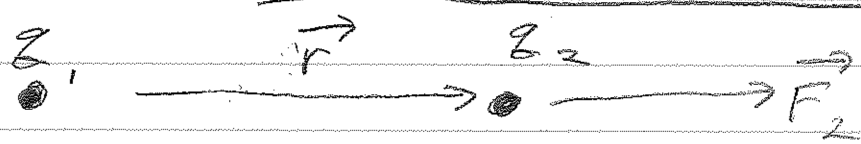
Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
October 6, 2020, 11am
On-line lecture

Administrative:
1. Homework 1 posted on
`faculty.washington.edu/ljrberg/AUT20_PHYS513`

Lecture: Electrostatics review I.
Coulomb's law.
The electric field E.
Gauss's law.
$\nabla\cdot\mathrm{E}$ & $\nabla\times\mathrm{E}$ (electrostatics).
The electric (scalar) potential Φ.
Boundary conditions and Green's theorem.
Surface singularities.

ELECTROSTATICS REVIEW I.
(THIS SHOULD BE FAMILIAR.)

START WITH COULOMB'S LAW IN VACUUM



$$\vec{F}_2 = \frac{1}{4\pi\varepsilon_0} q_1 q_2 \frac{1}{r^2} \hat{r}$$

$$= -\frac{1}{4\pi\varepsilon_0} q_1 q_2 \vec{\nabla}_{q_2} \frac{1}{|\vec{r}_2 - \vec{r}_1|}$$

SEE HOMEWORK FOR SIGN.

$\vec{r}$ IS THE "DISPLACEMENT VECTOR FROM q_1 TO q_2;

ε_0 IS THE "PERMITTIVITY OF FREE SPACE"

$$= \frac{10^7}{4\pi c^2} \ \frac{\text{FARADS}}{\text{METER}} \quad (\text{MKSA})$$

COULOMB'S LAW CONTAINS:

- LIKE CHARGES REPEL, OPPOSITES ATTRACT;
- VARIES WITH THE MAGNITUDE OF EACH CHARGE;
- VARIES AS $1/r^2$
- DIRECTED ALONG THE LINE JOINING THE CHARGES.

IT'S OBSERVED THE TOTAL FORCE CONTRIBUTED BY MANY CHARGES IS A VECTOR SUM.

(ELECTROSTATIC MAXWELL EQUATIONS FOLLOW DIRECTLY

$$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 \,; \quad \vec{\nabla} \times \vec{E} = 0,$$

WE'LL COME BACK TO THIS).)

A FEW DIFFERENTIAL - VECTOR IDENTITIES:

YOUR HOMEWORK HAS

$$\vec{\nabla} \frac{1}{|\vec{r}-\vec{r}\,'|} = -\vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}\,'|}$$

WE'LL ALSO USE

$$\nabla^2 \frac{1}{r} = -4\pi\,\delta(\vec{r})\,;$$

$$\vec{\nabla}\{g(|\vec{r}-\vec{r}\,'|)\} = -\vec{\nabla}'\{g(\vec{r}-\vec{r}\,')\}$$

$$\vec{\nabla}\,|\vec{r}-\vec{r}\,'| = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} \quad \{\nabla r = -\vec{\nabla}' r = \hat{r}\}$$

$$\vec{\nabla}\cdot(\vec{r}-\vec{r}\,') = 3 \quad \{\vec{\nabla} r = 3\}$$

ELECTRIC FIELD $\vec{E}$ IN VACUUM

$\vec{E}$ IS DEFINED IN TERMS OF THE FORCE ON A TEST CHARGE VIA

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}}{q} \quad \text{(FORCE/ CHARGE)}.$$

THIS EXPRESSION IS THE SAME IN ALL SYSTEMS OF UNITS, BUT UNITS OF $\vec{E}$ DIFFER.

$$\vec{E}\,[\text{VOLTS/METER}] \quad \text{(MKSA)}.$$

THE $\lim_{q \to 0}$ ENSURES THE TEST CHARGE WON'T DISRUPT THE SOURCES OF THE FIELD. THERE CAN CERTAINLY & OBVIOUSLY BE AN $\vec{E}$-FIELD WHERE THERE'S NO CHARGE ρ: q IS HOW YOU MEASURE $\vec{E}$.

NOTICE $\lim_{q \to 0}$ IN THE DEFINITION OF $\vec{E}$ CAN CAUSE PROBLEMS IN DESCRIBING ELEMENTARY CHARGES. FOR SUCH PROCESSES FIELDS ARE "DESCRIBED" IN TERMS OF ITS SOURCES, "ASSUMING" MACROSCOPIC LAWS REMAIN VALID.

FOR THOSE REFERENCING LANDAU & LIFSHITZ, THEY USE $\vec{e}$ FOR THE MICROSCOPIC FIELD AND $\vec{E}$ FOR THE MACROSCOPIC FIELD. THEY INTRODUCE DIELECTRICS IN THIS WAY.

WITH q A SCALAR, AS EXPERIMENT SUGGESTS, $\vec{E}$ IS A VECTOR.

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} q \frac{\hat{r}}{r^2} = -\frac{1}{4\pi\varepsilon_0} \vec{\nabla} \frac{1}{r}$$

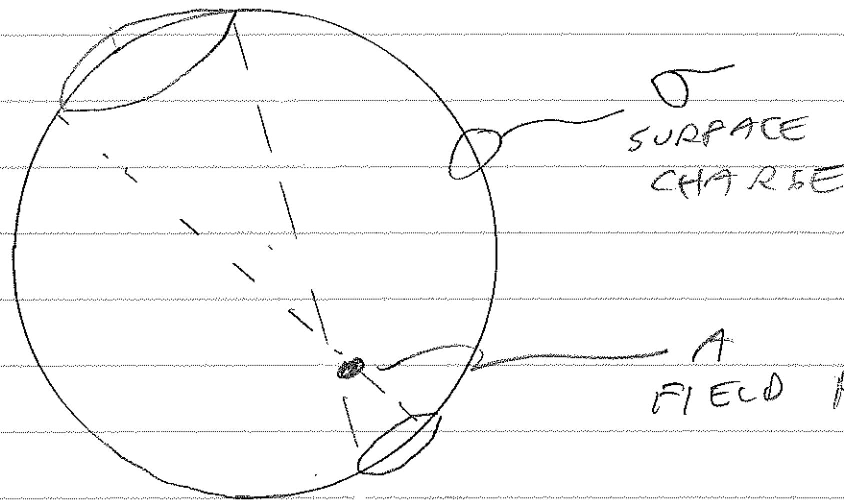
FOR DISTRIBUTED POINT CHARGES

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_i q_i \frac{\hat{r}_i}{r_i^2}$$

FOR A CONTINOUS CHARGE DISTRIBUTION

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\text{ALL SPACE}} \rho(\vec{r}\,') \frac{\hat{r}(\vec{r}, \vec{r}\,')}{r^2(\vec{r}, \vec{r}\,')} \, d\tau'$$

SIMPLE EXAMPLE: $\vec{E}$ INSIDE A UNIFORMLY-CHARGED SHELL

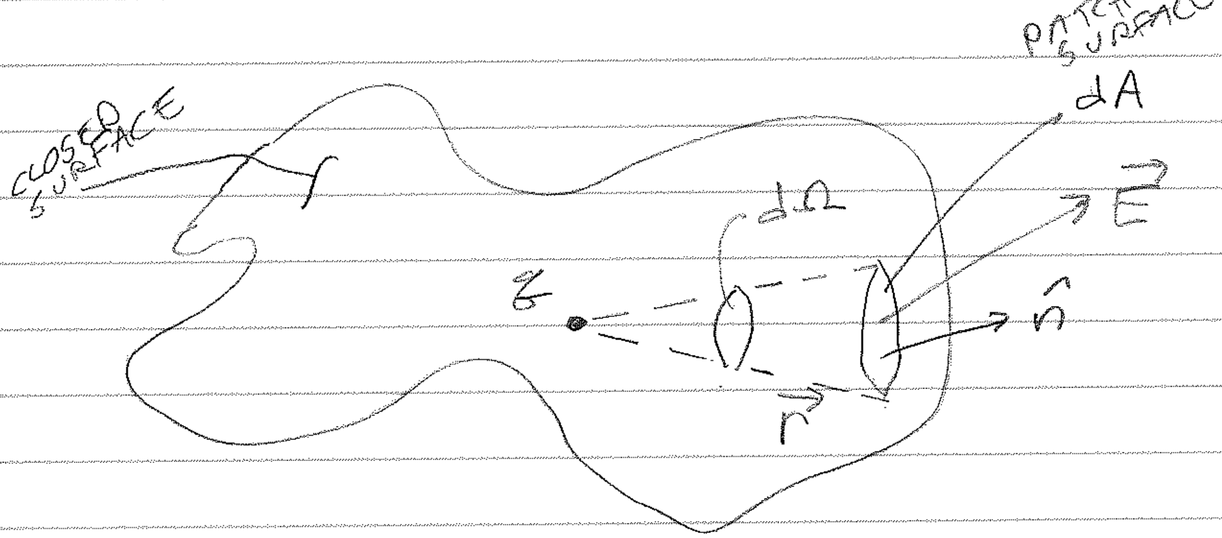


METHOD 1. EVALUATE THE NET FORCE FROM TWO OPPOSITE PATCHES. THE SURFACE CHARGE GROWS AS DISTANCE^2 FROM THE FIELD POINT; ALSO THE FORCE FALLS AS $1/\text{DISTANCE}^2$; THE NET FORCE IS ZERO. $E = 0$ INSIDE, MORE ON THIS LATER.

METHOD 2. IF INDEED THERE WERE AN $\vec{E}$ FIELD INSIDE, IT MUST BE RADIAL AND NOT DEPEND ON ANGULAR VARIABLES θ & ϕ.

BUT SUCH A FIELD WOULD REQUIRE CHARGE INSIDE. SO $E = 0$ INSIDE.

THIS BRINGS US TO GAUSS'S LAW.
WE'LL CONSIDER A POINT CHARGE q INSIDE AN ARBITRARY CLOSED SURFACE ("THE SURFACE OF A POTATO"):



WHAT IS THE SOLID ANGLE $d\Omega$ SUBTENDED BY dA?

$$d\Omega = \frac{dA}{r^2} \underbrace{\hat{r} \cdot \hat{n}}_{\text{ACCOUNTS FOR ORIENTATION OF PATCH.}}$$

NOW, EVALUATE THE ELECTRIC FLUX LEAVING dA

$$\vec{E} \cdot \hat{n}\, dA = \frac{1}{4\pi\varepsilon_0} q \frac{1}{r^2} dA\, \hat{r} \cdot \hat{n}.$$

$$= \frac{1}{4\pi\varepsilon_0} q\, d\Omega$$

NOW, FIND THE TOTAL ELECTRIC FLUX LEAVING THE ENTIRE SURFACE

$$\oiint \vec{E} \cdot \hat{n}\, dA = \oiint \frac{1}{4\pi\epsilon_0}\, q\, d\Omega$$

$$= q/\epsilon_0$$

THIS IS OBVIOUSLY THE INTEGRAL FORM OF GAUSS'S LAW: IT FOLLOWS FROM COULOMB'S LAW. NOTICE GAUSS'S LAW IS VALID FOR ANY CLOSED SURFACE, EVEN MULTIPLY-CONNECTED ONES.

IF $\vec{r}$ IS OUTSIDE THE SURFACE, THE INTEGRAL VANISHES. (THIS IS OBVIOUS FOR THE SPHERICAL SURFACE IN "METHOD 1" EARLIER.)

FOR THE DIFFERENTIAL FORM OF GAUSS'S LAW, APPLY THE DIVERGENCE THEOREM.

$$\oiint \vec{E}\cdot\hat{n}\,da' = \iiint \vec{\nabla}\cdot\vec{E}\,d\tau'$$

$$= \frac{1}{\epsilon_0}\iiint \rho\,d\tau'$$

SINCE THIS HOLDS FOR ANY SURFACE, SET INTEGRANDS EQUAL:

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

THIS IS ONE OF MAXWELL'S EQUATIONS. NOTICE TO FIND $\vec{E}$ YOU NEED ρ AND THE BOUNDARY CONDITIONS ... MORE ON THIS LATER.

NOW RECALL $\vec{E} = -\frac{1}{4\pi\epsilon_0} q \vec{\nabla} \frac{1}{r}$

ALSO RECALL THE CURL OF THE GRADIENT OF A SCALAR VANISHES, SO

$$\vec{\nabla} \times \vec{E} = 0 \quad \text{(STATICS)}$$

THIS IS ANOTHER MAXWELL EQUATION, A TERM IN FARADAY'S LAW.

YOUR HOMEWORK INCLUDES THE HELMHOLTZ THEOREM: A VECTOR FIELD IS (ALMOST) COMPLETELY DETERMINED BY ITS DIVERGENCE AND CURL. SINCE $\vec{\nabla} \times \vec{E} = 0$, THE ELECTROSTATIC FIELD FROM $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ IS (ALMOST) COMPLETELY DETERMINED BY ρ.

WE CAN ALSO FIND THE STATIC EQUATIONS BY BRUTE FORCE DIRECTLY FROM COULOMB'S LAW:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iiint \rho(\vec{r}\,') \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \, dv'$$

$$= -\frac{1}{4\pi\varepsilon_0} \iiint \vec{\nabla} \frac{1}{|\vec{r}-\vec{r}\,'|} \, dv'$$

WITH $\vec{\nabla}$ ACTING ON FIELD POINTS.

$$\vec{\nabla} \cdot \vec{E} = -\frac{1}{4\pi\varepsilon_0} \iiint \rho \nabla^2 \frac{1}{|\vec{r}-\vec{r}\,'|} \, dv'$$

SINCE $\nabla^2 \frac{1}{|\vec{r}-\vec{r}\,'|} = -4\pi\delta(\vec{r}-\vec{r}\,')$

$$= \frac{1}{\varepsilon_0} \iiint \rho \, \delta(\vec{r}-\vec{r}\,') \, dv'$$

$$= \rho / \varepsilon_0$$ GAUSS'S LAW.

SIMILARLY

$$\vec{\nabla} \times \vec{E} = -\frac{1}{4\pi\varepsilon_0} \iiint \rho \, \vec{\nabla} \times \vec{\nabla} \frac{1}{|\vec{r}-\vec{r}\,'|} \, d\tau'$$

$= 0$ SINCE THE CURL OF THE GRADIENT VANISHES.

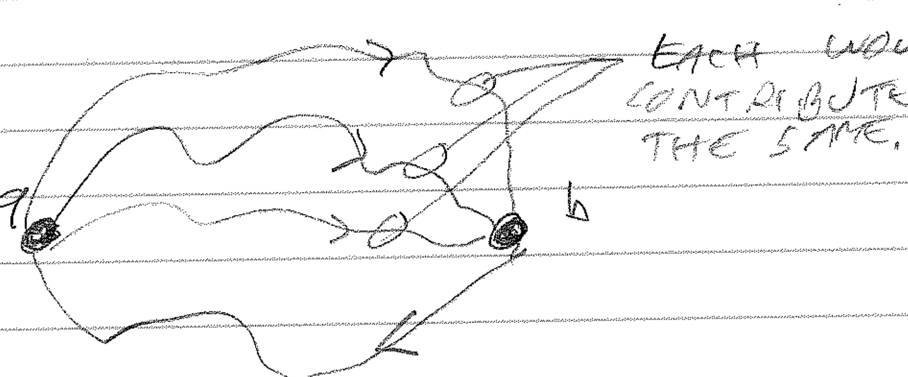
NOTICE $\vec{\nabla} \times \vec{E} = 0$ PLUS STOKE'S THEOREM GIVES

$$\oint \vec{E} \cdot \vec{dl} = 0 \qquad \text{(STATICS)}.$$

SINCE $\vec{F} = q\vec{E}$,

$$\oint \vec{F} \cdot \vec{dl} = 0 \quad \text{(STATICS)},$$

SO $\int_a^b \vec{F} \cdot \vec{dl}$ IS PATH INDEPENDENT



"ELECTROSTATIC FIELDS ARE CONSERVATIVE".

OBVIOUSLY, CHARGED PARTICLES CAN BE GIVEN ENORMOUS KINETIC ENERGIES AND RETURNED TO THEIR STARTING POINT (IN, e.g., A CYCLOTRON, OR A BATTERY); SO THERE EXIST (NON-STATIC) NON-CONSERVATIVE ELECTRIC FIELDS.

ELECTROSTATIC POTENTIALS

SINCE $\vec{\nabla} \times \vec{E} = 0$ (STATICS), WE HAVE $\vec{E} = -\vec{\nabla}\Phi$

Q: WHY? A: HELMHOLTZ THEOREM.

$$\Phi = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho}{|\vec{r}-\vec{r}'|} d\tau'$$

Q: WHY? A: HELMHOLTZ THEOREM.

FOR LATER IN MAGNETOSTATICS

$$\vec{A} = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}}{|\vec{r}-\vec{r}'|} d\tau'$$

Q: WHY? A: HELMHOLTZ THEOREM.

ALSO, SINCE $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$,

$\nabla^2 \Phi = -\rho/\epsilon_0$ POISSON'S EQUATION

OR $\nabla^2 \Phi = 0$ (WHERE $\rho = 0$; LAPLACE'S EQUATION).

MUCH OF ELECTROSTATICS COMES DOWN TO FINDING SOLUTIONS TO LAPLACE'S OR POISSON'S EQUATION (AND IF DESIRED $\vec{E}$ FROM $-\nabla\Phi$.)

<u>CLASSIC BOUNDARY-VALUE PROBLEM</u>.

FIND Φ IN TERMS OF ρ WITHIN A VOLUME PLUS EITHER Φ_s OR $\partial\Phi/\partial n|_s$ (THE SURFACE POTENTIAL OR THE GRADIENT OF THE SURFACE POTENTIAL).

SPECIFY $\Phi_s \rightarrow$ DIRICHLET PROBLEM

SPECIFY $\partial\Phi/\partial n|_s \rightarrow$ NEUMANN PROBLEM

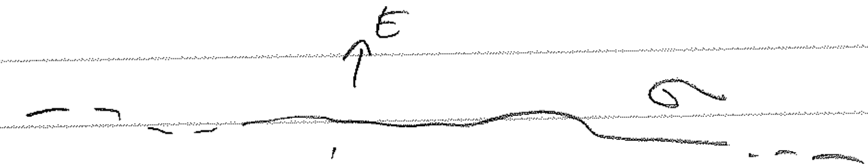
SPECIFYING Φ_s ON ON PART OF THE SURFACE AND $\partial\Phi/\partial n|_s$ ON THE REMAINDER IS LOGICALLY SOUND, BUT VERY CHALLENGING (IT'S SEEN IN DIFFRACTION PROBLEMS).

SPECIFYING Φ_s AND $\partial\Phi/\partial n|_s$ ON THE SAME SURFACE IS USUALLY, BUT NOT ALWAYS, OVER-SPECIFYING THE PROBLEM.

ANOTHER CLASSIC PROBLEM IS FINDING Φ ARISING FROM CHARGE AND DIPOLE DISCONTINUITIES. THESE ARE RELATED TO THE PREVIOUS PROBLEM (SEE HOMEWORK FOR THE DIPOLE LAYER PROBLEM).

Q: WHY IS A CHARGE LAYER RELATED TO THE NEUMANN BOUNDARY-VALUE PROBLEM?

A:



HEURISTICALLY

$$\vec{E} = -\vec{\nabla}\Phi,$$

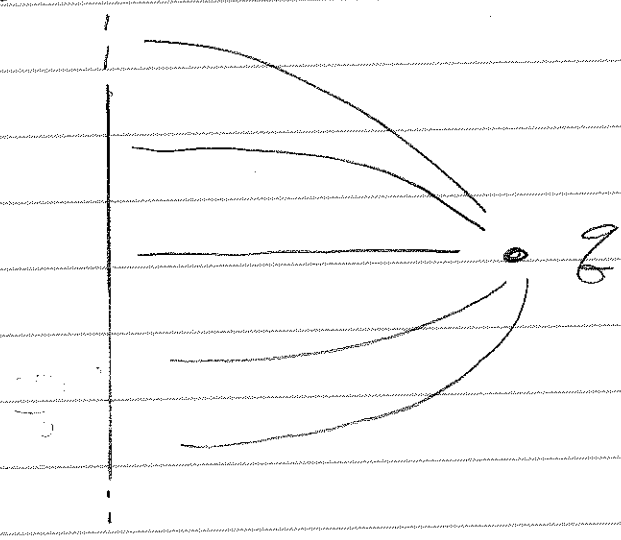
BUT FROM GAUSS'S LAW

$$\vec{E}\cdot\hat{n} = \tfrac{1}{\epsilon_0}\,\sigma, \quad \text{SO}$$

$$\text{SO } \frac{1}{\epsilon_0}\vec{E}\cdot\hat{n} = -\hat{n}\cdot\vec{\nabla}\Phi \sim \left.\partial\Phi/\partial n\right|_S$$

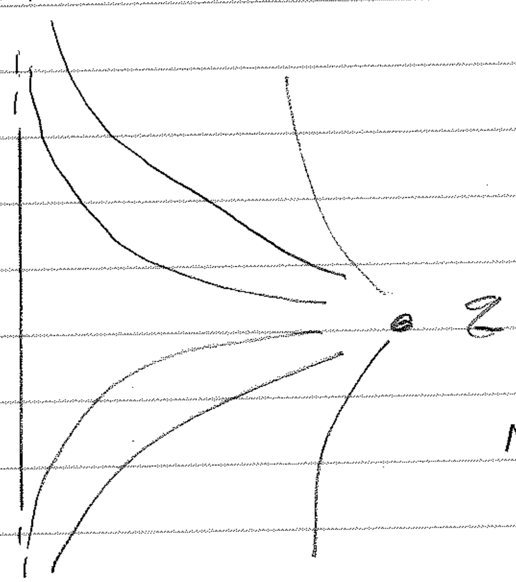
SOME COMMENTS ON BOUNDARY CONDITIONS:

I. DIRICHLET PROBLEM (Φ_S)



$\Phi_S = 0$, e.g. NOTICE $E_{\parallel}\big|_S = 0$.

II NEUMANN PROBLEM ($\partial\Phi/\partial n\big|_S$)



NOTICE

$E_{\perp}\big|_S = 0$

$\partial\Phi/\partial n\big|_S = 0$, e.g.

SURFACE SINGULARITIES

(ALSO SEE HOMEWORK).

RECALL $\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\substack{\text{ALL}\\ \text{SPACE}}} \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, dv'$

THIS HAS DISTRIBUTED SOURCES ρ, BUT NO BOUNDARY. SUPPOSE THE INTEGRATION ONLY EXTENDS OUT TO A CLOSED BOUNDARY. THAT IS, FOR THE CLOSED SURFACE, WE DESIRE Φ IN TERMS OF SOURCES ρ WITHIN THE SURFACE PLUS Φ_S OR $\partial\Phi/\partial n|_S$ ON THE SURFACE.

WE'LL USE GREEN'S THEOREM (JACKSON EQN 1.35)

$$\iiint (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dv$$

$$= \oint\!\!\!\oint (\phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi) \cdot \hat{n} \, dA'$$

WE ASSIGN $\phi \rightarrow \Phi$, $\psi \rightarrow \frac{1}{r}$,
THIS LEADS TO TERMS IN GREENS THEOREM

$$\nabla^2 \frac{1}{r} = -4\pi \delta(\vec{r}),$$

$$\nabla^2 \Phi = -\rho/\epsilon_0 ,$$ SO

GREEN'S THEOREM READS

$$-4\pi \iiint \left(\Phi \delta(\vec{r}) - \frac{1}{r} \frac{\rho}{4\pi\epsilon_0} \right) d\tau'$$

$$= \oiint \left(\Phi \frac{\hat{r}}{r^2} - \frac{\vec{\nabla}\Phi}{r} \right) \cdot \hat{n} \, dA'$$

HENCE

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d\tau'$$

$$- \frac{1}{4\pi} \oiint \Phi \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \cdot \hat{n} \, dA'$$

$$+ \frac{1}{4\pi} \oiint \frac{\vec{\nabla}\Phi \cdot \hat{n}}{|\vec{r}-\vec{r}'|} dA'$$

RECALL $\vec{\nabla}\Phi \cdot \hat{n} \Big|_S = \frac{\partial \Phi}{\partial n} \Big|_S$

ON HOMEWORK, YOU'LL APPLY THIS TO A DIPOLE LAYER WITH DIPOLE MOMENT PER UNIT AREA $\vec{\tau}$.

RECALL THE POTENTIAL ARISING FROM A "POINT" DIPOLE

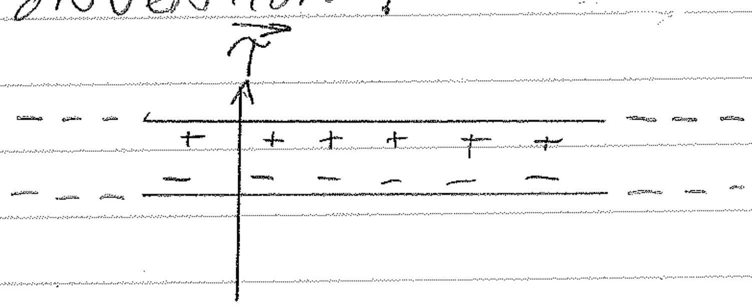
$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\vec{p}\cdot\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

(GOES AS "$1/r^2$")

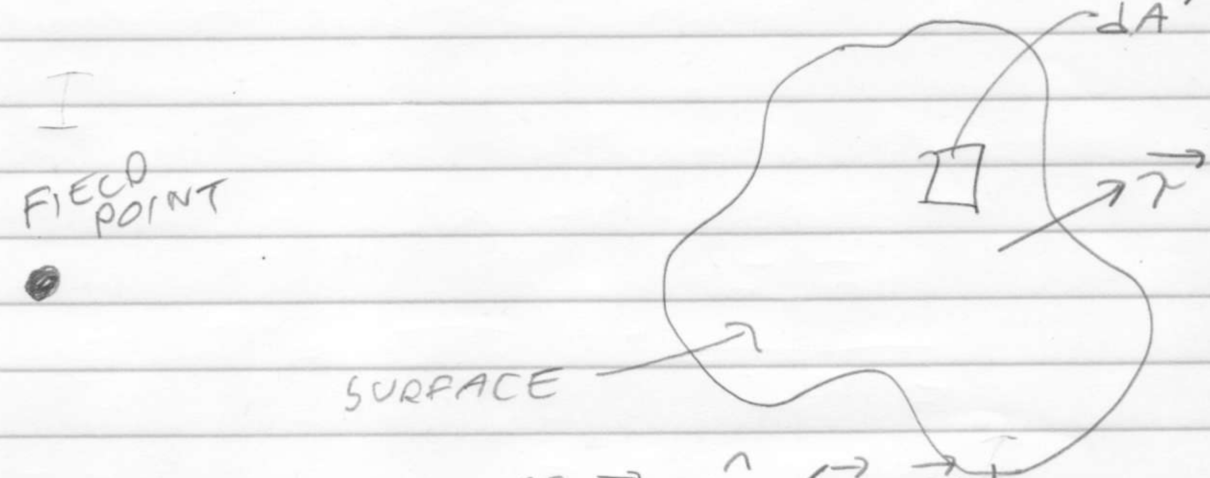
FOR A DISTRIBUTED SURFACE ("LAYER") OF DIPOLES

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iint \vec{\tau}\cdot\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dA'$$

THIS IS PRETTY COMPLICATED, THE HOMEWORK LOOKS AT A LAYER WITH $\vec{\tau}$ CONSTANT AND ALIGNED ALONG $\hat{n}$. I PICK THIS SIGN CONVENTION:



SKIPPING STEPS, THE RESULTING POTENTIAL FOR A SURFACE IS



$$\Phi(\vec{r}) = -\frac{1}{4\pi\varepsilon_0}\iint |\vec{\tau}|\,\frac{\hat{n}\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dA'$$

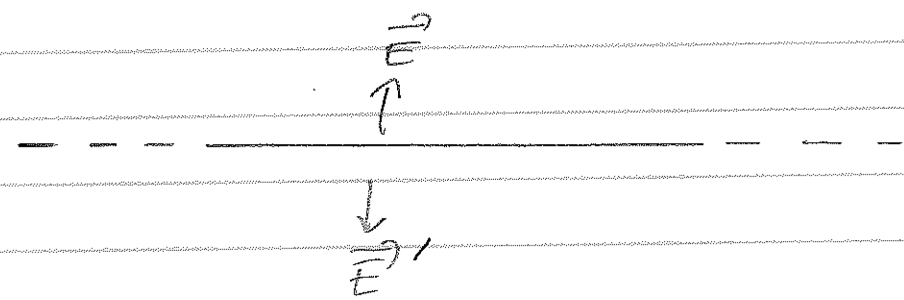
WE'VE SEEN PART OF THE INTEGRAND BEFORE: $\frac{\vec{r}\cdot\hat{n}}{r^2}\,dA = d\Omega$

SO $\Phi(\vec{r}) = -\frac{1}{4\pi\varepsilon_0}|\vec{\tau}|\,\Omega$

NOTICE A GEOMETRIC DISCONTINUITY: THE SOLID ANGLE Ω CHANGES BY 4π AS YOU CROSS THE SURFACE (-2π IN FRONT OF YOU TO $+2\pi$ IN BACK OF YOU). HENCE Φ IS DISCONTINUOUS BY $|\vec{\tau}|/\varepsilon_0$ ON CROSSING THE DIPOLE LAYER.

WHAT ABOUT THE CHANGE IN POTENTIAL ON CROSSING A SURFACE-CHARGE LAYER σ?

APPLY GAUSS'S LAW TO A SMALL PIECE OF THE SURFACE;



$$\Delta \vec{E} \cdot \hat{n} = \sigma / \varepsilon_0$$

$$\Delta \left[(-\nabla \Phi) \cdot \hat{n} \right]_S = \sigma / \varepsilon_0$$

$$-\Delta \left. \frac{\partial \Phi}{\partial n} \right|_S = \sigma / \varepsilon_0$$

HERE, IT'S THE NORMAL DERIVATIVE OF Φ THAT PICKS UP A DISCONTINUITY σ/ε_0 ON CROSSING THE SURFACE-CHARGE LAYER

GREEN'S THEOREM THEREFORE GIVES US IN TERMS OF σ & $\vec{\tau}$

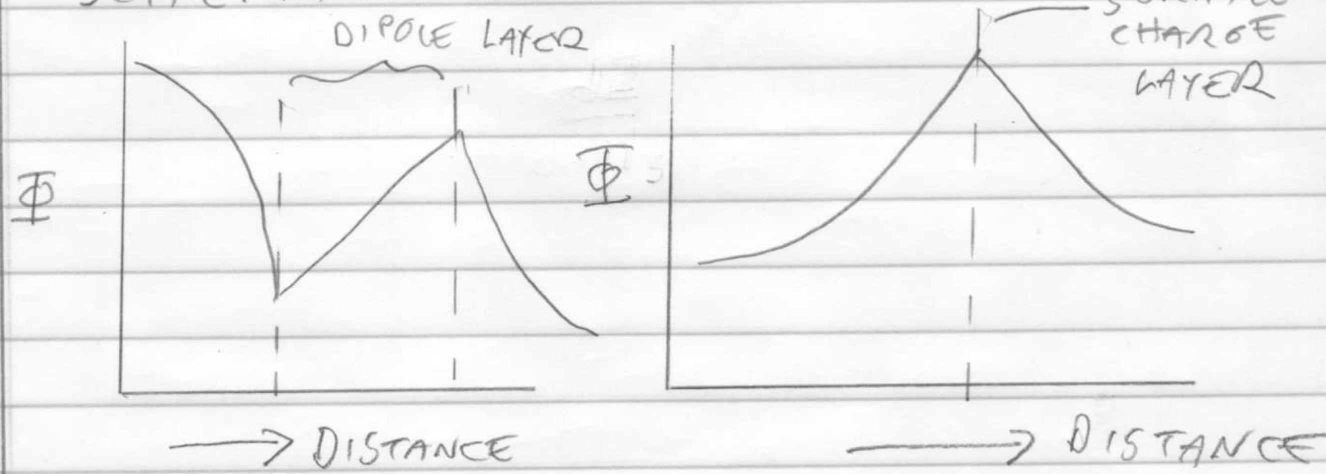
$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV'$$

$$+ \frac{1}{4\pi\varepsilon_0}\iint |\vec{\tau}|\,\frac{(\vec{r}-\vec{r}\,')\cdot\hat{n}}{|\vec{r}-\vec{r}\,'|^3}\,dA'$$

$$+ \frac{1}{4\pi\varepsilon_0}\iint \frac{\sigma(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dA'$$

WITH $\Phi_S = |\vec{\tau}|/\varepsilon_0$, $\left.\partial\Phi/\partial n\right|_S = \sigma/\varepsilon_0$.

SCHEMATICALLY



WE'LL CONSIDER VOLUME DISTRIBUTIONS OF POLARIZATIONS IN THE CONTEXT OF DIELECTRICS.