PHYS 513
OCT 11, 2019

SPECIAL LECTURE I,

EXAMPLE OF SOLUTION OF 2D LAPLACE'S EQUATION VIA COMPLEX ANALYSIS.

SOME POINTS FROM COMPLEX ANALYSIS:

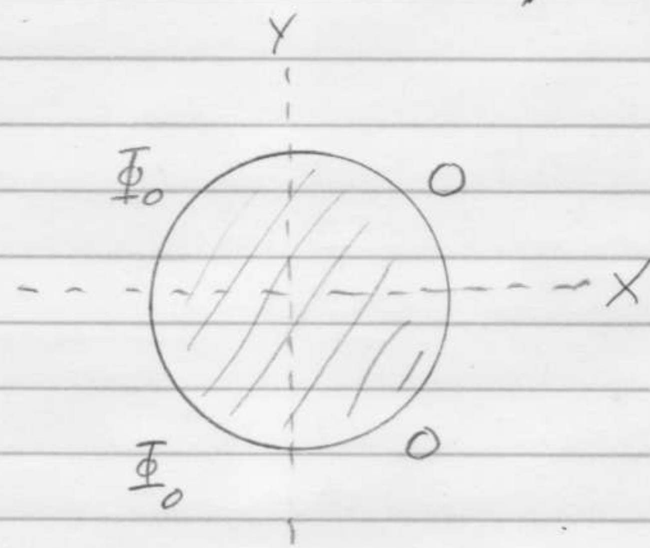
- COMPLEX, ANALYTIC, FUNCTIONS
  $f(z) = \mathrm{Re} f + i \mathrm{Im} f = f_x + i f_y$
  SATISFY A 2D LAPLACE'S EQUATIONS
  $$\frac{\partial^2}{\partial x^2} f_x = \frac{\partial^2}{\partial y^2} f_y = 0$$ (FROM DIFFERENTIATING THE CAUCHY-REIMANN EQUATIONS)

- THERE EXIST "CONFORMAL MAPPINGS" THAT TAKE $z \to z'$; THE NEW FUNCTION f' ALSO SATISFIES THE 2D LAPLACE'S EQUATION.

- HENCE, IF YOU HAVE f, YOU CAN FIND f', AND VICE VERSA.

EXAMPLE: "SPLIT" CYLINDER; ONE HEMI-CIRCLE AT $\Phi = \Phi_0$, THE OTHER AT $\Phi = 0$. (UNIT RADIUS.)

THIS HAS GEOMETRY IN THE COMPLEX PLANE:

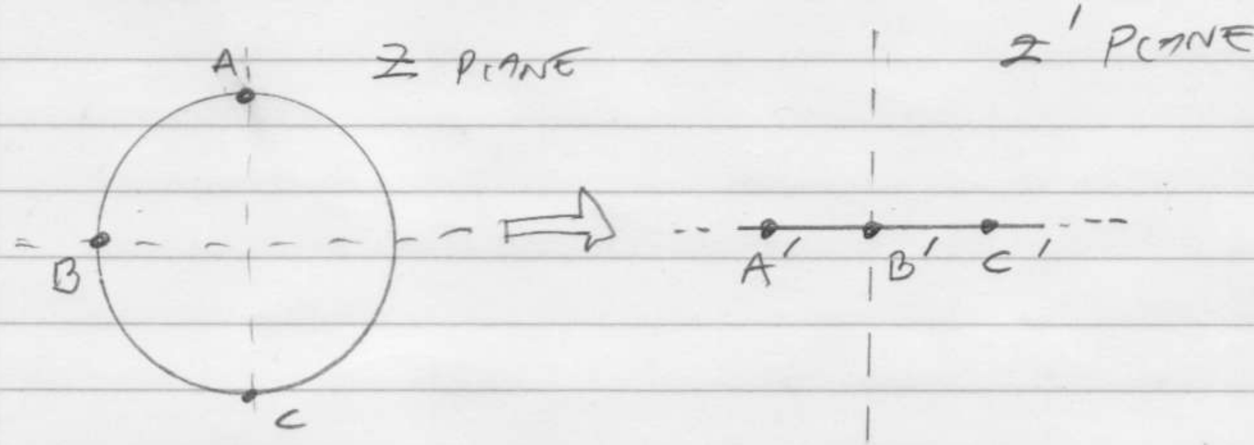


FINDING Φ, SAY, INSIDE, ISN'T TRIVIAL. (THOUGH WE WILL GET TO THIS VIA SEPARATION-OF-VARIABLES IN CYLINDRICAL COORDINATES.).

WITH A CONFORMAL MAP, WE'LL TRANSFORM $\Phi(x, y)$ INTO A SIMPLER PROBLEM $\Phi(x', y')$.

ATTACHED FIND SOME TRANSFORMATIONS FROM A BOOK ON COMPLEX ANALYSIS. AT THE BOTTOM, THERE'S A

TRANSFORMATION THAT MAPS THE INNER CIRCLE TO THE UPPER-HALF-PLANE.
THAT IS



WHERE POINTS A, B, C ARE MAPPED AS SHOWN.

THE PLANAR-PROBLEM HAS SOLUTION

$$\Phi(x', y') \sim \frac{\Phi}{\pi} \operatorname{ARG} \frac{z'+i}{z'-i}(-i)$$

$$\left(\sim \frac{\Phi}{\pi} \tan^{-1} y'/x'\right)$$

THIS CAN BE INVERSE-TRANSFORMED INTO THE CIRCLE-PROBLEM

$$\Phi(x, y) \sim \frac{\Phi}{\pi} \tan^{-1} \frac{1-x^2-y^2}{2x}$$

FOR THE POTENTIAL INSIDE THE CYLINDER.

THERE ARE MANY OTHER 2D PROBLEMS THAT CAN BE ADDRESSED BY CONFORMAL MAPPING.

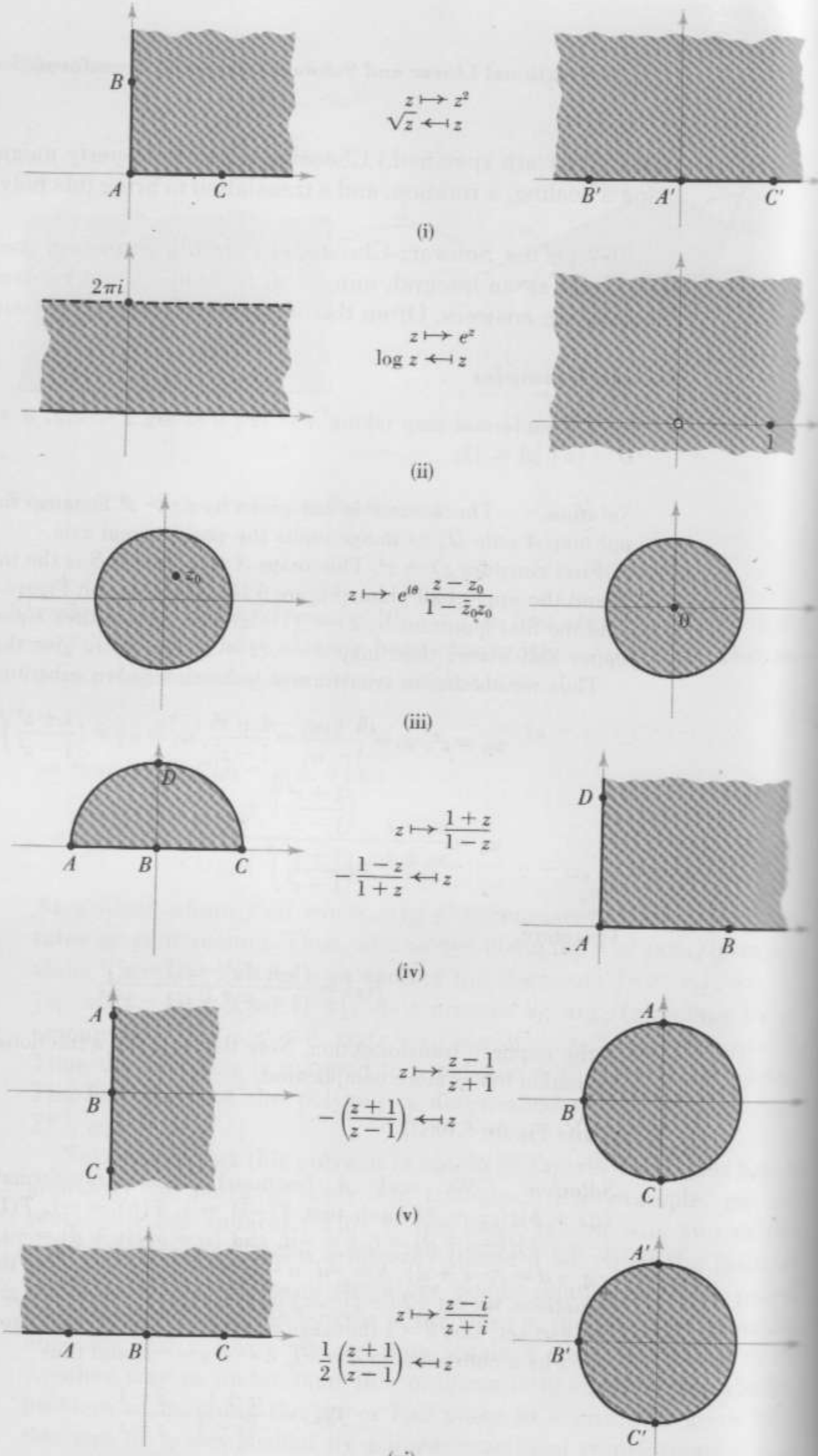


Figure 5.10 Some common transformations.

Figure 5.10