED SURFACE CHARGE ON A GROUNDED CONDUCTOR DUE TO A "UNIT" CHARGE.

THIS GIVES THE INTERPRETATION OF $\Phi(\vec{r}\,')$ AS A SUPERPOSITION: THE SURFACE POTENTIAL WEIGHTED BY THE AMOUNT OF SURFACE CHARGE.

ENERGY, IN VARIOUS FORMS. I

THE WORK SOME EXTERNAL AGENT NEEDS TO EXPEND TO MOVE A CHARGE q_i FROM ∞ TO A LOCATION IN SPACE WITH POTENTIAL Φ_i IS $q_i \Phi_i$

THE POTENTIAL Φ_i IS THAT FROM ALL THE OTHER CHARGES IN THE SYSTEM.

$$\Phi_i = \frac{1}{4\pi\varepsilon_0} \sum_j \frac{q_j}{|\vec{r}_i - \vec{r}_j|}$$

THE TOTAL WORK REQUIRED TO ASSEMBLE n CHARGES IS THEN

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}.$$

$$= \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \sum_{\substack{i \\ i \neq j}}^{n} \sum_{j}^{n} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

FOR A CONTINUOUS CHARGE DISTRIBUTION

$$W = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \iint \frac{\rho(\vec{r})\,\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d\vec{r}\, d\vec{r}\,'$$

RECALL $\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} dv'$, SO

$$W = \frac{1}{2} \int \rho(\vec{r})\, \Phi(\vec{r})\, dv$$

THE ABOVE APPROACH ARGUES THAT THE ELECTROSTATIC ENERGY IS SOMEHOW A PROPERTY OF THE CHARGES.

WE CAN FIND NEARLY EQUIVALENT EXPRESSIONS WHERE WE ARGUE THE ELECTROSTATIC ENERGY IS SOMEHOW A PROPERTY OF THE FIELD.

RECALL $\nabla^2 \Phi = -\rho/\varepsilon_0$, SO

$$W = \frac{1}{2}(-\varepsilon_0) \int \nabla^2 \Phi \;\Phi\, dv$$

INTEGRATE BY PARTS:

"U" $\rightarrow \Phi$, "V" $\rightarrow \vec{\nabla}\Phi$, SO

$$W = \frac{\varepsilon_0}{2} \int \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi \, dv$$

$$= \frac{\varepsilon_0}{2} \int E^2 \, dv$$

Q: WHAT HAPPENED TO THE SURFACE TERM?

SOMETHING STRANGE AROSE.

$$W = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \sum_{\substack{i,j \\ i \neq j}} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

CAN BE POSITIVE OR NEGATIVE.

$$W = \frac{\varepsilon_0}{2} \iiint E^2 \, d\tau$$

IS POSITIVE DEFINITE.

THERE'S A LONG DISCUSSION OF THIS IN JACKSON (P. 42). THE RESOLUTION IS THE TWO EXPRESSIONS FOR W ARE DIFFERENT. THE DISCRETE FORM EXCLUDES THE POTENTIAL AT A POSITION OF A CHARGE DUE TO THAT CHARGE. THE INTEGRAL EXPRESSION INCLUDES THE "SELF POTENTIAL". THIS AROSE IN THE TRANSITION

$$\frac{1}{2} \sum_i q_i \Phi_i \longrightarrow \frac{1}{2} \int \rho(\vec{r}) \Phi(\vec{r}) \, d\tau$$

"FREE" VS "TOTAL" ENERGY IN THERMODYNAMICS. MORE ON THIS IN THE NEXT LECTURE

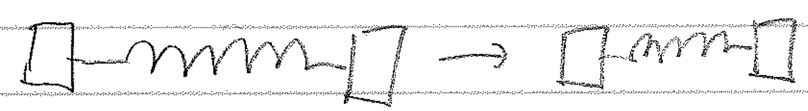
LONG BEFORE ELECTRODYNAMICS THERE WAS DEBATE ABOUT WHETHER THE ENERGY IS STORED IN THE CHARGES OR FIELDS. THAT IS, WHICH OF

$$W = \frac{1}{2}\iiint \rho \Phi \, dv$$

$$\text{OR } W = \frac{1}{2}\iiint E^2 \, dv$$

IS MORE PHYSICAL.

THE QUESTION AROSE BEFORE ELECTRODYNAMICS IN, e.g., MECHANICS, COMPRESS A SPRING WITH MASSES!



THIS REQUIRES WORK. IS THAT WORK STORED IN THE SPRING OR MASSES? YOU CAN CONCEPTUALIZE EXPERIMENTS TO RESOLVE THIS ("DROP IT IN ACID TO DISSOLVE THE SPRING, DOES THE ACID BATH HEAT UP?"). THE ANSWER IS: THERE'S NO EXPERIMENT THAT CAN TELL.

CAPACITANCE. REQUIRES TWO OR MORE CONDUCTORS. OFTEN ONE OF THE CONDUCTORS IS AN IMAGINARY CONDUCTING SURFACE AT ∞.

FOR THE COMMON CASE OF 2 CONDUCTORS, THE CAPACITANCE $\mathbb{C}$ IS

$$Q = \mathbb{C}\,\Delta\Phi$$ (SEE JACKSON PROBLEM 1.6)

WHERE $+Q$ IS CHARGE ON ONE CONDUCTOR AND $-Q$ IS CHARGE ON THE OTHER CONDUCTOR, AND $\Delta\Phi$ IS THE POTENTIAL DIFFERENCE BETWEEN THE TWO CONDUCTORS.

IN MKSA UNITS $\mathbb{C}$ [FARAD] OR $\mathbb{C}$ [COULOMB/VOLT].

SOMETIMES WE SAY "WHAT IS THE CAPACITANCE OF" A SINGLE OBJECT. WHAT THAT MEANS IS WHAT'S THE CAPACITANCE OF THAT OBJECT PLUS ALL OTHER OBJECTS IN ITS ENVIRONMENT.

TRIVIAL EXAMPLE. WHAT'S THE CAPACITANCE OF YOUR HEAD? WE CONCEPTUALIZE A SPHERE WITH ABOUT THE DIMENSIONS OF YOUR HEAD (MAYBE RADIUS 10 cm ?); THE SECOND CONDUCTOR IS THE SPHERE AT ∞. PUT CHARGE $+Q$ ON YOUR HEAD, $-Q$ ON THE SPHERE AT ∞. INTEGRATE THE RESULTING $E \cdot d\ell$ FROM THE HEAD TO ∞.

IT HAPPENS IN CGS "GAUSSIAN" UNITS THE UNITS OF CAPACITANCE C IS LENGTH [cm]. IN THESE UNITS THE CAPACITANCE OF YOUR HEAD IS ABOUT THE SIZE OF YOUR HEAD.

IF YOU INSIST! THE CAPACITANCE OF YOUR HEAD IN MKSA UNITS IS $1/4\pi\epsilon_0 \times C[\mathrm{cm}]$ OR $\sim 1/3 \times 10^{-10}$ F.