Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
November 5, 2020, 11am
On-line lecture

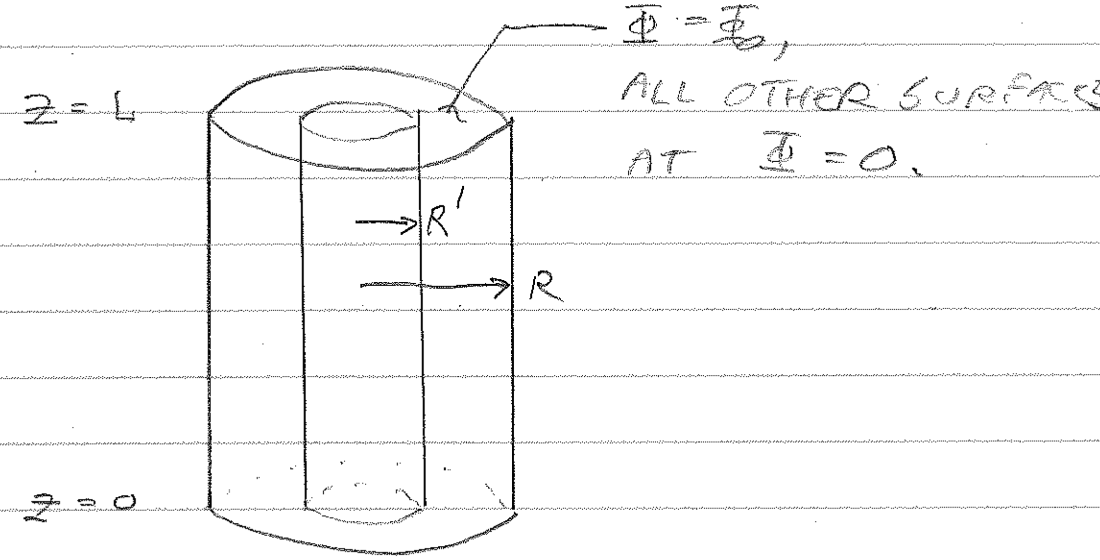
Administrative:

1. No homework this week
2. Ensure you're getting graded homework back.
3. Draft of this lecture is posted at
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
4. Office hours today after class at 12:30.
5. Exam will be posted on the course website this Friday 4 pm PST, it's due this Monday 4 pm PST. See submittal instructions on the exam. See exam information on course website.

Lecture: Multipoles, dielectrics. (Jackson chapter 4).

Section 4.1-2: Comment on the multipole expansion in magnetostatics.
Section 4.3: Macroscopic electric media (dielectrics).
The displacement vector D. Constitutive relations and linear electric media. Boundary conditions at dielectric media.

EXAMPLE: CO-AXIAL CYLINDERS. A CHALLENGING PROBLEM.



LET'S START WITH SOMETHING WRONG. JACKSON EQN. 3.105a

$$\Phi(\rho, \phi, z) = \sum_m \sum_n J_m(k_{mn}\rho) \cdot \left\{ a_{mn} \sinh k_{mn} z + b_{mn} \cosh k_{mn} z \right\} \cdot \left\{ c_{mn} \sin m\phi + d_{mn} \cos m\phi \right\}$$

AZIMUTHAL SYMMETRY $\Rightarrow m = 0$;
BOUNDARY CONDITION AT $Z = 0 \Rightarrow$ SINH

$$\Phi(\rho, z) = \sum_n a_n J_0(k_n \rho) \sinh(k_n z)$$

BOUNDARY CONDITION AT $\rho = R'$;

$$\Phi(\rho = R', z) = 0 = \sum_n a_n J_0(K_n R') \sinh(K_n z)$$

$$\Rightarrow K_n = \frac{x_{0n}}{R'}$$

$$\Phi(\rho, z) = \sum_n a_n J_0\left(\frac{x_{0n}}{R'}\rho\right) \sinh\left(\frac{x_{0n}}{R'} z\right).$$

BOUNDARY CONDITION AT $\rho = R$;

$$\Phi(\rho = R, z) = 0 = \sum_n a_n J_0\left(\frac{x_{0n}}{R'} R\right) \sinh\left(\frac{x_{0n}}{R'} z\right)$$

$$\Rightarrow \frac{x_{0n}}{R'} R = \text{A ZERO OF } J_0.$$

UNFORTUNATELY, THIS ONLY OCCURS FOR SPECIAL VALUES OF R'/R, NOT IN GENERAL.

SO, WE CAN'T SATISFY THE BOUNDARY CONDITIONS.

Q: WHERE DID WE GO WRONG?

(c)

A: THE REGION OF INTEREST EXCLUDES $\rho = 0$, WE CAN THEREFORE BRING IN IRREGULAR SOLUTIONS

$$\Phi(\rho, z) = \sum_n \sinh(k_n z)\left\{a_n J_0(k_n \rho) + b_n N_0(k_n \rho)\right\}$$

$$\Phi(\rho = R', z) = 0 = \sum_n \sinh(k_n z)\left\{a_n J_0(k_n R') + b_n N_0(k_n R')\right\}$$

$$\Rightarrow a_n J_0(k_n R') + b_n N_0(k_n R') = 0.$$

SIMILARLY $\Phi(\rho = R, z) = 0$ LEADS TO

$$\Rightarrow a_n J_0(k_n R) + b_n N_0(k_n R) = 0.$$

COMBINING THEM LEADS TO

$$N_0(k_n R) J_0(k_n R') - J_0(k_n R) N_0(k_n R') = 0.$$

THIS IS A TRANSCENDENTAL FUNCTION, I DON'T KNOW AN ANALYTIC WAY TO SOLVE IT, BUT CALL THE ROOTS k_n.

NOW APPLY THE LAST BOUNDARY CONDITION AT $z = L$, INVOKE ORTHOGONALITY TO FIND a_n AND b_n (via

Q: HOW DO YOU

FINISHING UP THE DISCUSSION OF THE MULTIPOLE EXPANSION.

THIS EXPANSION IS MORE GENERAL THAN ELECTROSTATICS. YOU CAN APPLY IT TO VECTORS. FOR INSTANCE, RECALL FROM MAGNETOSTATICS

$$\vec{A} = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, dv'$$

$$\sim \frac{1}{R} \iiint \vec{J}(\vec{r}\,') \, dv' - \left[\frac{d}{dx_\alpha'}\left(\frac{1}{r}\right)\right]_R \iiint x_\alpha' \, \vec{J}(\vec{r}\,') \, dv' + \ldots$$

IF $\vec{J}$ REPRESENTS STATIONARY (NON-TIME-DEPENDENT) CURRENTS, THERE'S NO MONOPOLE TERM.

YOU COULD INTUIT THIS BY CONCEPTUALLY BY DIVIDING THE CURRENT SYSTEM INTO CURRENT LOOPS:

$$\iiint \vec{J}(\vec{r}\,') \, dv' \Longrightarrow \sum_{\text{LOOPS}} J_{\text{LOOP}} \oint d\vec{\ell}$$

THIS IS ZERO SINCE $\oint d\vec{\ell} = 0$.

MORE FORMALLY

FOR STATIONARY CURRENTS $\vec{\nabla}\cdot\vec{J} = 0$.
Q: WHY? HENCE WE CAN WRITE

$$0 = \iiint (\vec{\nabla}\cdot\vec{J}) x'_\alpha \, dv'$$

$$= \iiint \vec{\nabla}\cdot(x'_\alpha \vec{J}) \, dv' - \iiint (\vec{J}\cdot\vec{\nabla} x'_\alpha) \, dv'$$

$$= \oiint (x'_\alpha \vec{J})\cdot\hat{n}\, dA - \iiint J_\beta \frac{\partial}{\partial x'_\beta} x'_\alpha \, dv'$$

WITH CURRENTS BOUNDED, THE SURFACE INTEGRAL VANISHES.
WITH $\partial x'_\alpha / \partial x'_\beta = \delta_{\alpha\beta}$,

$$\iiint J_\alpha(\vec{r}') \, dv' = 0$$

I'LL LEAVE THIS HERE FOR NOW. WHEN WE GET TO RADIATION AND NON-STATIONARY CURRENTS WE'LL READDRESS THIS.

IT MAY BE THE CASE THAT THERE'S A MONOPOLE VECTOR POTENTIAL, BUT ITS NON-ROTATIONAL AND HENCE WON'T REPRESENT FIELDS.

MACROSCOPIC MEDIA. JACKSON §4.3.
WE STEP BACK FROM THE ATOMIC SCALE.

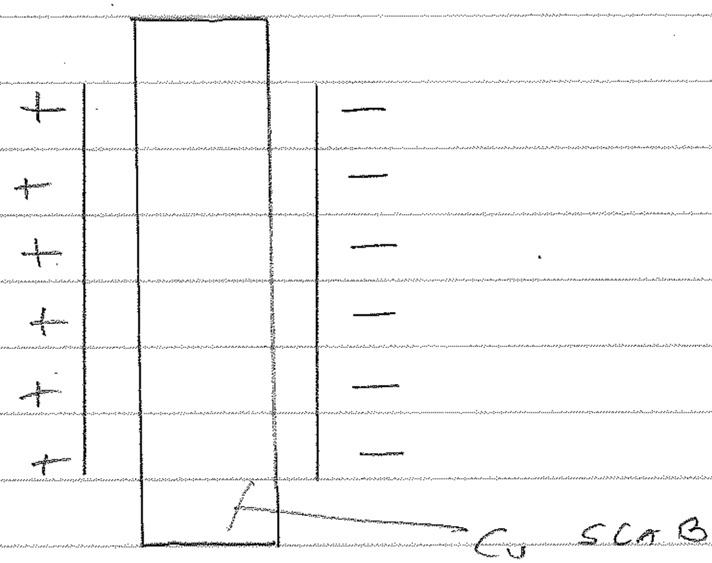
THE GENERAL IDEA IS TO TREAT DISTRIBUTED DIPOLE MOMENTS THROUGHOUT SOME VOLUME AS A SPECIAL SOURCE TERM IN POISSON'S EQUATION; IN OTHER WORDS, WE'D LIKE TO DESCRIBE THE GROSS BEHAVIOR OF A MEDIUM IN AN ELECTRIC FIELD IN TERMS OF ITS "POLARIZATION" (ITS DIPOLE MOMENT PER UNIT VOLUME)

WE START THIS WITH THE CONCEPTUALLY TRICKY CONCEPT OF SEPARATING THE TOTAL CHARGE DENSITY ρ INTO

- A "TRUE" OR "FREE" ρ_F; AND
- A "BOUND" POLARIZATION CHARGE DENSITY ρ_P.

AND, YES, ANY PARTICULAR SEPARATION CONTAINS ARBITRARINESS; SOME OTHER OBSERVER MIGHT CONSIDER SOME ρ_P TO BE ρ_F, ETC.

SOME TEXTS INTRODUCE THIS ARBITRARINESS WITH THE CONCEPTUAL PROBLEM OF SLIPPING A COPPER SLAB BETWEEN CAPACITOR PLATES!



THE FIELD BETWEEN THE PLATES CAN BE DESCRIBED AS DUE TO FREE CHARGES IN THE SLAB, OR, EQUIVALENTLY, IN TERMS OF THE POLARIZATION OF THE COPPER.

IF YOU REPLACE THE COPPER WITH A DIELECTRIC, WE'D ALMOST SURELY PREFER THE POLARIZATION VIEWPOINT (UNLESS WE'RE INTERESTED IN FIELDS AT THE ATOMIC SCALE).

THE "DISPLACEMENT" VECTOR $\vec{D}$
(JACKSON EQN. 4.34).

START WITH STATIC MAXWELL EQUATIONS

$$\nabla^2 \Phi = \frac{-\rho_E - \rho_P}{\varepsilon_0}; \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho_E + \rho_P}{\varepsilon_0}.$$

NOW RECALL FROM LAST LECTURE THE POTENTIAL FROM A POINT DIPOLE $\vec{p}$ AT $\vec{r}\,'$:

$$\Phi^{(2)}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}\,'|}$$

NOW DISTRIBUTE $\vec{p}$ OVER A VOLUME. AT EACH POINT THERE'S A DIPOLE DENSITY $\vec{P}$:

$$\Phi^{(2)}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint \vec{P}(\vec{r}\,')\cdot\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}\,'|}\,dv'$$

APPLY THE IDENTITY

$$\vec{\nabla}'\cdot\left(\frac{\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) = \frac{1}{|\vec{r}-\vec{r}\,'|}\vec{\nabla}'\cdot\vec{P}(\vec{r}\,') + \vec{P}(\vec{r}\,')\cdot\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}\,'|}$$

NOTICE A TOTAL DIVERGENCE ARISES;

$$\Phi^{(2)}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \oiint \frac{\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \cdot \hat{n}\, dA$$

$$- \frac{1}{4\pi\varepsilon_0} \iiint \frac{1}{|\vec{r}-\vec{r}\,'|} \vec{\nabla}' \cdot \vec{P}(\vec{r}\,')\, dv'.$$

THIS FORM IS FAMILIAR FROM

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \oiint \frac{\sigma(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} dA$$

$$- \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dv'.$$

WE CAN READ OFF THE VOLUME AND SURFACE POLARIZATION CHARGES

$$\sigma_p = \vec{P} \cdot \hat{n}; \quad \rho_p = -\vec{\nabla} \cdot \vec{P}.$$

NOW RECALL A MAXWELL EQUATION

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f + \rho_p}{\varepsilon_0}, \quad \text{so}$$

$$\vec{\nabla} \cdot \left\{ \vec{E} + \frac{\vec{P}}{\varepsilon_0} \right\} = \frac{\rho_f}{\varepsilon_0}$$

WE DEFINE THE "DISPLACEMENT" VECTOR $\vec{D}$

$$\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}.$$

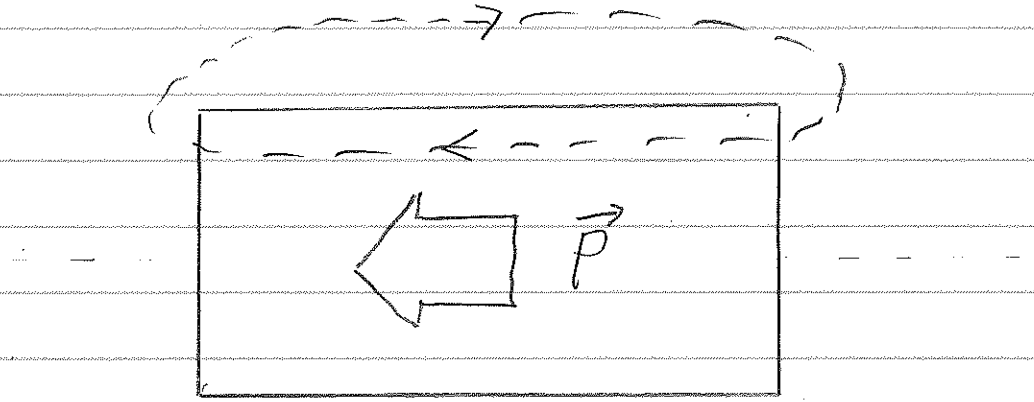
THE ELECTROSTATIC MAXWELL EQUATIONS ARE THEN

$$\vec{\nabla}\cdot\vec{D} = \rho_f \; ; \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho_f + \rho_p}{\varepsilon_0} .$$

$\vec{D}$ HAS SOME PROPERTIES AKIN TO $\vec{E}$, BUT TAKE CARE IN APPLYING GAUSS'S LAW, AND IT'S NOT ALWAYS THE CASE THAT $\vec{\nabla}\times\vec{D} = 0$.

RECALL THE HELMHOLTZ THEOREM FROM THE FIRST PROBLEM SET: A VECTOR FIELD IS (ALMOST) SPECIFIED BY ITS DIVERGENCE AND CURL. FOR $\vec{E}$, THERE IS NO CURL. FOR $\vec{D}$ THERE MAY BE A CURL.

EXAMPLE: BAR ELECTRET (THE ELECTRIC DUAL OF A BAR MAGNET). ASSUME THE BAR HAS UNIFORM AXIAL POLARIZATION $\vec{P}$.



Q: IS $\vec{P}$ "ROTATIONAL"? (DOES IT HAVE A CURL?)

A: OBVIOUSLY $\oint \vec{P} \cdot d\vec{l} \neq 0$ FOR THE DOTTED PATH ABOVE. HENCE $\vec{\nabla} \times \vec{P} \neq 0$ SOMEWHERE.

SINCE $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ AND $\vec{\nabla} \times \vec{E} = 0$, IT MUST BE $\oint \vec{D} \cdot d\vec{l} \neq 0$ ON THAT DOTTED PATH. HENCE $\vec{\nabla} \times \vec{D} \neq 0$ SOMEWHERE.

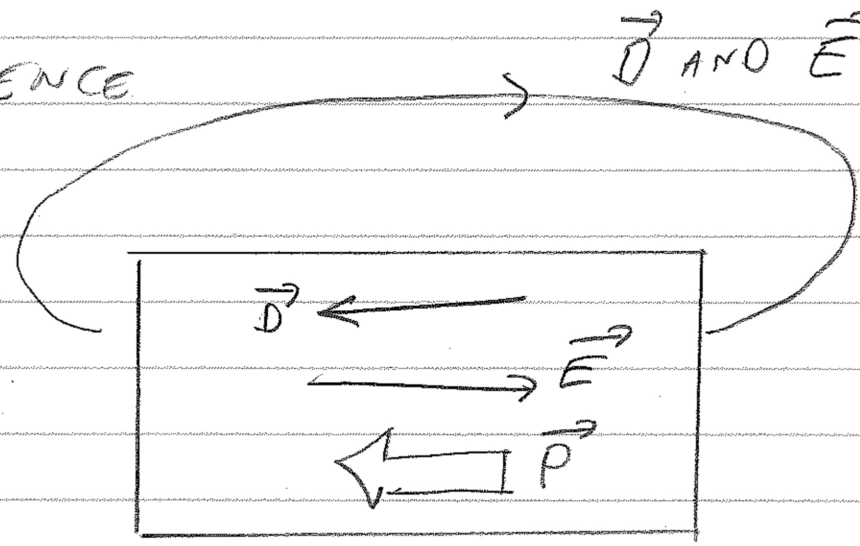
WHICH WAY DO $\vec{E}$ AND $\vec{D}$ POINT?

OUTSIDE THE BAR $\vec{P}=0$ AND $\vec{E}$ AND $\vec{D}$ ARE THE SAME (UP TO ε_0).

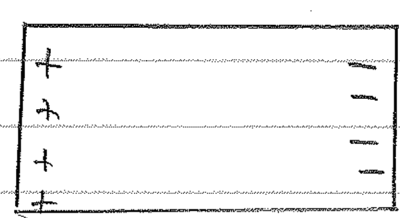
WE MUST HAVE $\oint \vec{E}\cdot d\vec{l} = 0$ (STATICS), SO THE CONTRIBUTION INSIDE THE BAR CANCELS THAT OUTSIDE THE BAR.

Q: DOES $\vec{D}$ CHANGE SIGN AT THE ENDS?
A: NO.

HENCE



THIS IS DUE TO THE POLARIZATION SURFACE CHARGE

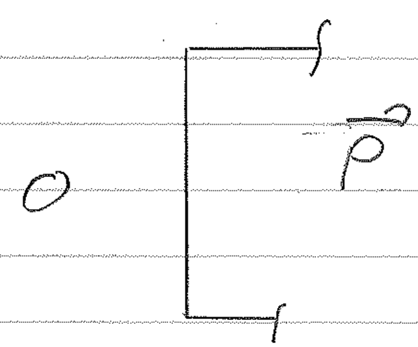


HERE WE HAVE $\oint \vec{D} \cdot d\vec{l} \neq 0$, SO $\vec{\nabla} \times \vec{D} \neq 0$ SOMEWHERE. WHERE IS $\vec{\nabla} \times \vec{D} \neq 0$.

MAYBE INSIDE?
INSIDE $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$
THIS HAS $\vec{\nabla} \times \vec{D} = 0$
SINCE $\vec{\nabla} \times \vec{E} = 0$ AND
$\vec{P}$ IS CONSTANT.

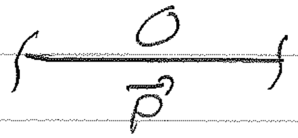
MAYBE OUTSIDE?
OUTSIDE $\vec{D} = \varepsilon_0 \vec{E}$ $(\vec{P} = 0)$,
THIS HAS $\vec{\nabla} \times \vec{D} = 0$.

MAYBE AT THE ENDS?



AS YOU MOVE AXIALLY, THE AXIAL VECTOR $\vec{P}$ CHANGES FROM $\vec{P}$ TO ZERO. BUT THERE'S NO CURL.
Q: WHY?

MAYBE THE CYLINDRICAL SIDES?



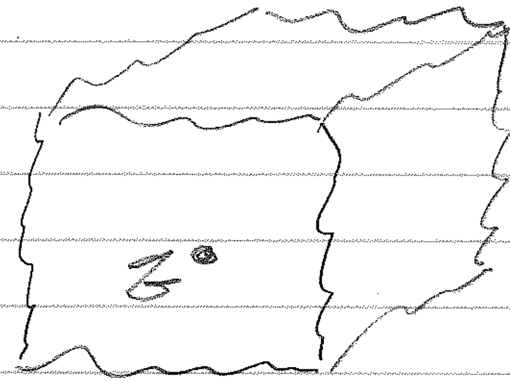
AS YOU MOVE VERTICALLY, THE AXIAL VECTOR CHANGES DRAMATICALLY AS YOU CROSS THE SURFACE. THIS INTRODUCES A CURL.

Q: WHY.

SO THE LOOP INTEGRAL $\oint \vec{D} \cdot d\vec{l}$ FOR THE DOTTED PATH SHOWN PREVIOUSLY IS THE BOUNDARY OF A SURFACE WITH A NON-ZERO CURL, AS YOU KNEW IT DID.

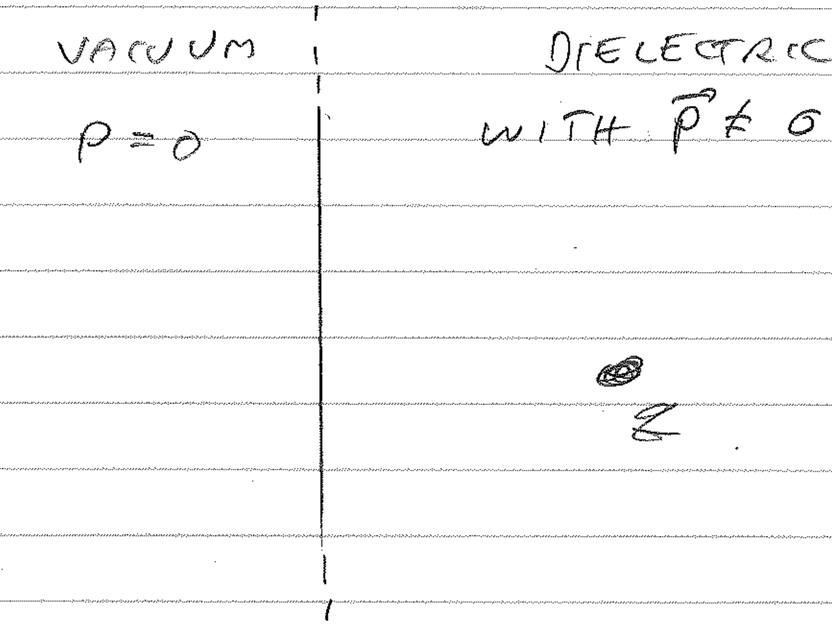
EXAMPLES OF GAUSS'S LAW IN DIELECTRICS

• POINT CHARGE IN INFINITE DIELECTRIC



FROM SYMMETRY, $\vec{D}$ IS RADIAL, SO $\vec{\nabla} \times \vec{D} = 0$. HERE, $\vec{D}$ IS (ALMOST) COMPLETELY SPECIFIED BY $\vec{\nabla} \cdot \vec{D}$, WHICH IS GAUSS'S LAW.

• POINT CHARGE IN SEMI-INFINITE DIELECTRIC.



HERE, YOU CAN'T INVOKE SYMMETRY TO ARGUE $\vec{D}$ IS RADIAL. IT'S TRUE $\vec{\nabla} \cdot \vec{D} = \rho_f$, BUT THIS DOESN'T FULLY SPECIFY $\vec{D}$.

WE'RE NOT QUITE DONE. WE NEED TO FIND FROM AN ENGINEER THE "CONSTITUTIVE RELATIONS" BETWEEN $\vec{D}(\vec{E})$ AND $\vec{E}$. THAT IS, HOW DOES $\vec{D}$ DEPEND ON THE EXTERNAL FIELD $\vec{E}$?

WE EARLIER LOOKED AT THE ELECTRET. IT HAS CONSTANT POLARIZATION:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}.$$

NOW, $\vec{P}$ COULD HAVE BEEN ZERO IN THE PAST, THEN $\vec{P}$ GOT FROZEN IN. TO FIND $\vec{D}$ HERE, YOU NEED THE HISTORY OF THE SYSTEM: SIMPLY KNOWING THE APPLIED $\vec{E}$ ISN'T ENOUGH.

THIS IS VERY DIFFERENT THAN THE SIMPLE "LINEAR DIELECTRIC".

THE LINEAR DIELECTRIC HAS PROPERTY

$\vec{P} = \varepsilon_0 \chi_E \vec{E}$;

$\vec{P} \sim \vec{E}$

χ_E IS AN "ENGINEERING" NUMBER, THE "ELECTRIC SUSCEPTIBILITY", NO TIME DEPENDENCE! TURN OFF $\vec{E}$ AND NO $\vec{P}$.

THIS WOULD NOT DESCRIBE THE ELECTRET.

THUS $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \underbrace{\varepsilon_0 (1 + \chi_E)}_{\text{FOR LINEAR DIELECTRIC}} \vec{E}$

THE "RELATIVE DIELECTRIC CONSTANT" (DIMENSIONLESS, THE SAME IN ALL SYSTEMS OF UNITS) IS

$\varepsilon_r = 1 + \chi_E$.

(SMYTHE CALLS THIS THE "SPECIFIC INDUCTIVE CAPACITY").

THE "DIELECTRIC CONSTANT" IS

$\varepsilon = \varepsilon_0 (1 + \chi_E)$.

THIS "LINEAR DIELECTRIC" FORMALISM IS SENSIBLE FOR (MOST) LIQUIDS AND AMORPHOUS SOLIDS AND CUBIC CRYSTALS. THAT IS, THE MEDIUM IS ISOTROPIC, AMONG OTHER THINGS.

BUT SOME MATERIALS, e.g., CRYSTALS WITH SYMMETRY LOWER THAN CUBIC, HAVE EACH SPATIAL COMPONENT A LINEAR DIELECTRIC BUT THE DIFFERENT DIRECTIONS COULD HAVE DIFFERENT DIELECTRIC CONSTANT. e.g.,

$$P_\alpha = \varepsilon_0 \chi_{E_{\alpha\beta}} E_\beta ,$$

χ_E IS A TENSOR.

THIS ALLOWS FOR $\vec{P}$ TO BE IN A DIFFERENT DIRECTION THAN $\vec{E}$. YOU MAY HAVE SEEN SIMILIAR OCCURANCES IN THE ELASTIC PROPERTIES OF SOLIDS.

AND FOR THE SAME REASON THAT THE ELASTIC MODULUS IN MECHANICS IS SYMMETRIC, THE SUSCEPTIBILITY TENSOR IS SYMMETRIC.

WE'LL COME BACK TO THIS.

BOUNDARY-VALUE PROBLEMS WITH DIELECTRICS. JACKSON §4.4

BOUNDARY CONDITIONS: LIKE THE CASE OF FIELDS NEAR CHARGE DENSITIES, WHERE WE DIVIDED CHARGES INTO DISCONTINUOUS AND CONTINUOUS VARIATIONS (AT, e.g., THE SURFACE AND IN THE BULK VOLUME), χ_E AND ε MAY BE ARBITRARY FUNCTION OF COORDINATES: THEY MAY CHANGE CONTINUOUSLY OR ABRUPTLY. AN IMPORTANT CASE IS WHERE ε CHANGES DISCONTINUOUSLY, SAY AT A BOUNDARY BETWEEN TWO DIFFERENT DIELECTRICS.