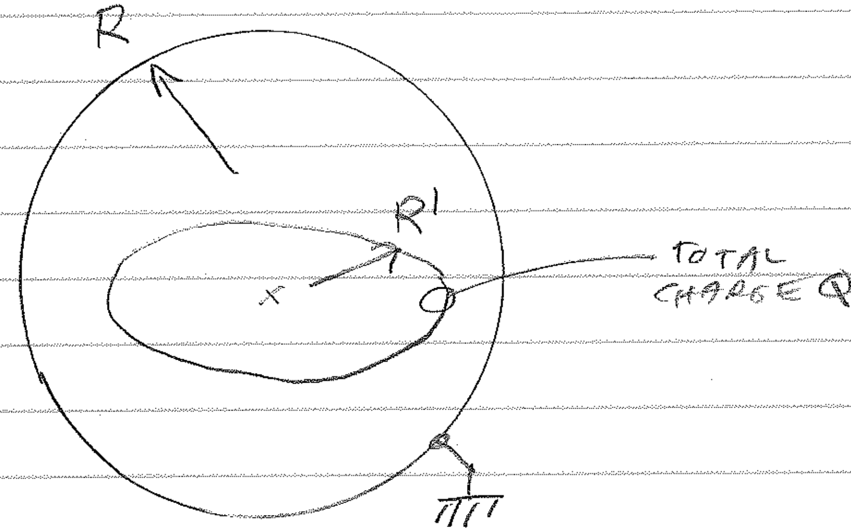
Physics 513, Electrodynamics I
Department of Physics, University of Washington
Autumn quarter 2020
November 3, 2020, 11am PST
On-line lecture

Administrative:

1. No homework this week.
2. Homework 4 draft solutions posted on
`faculty.washington.edu/ljrberg/AUT20_PHYS513`
3. Exam posted this Friday 4 pm PST, due this Monday 4 pm PST. You should spend 1 hour 20 minutes on the exam.
4. Exam information posted tomorrow, Wednesday.

Lecture: Multipoles, dielectrics. (Jackson chapter 4).
Section 4.1: Multipole expansion I: Rectangular-coordinate expansion. Forms of the quadrupole-moment tensor.
Section 4.1: Multipole expansion II: Spherical-coordinate expansion. Quadrupole-moment tensor revisited.

COMMENTS ON HW4, PROBLEM 1.



WE DID THIS IN LECTURE AND IT'S OUTLINED IN JACKSON AS A GREEN'S FUNCTION PROBLEM.

THE HOMEWORK CONSIDERS YOU APPROACH THIS WITH SUPERPOSITION. HERE ARE THE PIECES.

1. YOU ALREADY KNOW THE "INTERIOR" AND EXTERIOR" POTENTIAL OF THE CHARGED RING: WE DID IT IN LECTURE AND IT'S OUTLINED IN JACKSON.

2. THE GROUNDED SHELL IS A RESULT OF THE SUPERPOSITION OF TWO POTENTIALS; a) THE POTENTIAL OF THE RING AND (b) THE POTENTIAL DUE TO INDUCED SURFACE CHARGE ON THE SHELL.

OK. NOW TO CARRY THIS OUT.

THE "EXTERIOR" POTENTIAL OF DUE TO THE CHARGED RING IS

$$\Phi(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{\ell} \left(\frac{R'}{r}\right)^{\ell+1} P_\ell(0) P_\ell(\cos\theta) \qquad r > R'$$

$$P_\ell(0) = 0 \quad \text{FOR } \ell \text{ ODD}$$

$$P_\ell(0) = (-1)^{\ell/2} \frac{1 \cdot 3 \cdot \cdots (\ell-1)}{2 \cdot 4 \cdot \cdots \ell}$$

(SEE JACKSON P. 123)

Q: WHY IS $P_\ell(0) = 0$ FOR ℓ ODD?

(c)

THE "INTERIOR" POTENTIAL DUE TO THE INDUCED SURFACE CHARGE IS GENERICALLY

$$\Phi(r,\theta) = \frac{Q}{4\pi\varepsilon_0} \sum_{\ell} a_\ell \, r^\ell P_\ell(\cos\theta)$$

$a_\ell = 0$ FOR ℓ ODD

Q! WHY?

Q! WHAT HAPPENED TO $1/r^{\ell+1}$ TERMS?

Q! WHY ARE THESE $\{a_\ell\}$ THE SAME AS THOSE IN THE EXPANSION OF THE RING'S POTENTIAL?

NOT ONLY MUST THE SUM OF THE TWO POTENTIALS VANISH AT THE SURFACE, BUT THE SUM OF EACH PAIR OF TERMS IN THE EXPANSION MUST VANISH AS WELL.

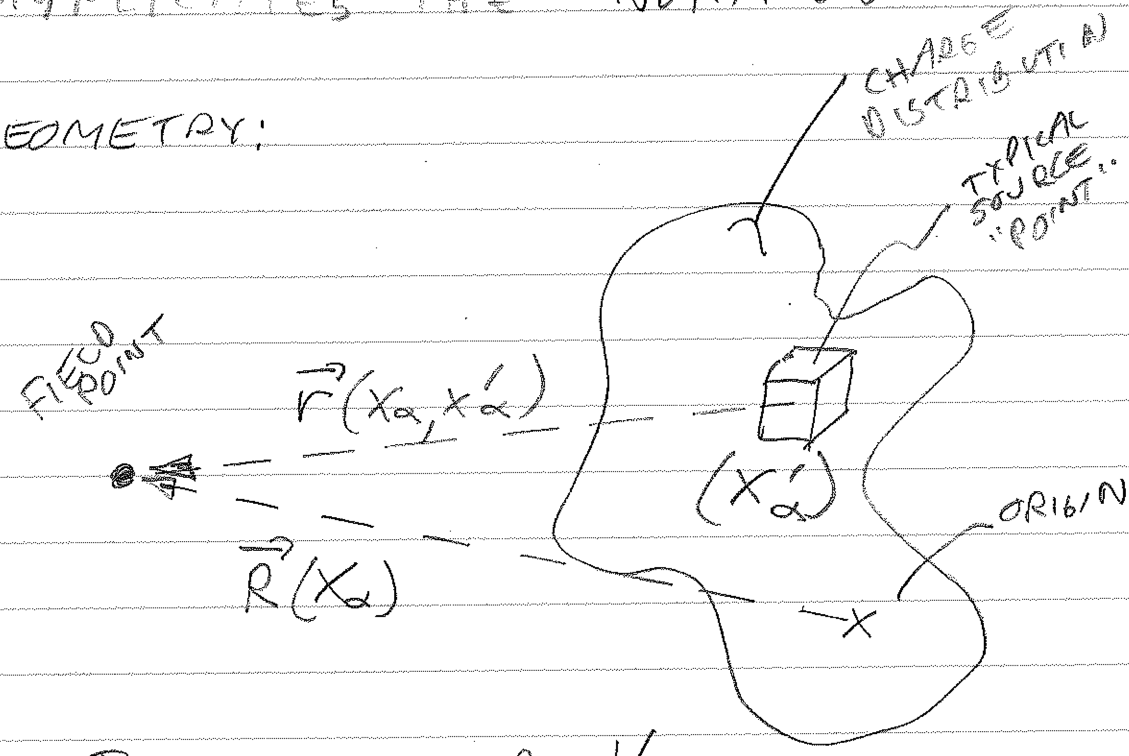
WE ARE DONE.

MULTIPOLE EXPANSION I: RECTANGULAR COORDINATES (TRADITIONAL APPROACH), JACKSON §4.1

THE POINT OF THIS SECTION IS TO APPLY THE $1/R$ EXPANSION TO A CHARGE DISTRIBUTION (AND, LATER, A CURRENT DISTRIBUTION).

THE RESULTING "MULTIPOLE MOMENTS" ARE EXPRESSED AS A TENSOR IN RECTANGULAR COORDINATES: THIS COMPLICATES THE NOTATION

GEOMETRY:



WE TAYLOR-EXPAND $1/r$ IN POWERS OF (x'_α). WE ASSUME THE FIELD POINT IS DISTANT.

IN TENSOR NOTATION:

$$\frac{1}{r} = \frac{1}{R} + x'_\alpha \left[\frac{\partial}{\partial x'_\alpha} \left(\frac{1}{r} \right) \right]_{r=R}$$

SUMMED

$$+ \frac{1}{2!} x_\alpha x_\beta \left[\frac{\partial^2}{\partial x_\alpha ' \partial x_\beta '} \left(\frac{1}{r} \right) \right]_{r=R}$$

$$+ \ldots$$

NOTICE "SUMMATION NOTATION". E.g.,

$$x'_\alpha \left[\frac{\partial}{\partial x_\alpha '} \left(\frac{1}{r} \right) \right]_{r=R}$$

$$\rightarrow \sum_\alpha x_\alpha ' \left[\frac{\partial}{\partial x_\alpha '} \left(\frac{1}{r} \right) \right]_{r=R}$$

"REPEATED INDICES ARE SUMMED."

$$\rightarrow x' \left[\frac{\partial}{\partial x'} \left(\frac{1}{r} \right) \right]_{r=R}$$

$$+ y' \left[\frac{\partial}{\partial y'} \left(\frac{1}{r} \right) \right]_{r=R}$$

$$+ z' \left[\frac{\partial}{\partial z'} \left(\frac{1}{r} \right) \right]_{r=R}.$$

OK, THIS IS A 1/r EXPANSION IN TENSOR NOTATION.

NOW APPLY THIS EXPATION TO THE POTENTIAL DUE TO A CHARGE DISTRIBUTION!

$$\Phi(x_\alpha) = \frac{1}{4\pi\epsilon_0}\iiint \frac{\rho(x'_\alpha)}{r}\,dV'$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{R}\iiint \rho(x'_\alpha)\,dV'$$

NOT SUMMED

$$+\frac{1}{4\pi\epsilon_0}\left[\frac{d}{dx'_\alpha}\left(\frac{1}{r}\right)\right]_{r=R}\iiint x'_\alpha\,\rho(x'_\alpha)\,dV'$$

SUMMED

$$+\frac{1}{4\pi\epsilon_0}\left[\frac{d^2}{dx'_\alpha\,dx'_\beta}\left(\frac{1}{r}\right)\right]_{r=R}\iiint x'_\alpha x'_\beta\,\rho(x'_\alpha)\,dV'$$

NOT SUMMED

Q: WHY ARE SOME QUANTITIES NOT SUMMED, ABOVE?

A: AFTER PERFORMING THE SUM IN THE $1/r$ EXPANSION, THERE ARE NO INDICIES. THIS IS "CONTRACTION". E.g., THIS OBJECT HAS NO INDICIES

$$x'_\alpha\left[\frac{d}{dx'_\alpha}\left(\frac{1}{r}\right)\right]_{r=R}.$$

SOME OF THE EXPANSION FACTORS LOOK FAMILIAR:

- $\iiint \rho(x_\alpha')\, dv'$ IS THE TOTAL CHARGE.

HENCE

$$\frac{1}{4\pi\varepsilon_0}\,\frac{1}{R}\iiint \rho(x_\alpha')\, dv'$$ IS THE

POTENTIAL DUE TO ALL THE CHARGE CONCENTRATED AT THE ORIGIN.

- $$\frac{1}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial x_\alpha'}\left(\frac{1}{r}\right)\right]_{r=R}\iiint x_\alpha'\,\rho(x_\alpha')\, dv'$$

$$\iiint x_\alpha'\,\rho(x_\alpha')\, dv'$$

IS A COMPONENT OF

THE ELECTRIC DIPOLE MOMENT

(THE CHARGE, WEIGHTED BY ITS POSITION), SO THE TERM IS

$$\frac{1}{4\pi\varepsilon_0}\,\vec{\nabla}\left(\frac{1}{r}\right)\cdot\vec{p}$$

NOTICE THE VOLUME INTEGRALS ARE JUST "NUMBERS"! THEY ARE CALLED "MOMENTS" OF THE CHARGE DISTRIBUTION. E.G.,

$\iiint \rho \, dv'$ IS THE TOTAL CHARGE (THE "MONOPOLE MOMENT").

$\iiint x_\alpha' \rho \, dv'$ IS THE α^{th} COMPONENT OF THE DIPOLE MOMENT (A VECTOR)

$\iiint x_\alpha' x_\beta' \rho \, dv'$ IS THE α^{th}, β^{th} COMPONENT OF THE QUADRUPOLE MOMENT (A TENSOR)

⋮

ETC.

JACKSON USES THE WORDS "DIPOLE", "QUADRUPOLE", ETC. IN DIFFERENT CONTEXTS:

- TO DESCRIBE A CERTAIN CHARGE (AND, LATER CURRENT) DISTRIBUTION.

- AS MOMENTS (AS ABOVE) OF A CHARGE (OR CURRENT) DISTRIBUTION.

WHAT, IN ALL THIS DISCUSSION, DEPENDS ON THE CHOICE OF ORIGIN

- TOTAL CHARGE (THE MONOPOLE MOMENT) IS INDEPENDENT OF THE CHOICE OF ORIGIN.
  Q: SHOW THIS (IT'S TRIVIAL).

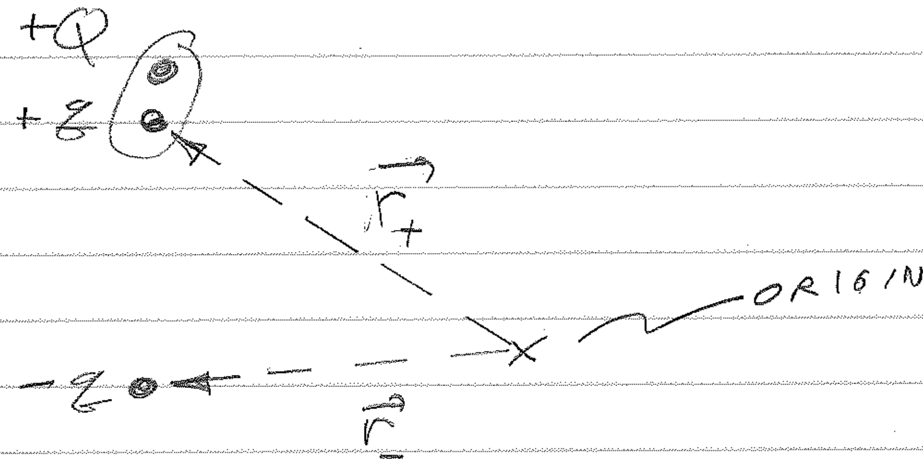
- THE DIPOLE MOMENT, IN GENERAL, DOES DEPEND ON THE CHOICE OF ORIGIN. BUT, IF THE TOTAL CHARGE VANISHES, THEN THE DIPOLE MOMENT IS INDEPENDENT OF ORIGIN.

- THE QUADRUPOLE MOMENT IS, IN GENERAL, DEPENDENT ON THE CHOICE OF ORIGIN. BUT IF THE TOTAL CHARGE AND DIPOLE MOMENT VANISH, THEN THE QUADRUPOLE MOMENT IS INDEPENDENT OF ORIGIN.

⋮

- THE LOWEST NON-ZERO MOMENT IS INDEPENDENT OF ORIGIN.

EXAMPLE: DIPOLE MOMENT AND CHOICE OF ORIGIN.



$$\vec{p} = \iiint \vec{r}\,' \rho(\vec{r}\,')\, dr'$$

$$= \iiint \vec{r}\,' \left\{ +q\,\delta(\vec{r}\,' - \vec{r}_+) + Q\,\delta(\vec{r}\,' - \vec{r}_+) - q\,\delta(\vec{r}\,' - \vec{r}_-) \right\} dr'$$

$$= q(\vec{r}_+ - \vec{r}_-) + Q\vec{r}_+$$

NOTICE $\vec{r}_+ - \vec{r}_-$ IS INDEPENDENT OF ORIGIN, WHILST $\vec{r}_+$ DEPENDS ON THE CHOICE OF ORIGIN.

THE DIPOLE MOMENT DEPENDS ON THE CHOICE OF ORIGIN UNLESS $Q = 0$.

THERE'S A MORE GENERAL PROOF OF THIS. START WITH SOME ORIGIN.

$$\vec{p} = \iiint \vec{r}' \rho(\vec{r}')\, dv'.$$

NOW, DISPLACE THAT ORIGIN BY $\vec{R}$:

$$\vec{p} = \iiint (\vec{r}' - \vec{R}) \rho(\vec{r}')\, dv'.$$

Q: WHY DID I WRITE IT THIS WAY, WITH $(\vec{r}' - \vec{R})$ AND $\rho(\vec{r}')$ AND dv'?

$$\vec{p} = \iiint \vec{r}' \rho(\vec{r}')\, dv'$$

$$- \vec{R} \iiint \rho(\vec{r}')\, dv'$$

THE 2^{ND} INTEGRAL IS THE TOTAL CHARGE. IF THE TOTAL CHARGE VANISHES, THE $\vec{p}$ IS UNCHANGED ON MOVING THE ORIGIN.

ON TO THE QUADRUPOLE AND THE CONFUSING ROLE OF PRINCIPAL AXES.

$$Q_{\alpha\beta} = \iiint x'_\alpha x'_\beta \, \rho(\vec{r}') \, dv'$$

CAVEAT! SOME MULTIPLY $Q_{\alpha\beta}$ BY $1/4\pi\epsilon_0$ THEN TAKE $1/4\pi\epsilon_0$ OUT OF THE TERM IN THE POTENTIAL.

Q! HOW MANY INDEPENDENT PARAMETERS ARE NEEDED TO FIX $Q_{\alpha\beta}$?

A! NOTICE $Q_{\alpha\beta}$ IS SYMMETRIC AND REAL (OR AT LEAST ALL COMPONENTS HAVE THE SAME PHASE) SO, A REAL SYMMETRIC TENSOR IS DESCRIBED BY SIX PARAMETERS. (Q! WHY?)

HOWEVER, JUST AS FOR THE QUADRUPOLE MOMENT TENSOR IN MECHANICS, THE TENSOR IS USUALLY EXPRESSED IN ITS PRINCIPAL AXES (OR, WE SAY IT'S "DIAGONALIZED").

Q: WITH $Q_{\alpha\beta}$ EXPRESSED IN ITS PRINCIPAL AXES FORM, HOW MANY INDEPENDENT PARAMETERS ARE NEEDED TO DESCRIBE $Q_{\alpha\beta}$?

A: IT'S TRICKY
— THE ANSWER IS 2.

Q: SUPPOSE THE CHARGE DISTRIBUTION HAS AN AXIS OF SYMMETRY. WITH $Q_{\alpha\beta}$ EXPRESSED IN ITS PRINCIPAL AXES FORM, HOW MANY INDEPENDENT PARAMETERS ARE NEEDED TO DESCRIBE $Q_{\alpha\beta}$?

A: IT'S TRICKY.
— THE ANSWER IS 1: THE ECCENTRICITY OF THE CHARGE DISTRIBUTION.

HOW DID WE ARRIVE AT THOSE CONCLUSIONS? (IT'S TRICKY.)

STEP I: SHOW THAT $Q_{\alpha\beta}$ CAN BE WRITTEN IN THE FORM (JACKSON EQN. 4.9 AND ON THE NEXT HOMEWORK)

$$Q_{\alpha\beta} = \frac{1}{3}\iiint \left(3X'_\alpha X'_\beta - r'^2 \delta_{\alpha\beta}\right) \rho(\vec{r}')\, dv'$$

(HINT: APPLY $\hat{X}_\alpha \hat{X}_\beta \delta_{\alpha\beta} = \hat{X}_\beta \hat{X}_\beta = 1$.)

HENCE, THE QUADRUPOLE TERM OF THE POTENTIAL IS

$$\Phi^{(4)} = \frac{1}{6}\left[\frac{d^2}{dX'_\alpha dX'_\beta}\left(\frac{1}{r}\right)\right]_{r=R} \cdot \left(3Q_{\alpha\beta} - \delta_{\alpha\beta}\, \mathrm{Tr}\, Q\right)$$

Q: HOW DID $\iiint r'^2 \delta_{\alpha\beta}\, \rho(\vec{r}')\, dv'$ BECOME $\delta_{\alpha\beta}\, \mathrm{Tr}\, Q$?

STEP 2: DOES THE $\delta_{\alpha\beta}$ TERM CONTRIBUTE TO $\Phi^{(4)}$?

IT HAD BETTER NOT! SETTING THIS TERM TO ZERO RETURNS THE ORIGINAL FORM OF $\Phi^{(4)}$.

$$\delta_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha' \partial x_\beta'} \left(\frac{1}{r}\right) = \nabla^2 \frac{1}{r} = 0.$$

SO, IT DOESN'T CONTRIBUTE.

HENCE, WE ARE FREE TO REDEFINE THE ~~A~~ QUADRUPOLE MOMENT TENSOR $Q_{\alpha\beta}'$ AS

$$\frac{Q_{\alpha\beta}'}{3} = 3Q_{\alpha\beta} - \delta_{\alpha\beta} \, \mathrm{TR}\, Q$$

NOTICE THIS IS TRACELESS.

REFERENCED TO ITS PRINCIPAL AXES, A SYMMETRIC TENSOR HAS 3 COMPONENTS. BUT THE TRACELESS REQUIREMENT ELIMINATES 1, SO THERE'S ONLY 2 INDEPENDENT COMPONENTS. IT'S TRICKY.

MULTIPOLE EXPANSION II: VIA SPHERICAL HARMONICS. (JACKSON EQNS. 4.3–6.)

AGAIN ASSUME THE FIELD POINT IS OUTSIDE OF AND FAR FROM THE CHARGE DISTRIBUTION.

THE GENERIC EXPANSION OF A POTENTIAL IS

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell m} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{1}{r^{\ell+1}} Y_\ell^m(\theta,\phi)$$

(SEE JACKSON EQN. 3.61; WE DISCARDED r^ℓ SOLUTIONS.)

AT THIS POINT, $\{q_{\ell m}\}$ ARE SIMPLY THE EXPANSION COEFFICIENTS. IN THE CONTEXT IN THIS CHAPTER, THIS FORM OF $\Phi(\vec{r})$ IS CALLED THE "MULTIPOLE EXPANSION".

WE'D LIKE TO FIND $\{q_{\ell m}\}$ IN TERMS OF THE CHARGE DISTRIBUTION (AND, LATER, CURRENT DISTRIBUTION).

WE'LL COMBINE THE 1/R EXPANSION (WHERE THE SOURCE POINTS $\vec{r}'$ ARE INSIDE THE CHARGE DISTRIBUTION)

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell m} \frac{4\pi}{2\ell+1} \frac{r'^{\ell}}{r^{\ell+1}} Y_\ell^{m*}(\theta',\phi') Y_\ell^m(\theta,\phi)$$

WITH

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \, dv'$$

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell m} \frac{4\pi}{2\ell+1} Y_\ell^m(\theta,\phi) \frac{1}{r^{\ell+1}}$$

$$\cdot \underbrace{\iiint Y_\ell^{m*}(\theta',\phi') \, r'^{\ell} \rho(\vec{r}') \, dv'}$$

WE IDENTIFY THE INTEGRAL AS $q_{\ell m}$.

$$q_{\ell m} = \iiint Y_\ell^{m*}(\theta',\phi') \, r'^{\ell} \rho(\vec{r}') \, dv'$$

IS THE "MULTIPOLE MOMENT" IN SPHERICAL COORDINATES.

SOME LOW-ORDER CASES

$$q_{00} = \iiint Y_0^{0*}(\theta', \phi')\, \rho(\vec{r}\,')\, dv'$$

LOOKING AT JACKSON P. 109,

$$Y_0^0 = 1/\sqrt{4\pi}$$

$$q_{00} = \frac{1}{\sqrt{4\pi}} \times \text{TOTAL CHARGE}$$

$$q_{10} = \iiint Y_1^{0*}(\theta', \phi')\, r'\, \rho(\vec{r}\,')\, dv'$$

$$= \iiint \sqrt{\frac{3}{4\pi}} \cos\theta'\, r'\, \rho(\vec{r}\,')\, dv'$$

$$= \iiint \sqrt{\frac{3}{4\pi}}\, z'\, \rho(\vec{r}\,')\, dv'$$

$$= \sqrt{\frac{3}{4\pi}}\, P_z$$

$$q_{11} = \iiint Y_1^{1*}(\theta', \phi')\, r'\, \rho(\vec{r}\,')\, dv'$$

$$= -\iiint \sqrt{\frac{3}{8\pi}} \sin\theta'\, e^{-i\phi'}\, r'\, \rho(\vec{r}\,')\, dv'$$

$$= -\iiint \sqrt{\frac{3}{8\pi}} \sin\theta' \left\{\cos\phi' - i\sin\phi'\right\} r' \rho(\vec{r}\,')\, dv'$$

$$= -\sqrt{\frac{3}{8\pi}}\, (P_x - iP_y)$$

etc. SEE JACKSON EQNS. 4.4-6.

NOTE ESPECIALLY THE $\ell = 2$ TERMS CONTAIN THE (TRACELESS) "QUADRUPOLE MOMENT TENSOR" (JACKSON EQN. 4, 9)

$$Q_{\alpha\beta} = \iiint \left(3x_\alpha' x_\beta' - r'^2 \delta_{ij}\right) \rho(\vec{r}\,') \, dV'$$

WE'VE SEEN THIS BEFORE. IT'S INTERESTING THAT EVEN THOUGH WE EXPANDED THE POTENTIAL IN SPHERICAL COORDINATES, THE MULTIPOLE MOMENTS $q_{\ell m}$ ARE MORE READILY EXPRESSED IN CARTESIAN FORM.

EXAMPLE: IN THE SPHERICAL MULTIPOLE FORMALISM, FIND THE FIELDS OF A DIPOLE.

IN GENERAL, OUTSIDE THE SOURCES

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell m}\frac{4\pi}{2\ell+1} q_{\ell m} Y_\ell^m(\theta,\phi)\frac{1}{r^{\ell+1}}.$$

THE FIELDS ARE OBTAINED FROM THE GRADIENT

$$\vec{\nabla}\Phi = \hat{r}\frac{\partial}{\partial r}\Phi + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\Phi + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\Phi.$$

THE GRADIENTS ARE (JACKSON EQN. 4.11)

$$E_r = \vec{\nabla}\Phi\cdot\hat{r} = \frac{1}{4\pi\epsilon_0}\sum_{\ell m}\frac{4\pi}{2\ell+1} q_{\ell m} Y_\ell^m(\theta,\phi)\frac{\ell+1}{r^{\ell+2}}$$

AND SIMILARLY FOR E_θ AND E_ϕ.

FOR A DIPOLE WITH $\vec{p}$ ALIGNED ALONG $\hat{z}$, $\ell=1$, $m=0$.

$q_{11} = 0$ (SINCE $p_x = 0$ AND $p_y = 0$);

$q_{1,-1} = 0$

$q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z$

(SEE JACKSON EQNS. 4.4–6.).

HENCE

$$E_r = \frac{1}{4\pi\varepsilon_0} \frac{4\pi}{3} \sqrt{\frac{3}{4\pi}}\, p_z \sqrt{\frac{3}{4\pi}} \cos\theta \frac{2}{r^3}$$

$$= \frac{2}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^3}.$$

SIMILARLY

$$E_\theta = \frac{1}{4\pi\varepsilon_0} \frac{p\sin\theta}{r^3}$$

$$E_\phi = 0.$$

THIS RESULT SHOULD BE FAMILIAR FROM UNDERGRADUATE ELECTRODYNAMICS.