Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
December 8, 2020, 11am PST
On-line lecture

Administrative:
1. 2nd midterm exam posted this Thursday, December 10; it's due by email this Friday, December 11, 4pm PST. For exam information see the course web site.

Lecture: (a) Magnetostatics, Faraday's Law, Quasi-Static Fields. (Jackson chapter 5) (b) Maxwell's Equations, Macroscopic E&M, Conservation Laws. (Jackson chapter 6)

1. Comment on the vector potential and gauge transformations.
2. Example: Long permeable cylindrical rod in an external B field (continued); a 2D magnetic boundary-value problem.
3. Faraday' law of induction. Faraday's observation and Maxwell's extension.
4. Energy in the magnetic field & self- and mutual-inductance.

EXERCISE: POTENTIAL DESCRIBING CONSTANT $\vec{B}$ FIELD, SAY $\vec{B_0} = B_0 \hat{X}$.

SINCE $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{A}$ IS ORTHOGONAL TO $\vec{B_0}$. BUT WHICH VECTOR FOR $\vec{A}$ IN THE PLANE PERPENDICULAR TO $\vec{B_0}$?

TRY $\vec{A} = B_0 Y \hat{Z}$; $\vec{\nabla} \times \vec{A} = B_0 \hat{X}$
IT'S A VALID POTENTIAL.

TRY $\vec{A} = -B_0 Z \hat{Y}$; $\vec{\nabla} \times \vec{A} = B_0 \hat{X}$.
IT'S ALSO A VALID POTENTIAL.

CONVINCE YOURSELF A VALID POTENTIAL CAN POINT ANYWHERE IN THE TRANSVERSE Y-Z PLANE.

HOW TO CONNECT THE VARIOUS VALID POTENTIALS.
CALL $\vec{A}^{(Z)} = B_0 Y \hat{Z}$ AND $\vec{A}^{(Y)} = -B_0 Z \hat{Y}$.
LET $\vec{\eta}$ BE THE VECTOR THAT TAKES $\vec{A}^{(Z)}$ TO $\vec{A}^{(Y)}$, THAT IS

$$\vec{\eta} = \vec{A}^{(Z)} - \vec{A}^{(Y)}$$
$$= B_0 Y \hat{Z} + B_0 Z \hat{Y}.$$

LET'S SEE IF $\vec{M}$ HAS A CURL!

$\vec{\nabla} \times \vec{M} = 0$ BY TRIVIAL DIRECT CALCULATION.

APPARENTLY $\vec{A}^{(z)}$ AND $\vec{A}^{(y)}$ DIFFER BY A QUANTITY THAT HAS NO CURL. HENCE BOTH DESCRIBE THE SAME FIELD. Q: WHY? THE ARGUMENT CAN BE EXTENDED TO ANY VECTOR IN THE TRANSVERSE Y-Z PLANE.

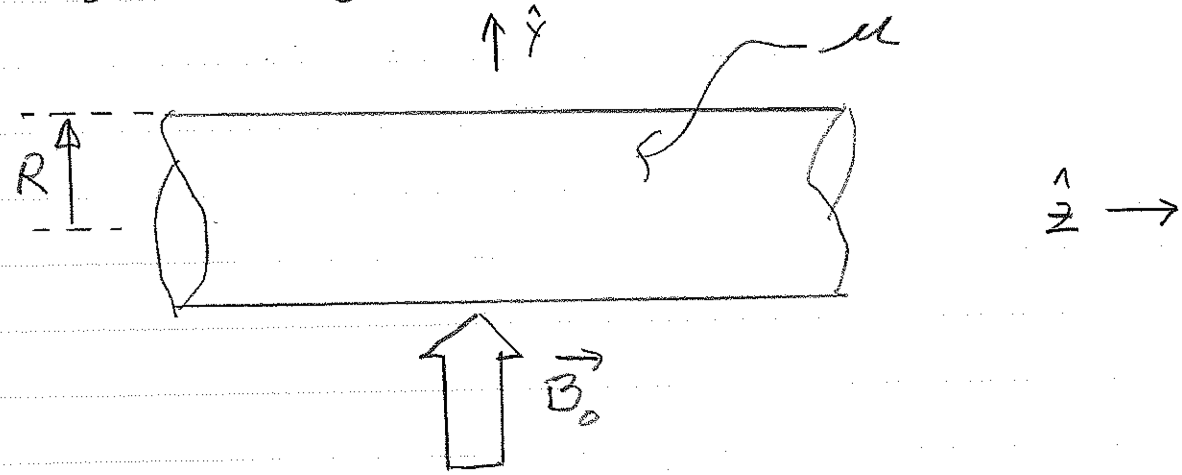
SINCE $\vec{\nabla} \times \vec{M} = 0$ EVERYWHERE, IT MUST BE DESCRIBABLE BY A POTENTIAL Λ. THAT IS

$$\vec{M} = \vec{\nabla}\Lambda .$$

THE PROCESS OF TRANSFORMING $\vec{A}^{(y)}$ TO $\vec{A}^{(z)}$ IS AN EXAMPLE OF A "GAUGE TRANSFORMATION". Λ IS THE (SCALAR) GAUGE FUNCTION.

WE'LL SEE NEXT QUARTER THAT WHEN $\vec{A} \rightarrow \vec{A}'$ BY ADDING $\vec{\nabla}\Lambda$, Φ MUST GO TO Φ' BY SUBTRACTING $\partial\Lambda/\partial t$.

BACK TO THE EXAMPLE: LONG PERMEABLE CIRCULAR ROD IN TRANSVERSE $\vec{B}$ FIELD.



FIND $\vec{A}$ EVERYWHERE.

THE SYSTEM IS TRANSLATIONALLY INVARINT: ALL d/dz VANISH.

$$\vec{B} = \vec{\nabla} \times \vec{A} \longrightarrow B_x = \frac{d}{dy} A_z, \quad B_y = -\frac{d}{dx} A_z.$$

Q: WHAT HAPPENED TO B_z?

A: THE SYMMETRY OF THE SYSTEM IS SUCH THAT, THE ROD, FLIPPED END-FOR-END, IS THE SAME SYSTEM. UNDER THIS OPERATION, $B_z \rightarrow -B_z$, A DIFFERENT MAGNETIC FIELD FOR THE SAME SYSTEM. HENCE, $B_z = 0$.

OR, ARGUE FROM $d\vec{A}/dz = 0$.

FOR A TRANSLATIONALLY INVARIANT ("2D") SYSTEM, LAPLACE'S AND POISSON'S EQUATIONS REDUCE TO

$$\nabla^2 A_z = 0, \quad \nabla^2 A_z = -\mu J_z .$$

FOR THE ROD POTENTIAL, BECAUSE OF THE BOUNDARY, THERE ARE SEPARATE SOLUTIONS FOR $r > R$ AND $r < R$.

INSIDE THE ROD, THE POTENTIAL HAS FORM (C.F., LECTURE OCT. 22 OR JACKSON EQN.S. 2.69);

$$A_z(r<R) = \sum_{\ell} (a_\ell \cos \ell\theta + b_\ell \sin \ell\theta)\, r^{\ell}.$$

Q: WHY ARE $r^{-\ell}$ TERMS ABSENT?

A: WE EXPECT A_z TO BE NON-SINGULAR FOR $r \to 0$.

OUTSIDE THE ROD: AT LARGE r, THE POTENTIAL SHOULD APPROACH THAT FOR A CONSTANT $\vec{B}$ FIELD. THAT IS,

$$A_z = B_0 y \quad (= B_0 r \sin\theta).$$

Q: WHY IS A_z IN THIS FORM?

A: $\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\partial A_z}{\partial y}\hat{x} = \vec{B}_0$.

THE OUTSIDE SOLUTION THEREFORE HAS FORM

$$A_z(r>R) = B_0 r \sin\theta + \sum_\ell (c_\ell \cos\ell\theta + d_\ell \sin\ell\theta)\, r^{-\ell}.$$

Q: WHY ARE $r^{+\ell}$ TERMS ABSENT?

A: WE'D LIKE THE POTENTIAL TO APPROACH $B_0 y\,\hat{z}$ FOR r LARGE WHETHER OR NOT THE ROD IS THERE.

Q: COULD WE HAVE FOUND $A_z(r>R)$ HAS TERM $B_0 r \sin\theta$ BY KEEPING THE $\ell = 1$ $r^{+\ell}$ TERMS?

A: YES.

APPLY MAGNETIC BOUNDARY CONDITIONS,

- $\vec{\nabla} \cdot \vec{B} = 0$; $\vec{B} \cdot \hat{n}$ IS CONTINUOUS ACROSS THE $r = R$ SURFACE. RECALL THIS IMPLIED THE MAGNETIC FLUX CROSSING THE SURFACE IS CONTINUOUS. FOR ANY CLOSED LOOP EMBEDDED IN THE SURFACE $\oint \vec{A} \cdot d\vec{l}$ $(= \Phi_M)$ IS CONTINUOUS ACROSS THE BOUNDARY. IF YOU CONSIDER A THIN RECTANGULAR LOOP EMBEDDED IN THE SURFACE, WITH LONG DIRECTION ALONG $\hat{z}$:

$$A_{z_{IN}}\Big|_{r=R} = A_{z_{OUT}}\Big|_{r=R}$$

WE JUST REDERIVED A BOUNDARY CONDITION ON $\vec{A}$ FROM LAST LECTURE, HERE APPLIED SPECIFICALLY TO A TRANSLATIONALLY-INVARIANT SYSTEM.

- $\vec{\nabla} \times \vec{H} = \vec{J}_{TRUE}$: FOR NO TRUE SURFACE CURRENTS,

$$\hat{n} \times \left(\frac{\vec{B}(r<R)}{\mu}\bigg|_{r=R} - \frac{\vec{B}(r>R)}{\mu_0}\bigg|_{r=R} \right) = 0.$$

(SEE LAST LECTURE OR JACKSON EQ'N. 5.87.)

THAT IS, THE TRANSVERSE (PARALLEL) COMPONENT OF $\vec{H}$ IS CONTINUOUS ACROSS THE $r = R$ SURFACE. HENCE

$$\frac{1}{\mu} \frac{d}{dr} A_z(r<R)\bigg|_{r=R} = \frac{1}{\mu_0} \frac{d}{dr} A(r>R)\bigg|_{r=R}.$$

Q: HOW DID THE BOUNDARY CONDITION REDUCE TO THIS FORM?

A: $\hat{n} \times \frac{\vec{B}(r<R)}{\mu}\Big|_{r=R}$ IS THE PARALLEL COMPONENT OF $\vec{H}$ AT $r=R$ (ROTATED 90° IN THE SURFACE). THAT IS

$$\frac{\vec{B}(r<R)}{\mu}\bigg|_{\substack{r=R \\ \parallel}} = \frac{\vec{B}(r>R)}{\mu_0}\bigg|_{\substack{r=R \\ \parallel}}.$$

$$\vec{B}(r<R)\Big|_{\substack{r=R\\ \parallel}} = \vec{\nabla}\times\vec{A}(r<R)\Big|_{\substack{r=R\\ \parallel}} .$$

BUT $\vec{A} \sim \hat{z}$, SO

$$\vec{\nabla}\times\vec{A} = \frac{1}{\mu}\frac{\partial A_z}{\partial\phi}\hat{r} - \frac{\partial A_z}{\partial r}\hat{\phi} .$$

WE'RE ONLY INTERESTED IN THE PARALLEL COMPONENT $\hat{\phi}$, NOT THE NORMAL COMPONENT $\hat{r}$. HENCE

$$\frac{1}{\mu}\frac{d}{dr}A_z(r<R)\Big|_{r=R} = \frac{1}{\mu_0}\frac{d}{dr}A_z(r>R)\Big|_{r=R}$$

AT THIS POINT THE STRUCTURE OF THE EQUATIONS IS THE SAME AS THAT OF THE DIELECTRIC ROD IN A TRANSVERSE $\vec{E}$ FIELD. ONLY $l=1$ TERMS REMAIN AND

$$A_z(r<R) = \frac{2\mu}{\mu+\mu_0}B_0 r\sin\theta$$

$$A_z(r>R) = \left(1 + \frac{\mu-\mu_0}{\mu+\mu_0}\left\{\frac{R}{r}\right\}^2\right)B_0 r\sin\theta ,$$

THE INSIDE POTENTIAL IN RECTANGULAR COORDINATES IS

$$A_z(r<R) = \frac{2\mu}{\mu+\mu_0} B_0 y,$$

WITH A MAGNETIC $\vec{B} = \vec{\nabla} \times \vec{A}$ FIELD

$$\vec{B}(r<R) = \frac{2\mu}{\mu+\mu_0} B_0 \hat{x}.$$

THIS IS A CONSTANT INNER FIELD
ANOTHER WAY TO WRITE THIS IS:

$$\vec{B}(r<R) = \frac{\mu-\mu_0}{\mu+\mu_0} B_0 \hat{x} + B_0 \hat{x},$$

THE INDUCED (AS OPPOSED TO THE TOTAL OR EXTERNAL FIELD) IS

$$\frac{\mu-\mu_0}{\mu+\mu_0} B_0 \hat{x}.$$

THE INDUCED FIELD IS IN THE SAME DIRECTION AS $\vec{B}_0$.

NOTICE HOW
DIFFERENT THIS IS COMPARED TO THE POLARIZATION OF DIELECTRICS IN AN EXTERNAL $\vec{E}$-FIELD

THE OUTSIDE FIELD IS

$$\vec{B}(r>R) = \vec{\nabla} \times \vec{A}(r>R)$$

RECALL $\vec{A} \sim \hat{z}$, SO

$$\vec{B}(r>R) = B_0 \hat{x} + \frac{\mu - \mu_0}{\mu + \mu_0} B_0 \frac{R^2}{r^2}\left\{ \cos\theta\,\hat{r} + \sin\theta\,\hat{\theta} \right\}$$

WE'VE SEEN SUCH A CHARACTER OF FIELDS BEFORE FOR THE DIELECTRIC ROD IN AN EXTERNAL $\vec{E}$ FIELD.

THE FIRST TERM IS THE EXTERNAL, CONSTANT $\vec{B}$ FIELD.

THE SECOND TERM IS THE 2D DIPOLE FIELD. (LECTURE OCT. 27, P. 21)

FARADAY'S LAW OF INDUCTION.

RECALL IN ELECTROSTATICS $\vec{\nabla} \times \vec{E} = 0$.
ALSO RECALL, TO SUPPORT STATIONARY CURRENTS THERE MUST BE ELECTRIC FIELDS FOR WHICH $\vec{\nabla} \times \vec{E} \neq 0$.

IT WAS OBSERVED THAT AN ELECTRIC FIELD FOR WHICH $\vec{\nabla} \times \vec{E} \neq 0$ ACCOMPANIES A TIME-VARYING MAGNETIC FIELD. MORE PRECISELY (FOR MAGNETIC FLUX Φ_M).
IN A CIRCUIT

$$-IR + \mathcal{E} = -\frac{d}{dt}\Phi_M. \quad \text{FARADAY'S LAW.}$$

$\mathcal{E}$ IS THE ELECTROMOTIVE FORCE IN THE CIRCUIT $\oint \vec{E} \cdot d\vec{l}$

AND NOTICE THE TOTAL TIME DERIVATIVE.

THIS RELATION IS IN NO WAY DERIVABLE FROM WHAT WE'VE SEEN UP TO NOW.

(IT'S ASSERTED, IN, e.g., GRIFFITHS, THAT FARADAY'S LAW IS A CONSEQUENCE OF CONSERVATION-OF-ENERGY APPLIED TO THE OVERALL ENERGY BALANCE OF CURRENTS IN MAGNETIC FIELDS. I'M PRETTY SURE THIS IS WRONG.)

$\frac{d}{dt}\Phi_M$ CAN HAVE CONTRIBUTIONS FROM CHANGING $\vec{B}$, CHANGING AREA OF CIRCUIT IN $\Phi_M = \iint \vec{B} \cdot \hat{n}\, da$, OR MOTION OF THE CIRCUIT.

MAXWELL VERY MUCH APPRECIATED THE SIGNIFICANCE OF FARADAY'S OBSERVATION, BUT MAXWELL WANTED TO CONSIDER A LOOP WITHOUT A WIRE. TO DO SO, MAXWELL NOTED THAT THE TANGENTIAL (PARALLEL) $\vec{E}$-FIELD IS CONTINUOUS ACROSS THE WIRE, SO THE EXPRESSION $-IR + \mathcal{E} = -\frac{d}{dt}\Phi_M$ AS WELL APPLIES TO THE REGION ADJACENT TO THE WIRE.

MAXWELL THEN NOTED ALL THE PHYSICAL CHARACTERISTICS OF THE WIRE (e.g., R) ARE EMBEDDED IN THE IR TERM, NOT THE $\oint \vec{E} \cdot d\vec{l}$ TERM. HENCE THE RELATION DOES NOT DEPEND ON THERE BEING A CURRENT-CARRYING CONDUCTOR AND

$$\mathcal{E} = -\frac{d}{dt} \Phi_M$$

IS A GENERAL PHYSICAL LAW RELATING THE $\vec{E}$ FIELD IN VACUUM TO A CHANGING $\vec{B}$-FIELD. THAT IS

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot \hat{n}\, da$$

$$\iint \vec{\nabla} \times \vec{E} \cdot \hat{n}\, da = -\frac{d}{dt} \iint \vec{B} \cdot \hat{n}\, da$$

FOR ANY SURFACE CONNECTED TO THE LOOP.

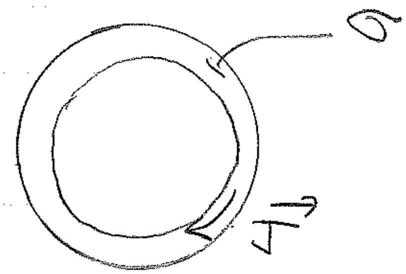
IF THE MEDIA IS STATIONARY

$$\frac{d}{dt} \rightarrow \frac{\partial}{\partial t} \quad \text{AND}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} B$$ IS A LAW OF NATURE.

FARADAY'S LAW.

TRIVIAL EXAMPLE: CIRCULAR LOOP CARRYING CURRENT.

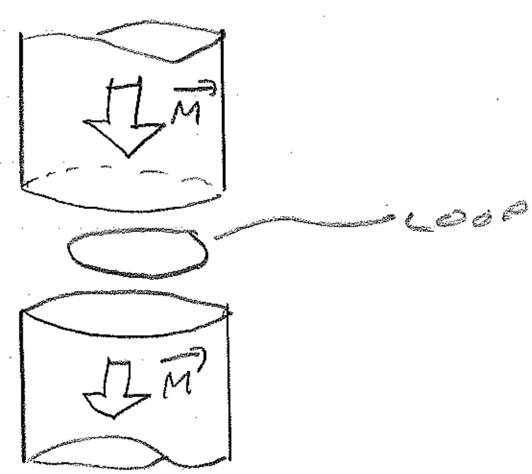


$\vec{J} = \sigma \vec{E}$, SO, OBVIOUSLY

$\oint \vec{E} \cdot d\vec{\ell} \neq 0$. $\vec{E}$ HAS A

CURL SOMEWHERE.

HOW WOULD YOU PRODUCE SUCH A CURRENT? BY, e.g., PUTTING THE LOOP BETWEEN THE POLES OF AN ELECTROMAGNETIC AND VARYING THE APPLIED FIELD:



HERE $\frac{\partial B}{\partial t} \neq 0$ THROUGH THE LOOP.

SO $\oint \vec{E} \cdot d\vec{l} \neq 0$ AROUND THE LOOP AND CURRENT FLOWS.

NOTICE IT DOESN'T MAKE PHYSICAL SENSE TO SAY THE $\mathcal{E}$ WAS PRODUCED AT A PARTICULAR PLACE IN THE CIRCUIT.

OUR MAXWELL EQUATIONS SO FAR

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \; ; \qquad \vec{\nabla} \cdot \vec{B} = 0.$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}\vec{B} = 0; \quad \underbrace{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}_{\text{THIS WILL NEED MODIFICATION.}}$$

Q: GUESS HOW YOU'D MODIFY $\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}\vec{B}$ TO ACCOUNT FOR MOVING MEDIA?

A: HINT: CONSIDER

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}.$$

ENERGY RELATIONS IN QUASI-STATIONARY ($v \ll c$) MOTION AND FIELD ($\frac{\partial D}{\partial t} \ll J$): CURRENT-CARRYING SYSTEMS.

CONSIDER A BATTERY (A "SOURCE" OF $\mathcal{E}$) DRIVING CURRENTS. THERE'S HEATING AND A $\vec{B}$ FIELD DUE TO CURRENTS.

RECALL $\vec{J} = \sigma(\vec{E}^I + \vec{E}^R)$ WITH $\vec{E}^I$ AND $\vec{E}^R$ IRROTATIONAL (CURL-LESS) AND ROTATIONAL (NON-ZERO CURL) FIELDS.

WE THEREFORE HAVE

$$\vec{J}\cdot\vec{J} = \sigma(\vec{E}^I\cdot\vec{J} + \vec{E}^R\cdot\vec{J}), \text{ OR}$$

$$\vec{E}^R\cdot\vec{J} = \frac{J^2}{\sigma} - \vec{E}^I\cdot\vec{J}.$$

THE $\vec{E}^R\cdot\vec{J}$ (AS WE SAW BEFORE) HAS UNITS OF POWER/VOLUME: THE ROTATIONAL FIELDS ARE SUPPLIED BY THE BATTERY. SO $\vec{E}^R\cdot\vec{J}$ IS THE TIME RATE PER VOLUME OF WORK DONE BY THE BATTERY.

WE'VE SEEN $\frac{J^2}{\sigma}$ IN UNDERGRADUATE ELECTRODYNAMICS, IT'S JOULE HEATING.

THE REMAINING TERM $-\vec{E}^I \cdot \vec{J}$ THEREFORE SEEMS TO BE THE TIME RATE PER VOLUME OF ENERGY FEEDING THE MAGNETIC FIELD.

Q: WHY CAN WE NEGLECT THE DISPLACEMENT CURRENT IN $\vec{J}$?

NOW, INTEGRATE THE LOCAL ENERGY BALANCE FROM THE PREVIOUS PAGE OVER ALL SPACE.

FOR $\vec{\nabla} \times \vec{H} = \vec{J}$,

$$\iiint \vec{E}^R \cdot (\vec{\nabla} \times \vec{H})\, dv$$

$$= \iiint \frac{(\vec{\nabla} \times \vec{H})^2}{\sigma}\, dv - \iiint \vec{E}^I \cdot (\vec{\nabla} \times \vec{H})\, dv.$$

STUDY THE LAST TERM $\iiint \vec{E}^{I} \cdot (\vec{\nabla} \times \vec{H})\, dv$.
IT LOOKS LIKE PART OF

$$\vec{\nabla} \cdot (\vec{E}^{I} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}^{I}) + \vec{E}^{I} \cdot (\vec{\nabla} \times \vec{H}).$$

HOWEVER $\iiint \vec{\nabla} \cdot (\vec{E}^{I} \times \vec{H})\, dv$

$$= \oiint \vec{E}^{I} \times \vec{H} \cdot \hat{n}\, da \longrightarrow 0$$

FOR LOCALIZED SOURCES.

COMMENT: WE'LL NEED TO BE MORE CAREFUL LATER WHEN WE CONSIDER RADIATION.

FINALLY RECALL $\vec{\nabla} \times \vec{E} = -\frac{d}{dt} \vec{B}$.
HENCE

$$\iiint \vec{E}^{I} \cdot (\vec{\nabla} \times \vec{H})\, dv$$

$$= \iiint \frac{(\nabla \times H)^2}{\sigma}\, dv + \iiint \vec{H} \cdot \frac{d}{dt} \vec{B}\, dv.$$

THE LAST TERM IS CLEARLY THE RATE PER VOLUME ENERGY IS IS FEEDING THE MAGNETIC FIELD. THAT IS, WE IDENTIFY THE MAGNETIC FIELD FREE-ENERGY

$$\delta U_M = \iiint \vec{H} \cdot \delta \vec{B}\, dv.$$

NOTICE THE SLIGHT DIFFERENCE AS COMPARED TO THE ELECTRIC FREE ENERGY

$$\delta U_E = \iiint \vec{E} \cdot \delta \vec{D} \, dv.$$

JUST LIKE FOR LINEAR DIELECTRICS WE FOUND THIS FORM OF δU_E LED TO FREE ENERGY

$$U_E = \frac{1}{2} \iiint \vec{E} \cdot \vec{D} \, dv,$$

FOR PERMEABLE MEDIA $\vec{B} = \mu \vec{H}$,

$$U_M = \frac{1}{2} \iiint \vec{B} \cdot \vec{H} \, dv.$$

THIS IS ENERGY IN TERMS OF FIELDS. WHAT ABOUT ENERGY IN TERMS OF SOURCES AND POTENTIALS?

IN U_M RECALL $\vec{B} = \vec{\nabla} \times \vec{A}$.
FURTHER RECALL

$$\vec{\nabla} \cdot (\vec{A} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{H}).$$

Q: WHY DOES THE TOTAL DIVERGENCE VANISH?

HENCE $U_M = \frac{1}{2}\iiint \vec{A} \cdot (\vec{\nabla} \times \vec{H})\, dv$ OR

$$U_M = \frac{1}{2}\iiint \vec{A} \cdot \vec{J}\, dv.$$

Q: COULD YOU HAVE GUESSED THIS FORM OF U_M?

A: HINT: RECALL $U_E = \frac{1}{2}\iiint \Phi \rho\, dv$.

RECALL FROM ELECTROSTATICS: THE FACTOR OF 1/2 TAKES INTO ACCOUNT THAT THE POTENTIALS (Φ AND $\vec{A}$) INCLUDE THE FIELDS OF THE SOURCES (ρ, $\vec{J}$) THEMSELVES.

YOU MIGHT WANT TO ASK WHAT THE INTERACTION ENERGY IS OF SOURCES IN EXTERNAL FIELDS. IN THIS CASE

$$U = \iiint \vec{J} \cdot \vec{A}_{EXT}\, dv + \iiint \rho \Phi_{EXT}\, dv.$$