Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
October 6, 2020, 11am
On-line lecture

Administrative:
1. Homework 1 posted on faculty.washington.edu/ljrberg/AUT20_PHYS513

Lecture: Electrostatics review I.
Coulomb’s law.
The electric field E.
Gauss’s law.
\( \nabla \cdot \mathbf{E} \) & \( \nabla \times \mathbf{E} \) (electrostatics).
The electric (scalar) potential \( \Phi \).
Boundary conditions and Green’s theorem.
Surface singularities.
Electrostatics Review I

(this should be familiar)

Start with Coulomb's Law in Vacuum

\[ F_2 = \frac{1}{4\pi \varepsilon_0} \| q_1 q_2 \| \frac{\vec{r}}{r^2} \]

SEE HOMEWORK FOR SIGN.

\( \vec{r} \) is the "displacement vector from \( q_1 \) to \( q_2 \).

\( \varepsilon_0 \) is the "permittivity of free space"

\[ \varepsilon_0 = \frac{10^{-7} \text{ farads}}{\text{meter (MKSA)}} \]

\( 4\pi c^2 \)

Coulomb's Law contains:

- Like Charges Repel, Opposite Attract,
- Varies with the magnitude of each charge,
- Varies as \( \frac{1}{r^2} \),
- Directed along the line joining the charges.
It's observed the total force contributed by many charges is a vector sum.

\( \nabla \cdot \vec{E} = \rho/\varepsilon_0; \quad \nabla \times \vec{E} = 0, \)

(We'll come back to this.)

A few differential vector identities:

Your homework has

\[ \nabla \frac{1}{|\vec{r} - \vec{r}'|} = -\nabla \frac{1}{|\vec{r} - \vec{r}'|} \]

We'll also use

\[ \nabla^2 \frac{1}{\vec{r}} = -4\pi \delta(\vec{r}) \]

\[ \nabla^2 \{ g(|\vec{r} - \vec{r}'|) \} = -\nabla \{ g(|\vec{r} - \vec{r}'|) \} \]

\[ \nabla |\vec{r} - \vec{r}'| = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}, \quad \{ \nabla r = -\nabla' r = r \} \]

\[ \nabla \cdot (\vec{r} - \vec{r}') = 3 \quad \{ \nabla r = 3 \} \]
Electric Field $\vec{E}$ in Vacuum

$\vec{E}$ is defined in terms of the force on a test charge via

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}}{q} \quad \text{(force/charge)}.$$

This expression is the same in all systems of units, but units of $\vec{E}$ differ.

$$\vec{E} \text{[volts/meter]} \quad \text{(MKSA)}.$$

The $\lim_{q \to 0}$ ensures the test charge won't disrupt the sources of the field. There can certainly be obviously an $E$-field where there's no charge $q$; $\vec{E}$ is how you measure $\vec{E}$.

Notice $\lim_{q \to 0}$ in the definition of $\vec{E}$ can cause problems in describing elementary charge.

For such processes fields are "described" in terms of its sources, "assuming" macroscopic laws remain valid.
For those referencing Landau & Lifshitz, they use $\mathbf{E}$ for the microscopic field and $\mathbf{E}$ for the macroscopic field. They introduce dielectrics in this way.

With $\mathbf{E}$ a scalar, as experiment suggests, $\mathbf{E}$ is a vector,
\[
\mathbf{E} = \frac{1}{4 \pi \epsilon_0} \mathbf{r} \frac{1}{r^2} = -\frac{1}{4 \pi \epsilon_0} \nabla \frac{1}{r}
\]

For distributed point charges
\[
\mathbf{E} = \frac{1}{4 \pi \epsilon_0} \sum_i z_i \mathbf{r}_i \frac{1}{r_i^2}
\]

For a continuous charge distribution
\[
\mathbf{E}(\mathbf{r}) = \frac{1}{4 \pi \epsilon_0} \int \int \int \mathbf{r}(\mathbf{r}) \frac{\mathbf{r}(\mathbf{r}_i \mathbf{r}_j)}{r^2(\mathbf{r}, \mathbf{r}_i \mathbf{r}_j)} \text{d}V
\]
Simple example: $E$ inside a uniformly-charged shell

Method 1. Evaluate the net force from two opposite patches. The surface charge grows as distance $r^2$ from the field point, also the force falls as $1/r^2$; the net force is zero. $E = 0$ inside, more on this later.

Method 2. If indeed there were an $E$ field inside, it must be radial and not depend on angular variables $\theta$ & $\phi$.

But such a field would require charge inside, so $E = 0$ inside.
This brings us to Gauss's Law.

We'll consider a point charge $Q$ inside an arbitrary closed surface ("the surface of a potato"): 

What is the solid angle $d\Omega$ subtended by $dA$?

$$d\Omega = \frac{dA \cdot \hat{\mathbf{n}}}{r^2}$$

accounts for orientation of patch.

Now, evaluate the electric flux leaving $dA$

$$E \cdot \hat{\mathbf{n}} \, dA = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \, dA \cdot \hat{\mathbf{n}}$$

$$= \frac{1}{4\pi \varepsilon_0} \, Q \, d\Omega$$
Now, find the total electric flux leaving the entire surface

\[ \oint E \cdot n \, dA = \oint \frac{1}{4\pi \varepsilon_0} \, \partial \cdot \Omega \]

\[ = \frac{q}{\varepsilon_0} \]

This is obviously the integral form of Gauss's law; it follows from Coulomb's law. Notice Gauss's law is valid for any closed surface, even multiply-connected ones.
If $\mathbf{E}$ is outside the surface, the integral vanishes. (This is obvious for the spherical surface in "Method 1" earlier.)

For the differential form of Gauss's Law, apply the Divergence Theorem.

$$ \iiint \mathbf{E} \cdot \mathbf{n} \, d\mathbf{a}' = \iiint \nabla \cdot \mathbf{E} \, dV' $$

$$ = \frac{1}{\varepsilon_0} \iiint \rho \, dV' $$

Since this holds for any surface, set integrals equal:

$$ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} $$

This is one of Maxwell's equations. Notice to find $\mathbf{E}$ you need $\mathbf{\rho}$ and the boundary conditions... more on this later.
Now recall \( E = -\frac{1}{4\pi\varepsilon_0} \nabla \cdot \frac{1}{r} \).

Also recall the curl of the gradient of a scalar vanishes, so

\[ \nabla \times \mathbf{E} = 0 \quad \text{(statics)} \]

This is another Maxwell equation, a term in Faraday's law.

Your homework includes the Helmholtz theorem: A vector field \( \mathbf{E} \) is (almost) completely determined by its divergence and curl, since \( \nabla \times \mathbf{E} = 0 \), the electrostatic field from \( \nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \) is (almost) completely determined by \( \rho \).
We can also find the static equations by brute force directly from Coulomb's law:

\[ E(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \iiint \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, d\mathbf{r}' \]

\[ = -\frac{1}{4\pi\varepsilon_0} \iiint \nabla \cdot \frac{1}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{r}' \]

with \( \nabla \) acting on field points.

\[ \nabla \cdot E = -\frac{1}{4\pi\varepsilon_0} \iiint \nabla \cdot \frac{1}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{r}' \]

Since \( \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\frac{4\pi\delta(\mathbf{r} - \mathbf{r}')}{\varepsilon_0} \)

\[ = \frac{1}{\varepsilon_0} \iiint \delta(|\mathbf{r} - \mathbf{r}'|) \, d\mathbf{r}' \]

\[ = \rho / \varepsilon_0 \quad \text{Gauss's law.} \]

Similarly

\[ \nabla \times E = -\frac{1}{4\pi\varepsilon_0} \iiint \rho(\mathbf{r}') \frac{\nabla \times \nabla}{|\mathbf{r} - \mathbf{r}'|^3} \, d\mathbf{r}' \]

\[ = 0 \quad \text{since the curl of the gradient vanishes.} \]
Notice $\nabla \times \vec{E} = 0$ plus Stoke's Theorem gives $\oint \vec{E} \cdot d\vec{r} = 0$ (statics).

Since $\vec{F} = q \vec{E}$,

$\oint \vec{F} \cdot d\vec{r} = 0$ (statics),

so $\oint \vec{F} \cdot d\vec{r}$ is path independent.

Each would contribute the same.

"Electrostatic fields are conservative".

Obviously, charged particles can be given enormous kinetic energies and returned to their starting point (in, e.g., a cyclotron, or a battery); so there exist (non-static) non-conservative electric fields.
**Electrostatic Potentials**

Since $\nabla \times \mathbf{E} = 0$ (statics), we have $\mathbf{E} = -\nabla \Phi$

**Q:** Why? **A:** **Heaviside** Theorem.

$$\Phi = \frac{1}{4\pi\varepsilon_0} \iiint \frac{2}{r^2} \, dV'$$

**Q:** Why? **A:** **Heaviside** Theorem. For later in magnetostatics

$$\mathbf{A} = \frac{\mu_0}{4\pi} \iiint \frac{\mathbf{J}}{r^2} \, dV'$$

**Q:** Why? **A:** **Heaviside** Theorem.

Also, since $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$,

$$\nabla^2 \Phi = -\rho/\varepsilon_0$$ Poisson's Equation

or $\nabla^2 \Phi = 0$ (where $\rho = 0$; Laplace's Equation).
Much of electrostatics comes down to finding solutions to Laplace's or Poisson's equation (and if desired \( \mathbf{E} \) from \(-\nabla \Phi\)).

**Classical Boundary-Value Problem.**
Find \( \Phi \) in terms of \( \Phi \) within a volume plus either \( \Phi \) or \( \frac{\partial \Phi}{\partial n} \) (the surface potential or the gradient of the surface potential).

Specify \( \Phi \) \( \Rightarrow \) **Dirichlet Problem**
Specify \( \frac{\partial \Phi}{\partial n} \) \( \Rightarrow \) **Neumann Problem**

Specifying \( \Phi \) on on part of the surface and \( \frac{\partial \Phi}{\partial n} \) on the remainder is logically sound, but very challenging (it's seen in diffraction problems).

Specifying \( \Phi \) and \( \frac{\partial \Phi}{\partial n} \) on the same surface is usually, but not always, over-specifying the problem.
Another classic problem is finding $\Phi$ arising from charge and dipole discontinuities. These are related to the previous problem (see homework for the dipole layer problem).

Q: Why is a charge layer related to the Neumann boundary-value problem?

A: \[ - \nabla \Phi \]

Heuristically:
\[ E^0 = -\nabla \Phi, \]

but from Gauss's law:
\[ E \cdot \hat{n} = \frac{1}{\varepsilon_0} \sigma, \] so
\[ \frac{1}{\varepsilon_0} E \cdot \hat{n} = -\hat{n} \cdot \nabla \Phi \sim \int_{\partial \Sigma} \frac{\Phi}{ds}. \]
Some comments on boundary conditions:

I. Dirichlet problem \((\Phi_s)\)

\[ \Phi_s = 0, \text{ e.g.} \]

Notice \(E_{ll} |_s = 0\).

II. Neumann problem \((\frac{d\Phi}{dn})_s\)

Notice

\[ E_{ll} |_s = 0 \]

\[ \frac{d\Phi}{dn} |_s = 0, \text{ e.g.} \]
**Surface Singularities**
(Also see homework.)

Recall \( \Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV' \) _all space_

This has distributed sources \( \rho \), but no boundary. Suppose the integration only extends out to a closed boundary. That is, for the closed surface, we desire \( \Phi \) in terms of sources \( \rho \) _within the surface plus \( \Phi \) _on the surface_.

We'll use Green's theorem
(Jackson Eqn 1.35)

\[
\iiint (\phi \nabla^2 \phi - \nabla \phi \cdot \nabla \phi) \, dV = \oint (\phi \nabla^2 \phi - \nabla \phi \cdot \nabla \phi) \cdot \mathbf{n} \, dA
\]
We assign \( \phi \to \Phi \), \( \psi \to \frac{1}{r} \).

This leads to terms in Green's Theorem:

\[
\nabla^2 \frac{1}{r} = -4\pi \delta(r^3)
\]

\[
\nabla^2 \Phi = -\frac{\rho}{4\pi \epsilon_0}
\]

so Green's Theorem reads:

\[
-4\pi \int \int \int (\Phi \delta(r^3) - \frac{1}{r} \frac{\rho}{4\pi \epsilon_0}) \, dV = \oint \oint \oint \left( \Phi \frac{r}{r^2} - \frac{\nabla \Phi}{r} \right) \cdot \hat{n} \, dA
\]

Hence:

\[
\Phi(r^3) = \frac{1}{4\pi \epsilon_0} \int \int \int \frac{\rho(r^1)}{|r^1 - r^1|} \, dV
\]

\[
-\frac{1}{4\pi} \oint \oint \oint \frac{\nabla \Phi}{|r^1 - r^1|^3} \cdot \hat{n} \, dA
\]

\[
+ \frac{1}{4\pi} \oint \oint \oint \frac{\nabla \Phi}{|r^1 - r^1|} \cdot \hat{n} \, dA
\]

Recall

\[
\nabla \Phi \cdot \hat{n} = \frac{d\Phi}{dn}
\]
On homework, you'll apply this to a dipole layer with dipole moment per unit area \( \vec{p} \).

Recall the potential arising from a "point" dipole:

\[
\Phi(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \frac{\vec{p} \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^{3}}
\]

(608 A^3 "1/\rho^2")

For a distributed surface ("layer") of dipole:

\[
\Phi(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \int \vec{p}' \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^{3}} \, dA',
\]

This is pretty complicated. The homework looks at a layer with \( \vec{p} \) constant and aligned according to \( \vec{n} \). I pick this sign convention:

\[
\vec{\gamma}
\]

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Skipping steps, the resulting potential for a surface is:

\[ \Phi(\mathbf{r}) = -\frac{1}{4\pi \varepsilon_0} \int \frac{\mathbf{\hat{n}} \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dA' \]

We've seen part of the integrand before: \( \mathbf{r} \cdot \mathbf{\hat{n}} \, dA = \rho \)

so

\[ \Phi(\mathbf{r}) = -\frac{1}{4\pi \varepsilon_0} \int \frac{|\mathbf{r}'| - \rho}{|\mathbf{r} - \mathbf{r}'|^2} \, dA' \]

Notice a geometric discontinuity: the solid angle \( \Omega \) changes by \( 4\pi \) as you cross the surface (\(-2\pi\) in front of you to \(+2\pi\) in back of you). Hence \( \Phi \) is discontinuous by \( |\mathbf{r}'|/\varepsilon_0 \) on crossing the dipole layer.
What about the change in potential on crossing a surface-charge layer $\sigma$?

Apply Gauss's law to a small piece of the surface:

$$\varepsilon \cdot \mathbf{E} = \sigma / \varepsilon_0$$

$$\Delta \varepsilon \cdot \mathbf{n} = \sigma / \varepsilon_0$$

$$\Delta \left[ (-\nabla \Phi) \cdot \mathbf{n} \right]_S = \sigma / \varepsilon_0$$

$$- \Delta \frac{d \Phi}{d n} \bigg|_S = \sigma / \varepsilon_0$$

Here, it's the normal derivative of $\Phi$ that picks up a discontinuity $\sigma / \varepsilon_0$ on crossing the surface-charge layer.
GREEN'S THEOREM THEREFORE GIVES US IN TERMS OF \( D \) AND \( T \):

\[
\Phi(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int_D \frac{\rho(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} \, dV'
+ \frac{1}{4\pi \varepsilon_0} \int_{\partial D} \left( \mathbf{r} - \mathbf{r}' \right) \cdot \hat{n} \, dA'
+ \frac{1}{4\pi \varepsilon_0} \int_{\partial D} \frac{\mathbf{E}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|^3} \, dA'
\]

WITH \( \Phi_s = \frac{\sigma}{\varepsilon_0}, \quad \frac{\partial \Phi}{\partial n}|_s = 0 \).

SCHEMATICALLY

Dipole layer

\( \Phi \)

\( \Theta \)

\rightarrow DISTANCE

SURFACE CHARGE LAYER

\( \Phi \)

\( \Theta \)

\rightarrow DISTANCE

WE'LL CONSIDER VOLUME DISTRIBUTIONS OF POLARIZATIONS IN THE CONTEXT OF DIELECTRICS.