

FUNDAMENTAL CONSTANTS

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2 \quad (\text{permittivity of free space})$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2 \quad (\text{permeability of free space})$$

$$c = 3.00 \times 10^8 \text{ m/s} \quad (\text{speed of light})$$

$$e = 1.60 \times 10^{-19} \text{ C} \quad (\text{charge of the electron})$$

$$m = 9.11 \times 10^{-31} \text{ kg} \quad (\text{mass of the electron})$$

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \quad \begin{cases} \hat{\mathbf{x}} = \sin \theta \cos \phi \hat{\mathbf{r}} + \cos \theta \cos \phi \hat{\boldsymbol{\theta}} - \sin \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin \theta \sin \phi \hat{\mathbf{r}} + \cos \theta \sin \phi \hat{\boldsymbol{\theta}} + \cos \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1} \left(\sqrt{x^2 + y^2} / z \right) \\ \phi = \tan^{-1}(y/x) \end{cases} \quad \begin{cases} \hat{\mathbf{r}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s \cos \phi \\ y = s \sin \phi \\ z = z \end{cases} \quad \begin{cases} \hat{\mathbf{x}} = \cos \phi \hat{\mathbf{s}} - \sin \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin \phi \hat{\mathbf{s}} + \cos \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \quad \begin{cases} \hat{\mathbf{s}} = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, & \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, & \mathbf{H} = \frac{1}{\mu} \mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

$$\text{Energy:} \quad U = \frac{1}{2} \int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau$$

$$\text{Momentum:} \quad \mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B}) d\tau$$

$$\text{Poynting vector:} \quad \mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})$$

$$\text{Larmor formula:} \quad P = \frac{\mu_0}{6\pi c} q^2 a^2$$

VECTOR IDENTITIES

Triple Products

$$(1) \quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$(2) \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Product Rules

$$(3) \quad \nabla(fg) = f(\nabla g) + g(\nabla f)$$

$$(4) \quad \nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

$$(5) \quad \nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$(6) \quad \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$(7) \quad \nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$

$$(8) \quad \nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Second Derivatives

$$(9) \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$(10) \quad \nabla \times (\nabla f) = 0$$

$$(11) \quad \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_a^b (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A}) d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}; \quad d\tau = dx dy dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin \theta dr d\theta d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$

Curl: $\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}}$
 $+ \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}}$

Laplacian: $\nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}; \quad d\tau = s ds d\phi dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{\mathbf{r}} \text{ Coulomb's Law, } \mathbf{F} = Q\mathbf{E} + Q\mathbf{v} \times \mathbf{B} \text{ Lorentz force}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} d\tau'$$

$$\oiint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{encl}} \text{ Gauss' Law (integral) } \oint \mathbf{E} \cdot d\mathbf{l} = 0 \text{ (statics)}$$

$$\Phi_{ab} = -\int_a^b \mathbf{E} \cdot d\mathbf{l} \text{ Electrostatic potential, and } \mathbf{E} = -\nabla\Phi \text{ (statics)}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \text{ and } \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')}{r} d\tau'$$

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} \text{ and } \frac{\partial\Phi_{\text{above}}}{\partial n} - \frac{\partial\Phi_{\text{below}}}{\partial n} = \frac{1}{\epsilon_0} \sigma \text{ boundary conditions}$$

$$D_{\text{above}}^\perp - D_{\text{below}}^\perp = \sigma_f$$

$$\Phi(b) - \Phi(a) = \frac{W}{q} \quad W = \frac{1}{2} \sum_i q_i \Phi_i \quad W = \frac{1}{2} \iiint \rho\Phi d\tau$$

$$Q = CV \quad W = \frac{1}{2} CV^2 \quad W = \frac{1}{2} Q^2 / C \text{ capacitors}$$

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \text{ Poisson's equation} \quad \nabla^2\Phi = 0 \text{ Laplace's equation}$$

$$\nabla'^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}')$$

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \iiint \rho(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') d^3x' + \frac{1}{4\pi} \oiint \left[G(\mathbf{x}, \mathbf{x}') \frac{\partial\Phi}{\partial n'} - \Phi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \right] da'$$

$\Phi(x, y) = \sum_n (a_n e^{+kx} + b_n e^{-kx}) (c_n \sin ky + d_n \cos ky)$ A solution to Laplace's equation in rectangular coordinates in two dimensions

$$\int_0^a \sin(n\pi \frac{y}{a}) \sin(n'\pi \frac{y}{a}) dy = \begin{cases} 0 & n \neq n' \\ \frac{a}{2} & n = n' \end{cases} \text{ orthogonality, e.g., of sines}$$

$$\int_0^a \sin(n\pi \frac{y}{a}) dy = \begin{cases} 0 & n \text{ even} \\ \frac{2a}{n\pi} & n \text{ odd} \end{cases} \quad e^{+kx} + e^{-kx} = 2 \cosh kx$$

$$\int_0^\pi \cos^2 \theta \sin \theta d\theta = 2/3$$

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = (3x^2 - 1)/2$
$P_3(x) = (5x^3 - 3x)/2$
$P_4(x) = (35x^4 - 30x^2 + 3)/8$
$P_5(x) = (63x^5 - 70x^3 + 15x)/8$

TABLE 3.1 Legendre Polynomials.

$$\Phi(r, \theta) = \sum_l (a_l r^l + \frac{b_l}{r^{l+1}}) P_l(\cos \theta)$$

A solution to Laplace's equation in spherical coordinates with azimuthal symmetry

$$\int_{-1}^{+1} P_{l'}(x) P_l(x) dx = \frac{2}{2l+1} \delta_{ll'} \quad \text{orthogonality}$$

$$\frac{1}{R} = \frac{1}{r} \sum_n \left(\frac{r'}{r}\right)^n P_n(\cos \theta) \quad 1/R \text{ expansion azimuthal symmetry}$$

$$\text{Expansion into spherical harmonics } \Phi(r, \theta, \phi) = \sum_{\ell, m} \left[a_{\ell, m} r^\ell + \frac{b_{\ell, m}}{r^{\ell+1}} \right] Y_{\ell, m}(\theta, \phi)$$

$$\frac{1}{R} = 4\pi \sum_{\ell, m} \frac{1}{2^{\ell+1}} \frac{r_{<}^\ell}{r_{>}^{\ell+1}} Y_{\ell, m}^*(\theta', \phi') Y_{\ell, m}(\theta, \phi) \quad 1/R \text{ expansion spherical harmonics}$$

Expansion into Bessel regular functions

$$\Phi(\rho, \phi, z) = \sum_{m, n} J_m(k_{m, n} \rho) \times (a_{m, n} e^{+k_{m, n} z} + b_{m, n} e^{-k_{m, n} z}) \\ \times (c_{m, n} e^{+im\phi} + d_{m, n} e^{-im\phi})$$

$$\text{Orthogonality } \int_0^a \rho J_\nu \left(x_{\nu n'} \frac{\rho}{a}\right) J_\nu \left(x_{\nu n} \frac{\rho}{a}\right) d\rho = \frac{a^2}{2} [J_{\nu+1}(x_{\nu n})]^2 \delta_{nn'}$$

Multipole expansion

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell m} \frac{4\pi}{2^{\ell+1}} q_{\ell m} \frac{Y_{\ell m}(\theta, \phi)}{r^{\ell+1}} \quad q_{\ell m} = \iiint Y_{\ell m}^*(\theta', \phi') r'^\ell \rho(\mathbf{r}') dv'$$

$$W = \frac{1}{2} \iiint \rho(\mathbf{r}) \Phi(\mathbf{r}) dv \quad W = \frac{\epsilon_0}{2} \iiint E^2 dv \quad W = \frac{1}{2} \iiint \mathbf{E} \cdot \mathbf{D} dv$$

$$\text{Boundary conditions } (\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n}_{21} = \sigma \quad (\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n}_{21} = 0$$