If you need more space than is available to answer any part of a problem, use the back side of the same page to complete your answer. Scratch paper will not be graded.

Show your work in enough detail so that the grader can follow your reasoning and your method of solution.

Feel free to ask for an equation. I have a copy of Jackson and can look up an equation for you.

POINTS

1. __________________________/25

2. __________________________/25

3. __________________________/25

4. __________________________/25

Total __________________________/100
I. (25 points) Magnetic field calculation.

a. Circular loop. Consider a circular loop of radius $R$ carrying current $I$. Find the magnetic field $B$ everywhere on the loop axis.

b. Solenoid. Consider a cylinder of length $L$ and radius $R$ carrying solenoidal surface currents $K$. Find the magnetic field $B$ everywhere on the solenoid axis. Express your answer in terms of $\theta_a$ and $\theta_b$ (the angles to the edges of the solenoid face as “seen” by the field point) shown below.
II. (25 points) Forces on dielectrics. Consider a “U-tube” containing a liquid linear dielectric of permittivity $\varepsilon$ and mass density $\rho$. One arm of the tube is between plates of a capacitor. (You can assume the capacitor is ideal, so the electric field $E_0$ between the plates is uniform.) See the sketch below.

![Sketch of U-tube with electric field](image)

a. With the cross-section of the U-tube circular, find the height the liquid rises $h-h_0$ when the electric field is applied. Hint: recall the electric field within a dielectric cylinder with axis at right angles to a uniform applied field $E_0$ is $2E_0/(1+\varepsilon/\varepsilon_0)$.

b. How would your result change if the U-tube cross section is a rectangle with a very long edge parallel to the applied field, and a very short edge normal to the electric field?

c. How would your result change if the U-tube cross section is a rectangle with a very short edge parallel to the applied field, and a very long edge normal to the electric field?
III. (25 points) Mutual Inductance. Consider two coaxial thin circular wires of radii $a$ and $b$ with their centers a distance $d$ apart and $d >> a$ and $b$. Find the mutual inductance.
IV. (25 points) Image currents, magnetic materials, inductance.

a. A long thin wire carrying current $I$ lies a distance $d$ in vacuum from the surface of a semi-infinite slab of linear permeable material (with permeability $\mu$, and as usual $\mu > \mu_0$). Find the force per unit length on the wire. Hint: Choose a sensible location of the image currents.

b. The wire is replaced with a circular loop of thin wire placed on the surface of the semi-infinite slab. What is the ratio of the loop’s self-inductance when on the slab surface to the loop’s self-inductance when in free space? Hint: (i) You may want to consider part a to infer the location of the image current. (ii) You may want to consider the location of sources of B field to infer the “shape” of the B field.
I. A piece of current \( I \) produces

Field \( dB = \frac{\mu_0}{4\pi} \frac{I}{r^2} \cos \theta \, dr \)

Hence \( B_z = \frac{\mu_0}{4\pi} \frac{I \pi R^2}{r^3} \cos \theta \)

\[
B_z = \frac{\mu_0 I R^2}{2 (R^2 + z^2)^{3/2}}
\]

\( \Box \)

b. We integrate the above expression

\[
B_z = \frac{\mu_0}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{K^2 R^2}{(R^2 + z^2)^{3/2}} \, dz
\]

But \( z = R \cot \theta \) so \( dz = \frac{-R}{\sin^2 \theta} \, d\theta \) and

Also \( \frac{1}{(R^2 + z^2)^{3/2}} = \frac{\sin^2 \theta}{R^3} \) so

\[
B_z = \frac{\mu_0}{2} K \int_{\theta_0}^{\theta_f} \frac{R^3 \sin^3 \theta}{R^3 \sin^2 \theta} (-R \, d\theta)
\]

\[
\frac{\theta_0}{\theta_0} \quad \frac{\theta_f}{\theta_f}
\]

\[
= \frac{\mu_0}{2} K \int_{\theta_0}^{\theta_f} \frac{\sin \theta}{\sin \theta} \, d\theta
\]

\[
B_z = \frac{\mu_0 K}{2} \left[ \sin \theta \right]_{\theta_0}^{\theta_f}
\]

OR \( B_z = \frac{\mu_0 K}{2} (\cos \theta_0 - \cos \theta_f) \)
II. We have several ways of getting this.
One approach is to take the (free) energy with and without dipoles:

\[ U = \frac{1}{2} \int \int \int \mathbf{D} \cdot \mathbf{E} \, d\mathbf{v} \text{ (with)}, \quad U_0 = \frac{1}{2} \int \int \int \mathbf{D} \cdot \mathbf{E} \, d\mathbf{v} \text{ (without)} \]

\[ U = U_0 + \frac{1}{2} \int \int \int \left\{ \mathbf{E} \cdot (\mathbf{D} - \mathbf{D}_0) + (\mathbf{E} - \mathbf{E}_0) \cdot \mathbf{D} \right\} \, d\mathbf{v} \]

Notice \( \int \int \int \mathbf{E} \cdot (\mathbf{D} - \mathbf{D}_0) \, d\mathbf{v} = -\int \int \int \nabla \cdot (\mathbf{D} - \mathbf{D}_0) \, d\mathbf{v} \]
\[ + \int \int \int \mathbf{D} \cdot (\mathbf{D} - \mathbf{D}_0) \, d\mathbf{v} \]

Vanishes: the first term is a total divergence with fields falling faster than \( \frac{1}{r^2} \). The second term vanishes since there's no free charge.

The same arguments lead to \( \int \int \int \mathbf{E}_0 \cdot (\mathbf{E} - \mathbf{E}_0) \, d\mathbf{v} = 0 \).

\[ U = U_0 + \frac{1}{2} \int \int \int (\mathbf{E} \cdot \nabla - \mathbf{E}_0 \cdot \nabla) \mathbf{E} \, d\mathbf{v} \]

For this liquid dielectric \( \mathbf{D} = \varepsilon \mathbf{E} \), so \( U = U_0 - \frac{1}{2} \int \int \int (\varepsilon - \varepsilon_0) \mathbf{E} \cdot \mathbf{E}_0 \, d\mathbf{v} \), or

\[ U = U_0 - \frac{1}{2} \int \int \int \varepsilon \mathbf{E} \cdot \mathbf{E}_0 \, d\mathbf{v} \]

When the column rise, gravitational energy \( g \Delta A (h - h_0)^2 \) is added, with a tube cross-section.
The energy expression now reads
\[ U = U_0 - \frac{1}{2} \iint \overrightarrow{P} \cdot \overrightarrow{E_0} \ dA - x + \rho g A (h - h_0)^2 \]

With equilibrium \( \frac{\partial U}{\partial h} = 0 \),
\[ h - h_0 = \frac{\overrightarrow{P} \cdot \overrightarrow{E_0}}{\rho g} \]

a. Circular cross section,
\[ \overrightarrow{E} = \frac{2 \overrightarrow{E_0}}{1 + \varepsilon / \varepsilon_0} \]
Hence
\[ \overrightarrow{P} = 2 \varepsilon \overrightarrow{E_0} \frac{\varepsilon - \varepsilon_0}{\varepsilon + \varepsilon_0} \]
and
\[ h - h_0 = \frac{\varepsilon_0 E_0^2 (\varepsilon - \varepsilon_0)}{\rho g} \frac{\varepsilon - \varepsilon_0}{2(\varepsilon + \varepsilon_0)} \]

b. Here, the boundary condition on the continuity of the tangential components of the electric field gives
\[ \overrightarrow{E} \sim \overrightarrow{E_0} \]
Hence
\[ \overrightarrow{P} = (\varepsilon - \varepsilon_0) \overrightarrow{E_0} \]
and
\[ h - h_0 = \frac{\varepsilon_0 E_0^2}{\rho g} \frac{1}{4} \left( \frac{\varepsilon - \varepsilon_0}{\varepsilon_0} - 1 \right) \]

c. Here, the boundary condition on the continuity of \( \overrightarrow{D} \) gives \( \overrightarrow{D} \sim \overrightarrow{D_0} \).
\[ \overrightarrow{P} = \varepsilon \overrightarrow{E} - \varepsilon_0 \overrightarrow{E_0} \]
\[ \overrightarrow{P} = \overrightarrow{D} = \varepsilon_0 \overrightarrow{E_0} \]
so
\[ h - h_0 = \frac{\varepsilon_0 E_0^2}{\rho g} \frac{1}{4} \left( 1 - \frac{\varepsilon_0}{\varepsilon} \right) \]
Another approach is to integrate the term in the volume-stress containing the gradient of the dielectric constant:

\[
\nabla V = \frac{\partial F}{\partial V} = -\frac{\varepsilon_0}{2} E^2 \frac{\partial}{\partial V} \varepsilon
\]

\[
\rho_b - \rho_a = \frac{\varepsilon_0}{2} \int_b^a E^2 \frac{\partial}{\partial V} \varepsilon \, dx
\]

where \(b\) and \(a\) are just on opposite sides of the boundary. Hence

\[
\rho_b - \rho_a = \frac{\varepsilon_0}{2} \int_b^a \left( E_t^2 + E_n^2 \right) \frac{\partial}{\partial V} \varepsilon \, dx
\]

where we decomposed \(E^2\) into parts tangential and normal to the boundary.

This is what we saw in class. In this problem \(E_n = 0\). After applying boundary conditions on the tangential components of \(E\):

\[
\rho_b - \rho_a = \frac{(\varepsilon - \varepsilon_0) E_t^2}{2}
\]

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With \(E\) the field in the dielectric, we saw this in class.

This pressure is balanced by the gravitational force leading to the previous results.
Another approach is to find the stress across the boundary from energy. On one side the energy is \( U_0 = \frac{\varepsilon_0}{2} E_0^2 \), on the other side its \( U = \frac{\varepsilon}{2} E^2 \). This leads to a pressure difference
\[
P_0 - P_a = \frac{\varepsilon - \varepsilon_0}{2} E^2,
\]
as before.

Griffiths basically uses this approach to find the dielectric rise in a capacitor by finding the change in the capacitor's stored energy and equating it the change to the gravitational potential.
We'll use Neumann's formula for $L$.

We'll need $d\mathbf{a} \cdot d\mathbf{b}$.

$d\mathbf{b} = b \, d\theta$ (see sketch above), so the $d\mathbf{a} \cdot d\mathbf{b}$ product is $b \cos \theta \, d\theta$.

Now we need $R_{ab}$, the distance from an element of loop $d\mathbf{a}$ to an element in $d\mathbf{b}$. It is:

$$R_{ab} = \left( d^2 + a^2 + b^2 - 2ab \cos \theta \right)^{\frac{1}{2}}$$

Hence, Neumann's formula in this geometry is:

$$L = \frac{1}{2} \mu_0 a b \left\{ \int_0^{2\pi} \frac{\cos \theta}{\sqrt{d^2 + a^2 + b^2 - 2ab \cos \theta}} \, d\theta \right\}$$

Now $d >> a$ or $b$, so

$$L \approx \frac{1}{2} \mu_0 a b \left\{ \int_0^{2\pi} \frac{\cos \theta}{d} \left[ 1 - \frac{1}{2} \left( 1 + \frac{a^2 + b^2}{d^2} \right) \right] \, d\theta \right\}$$

The terms $\frac{a^2}{d^2}$, $\frac{b^2}{d^2}$ integrate to zero, and $\int_0^{2\pi} \cos \theta \, d\theta = 0$. 
\[ L = \frac{1}{2} M_0 \left( \frac{ab}{d^3} \right)^2 \int_0^{2\pi} \cos^2 \theta \, d\theta \]

So

\[ L \approx \frac{1}{2} M_0 \left( \frac{ab}{d^3} \right)^2 \pi \]

(You can find a slightly better approximation by keeping low-order terms in \( a \) and \( b \).)

(I do not believe that for the general case where \( d \sim a \) and \( b \), \( L \) can be expressed in terms of simple functions. Usually general expressions involving loops bring in elliptic integrals.)
IIIa. Two views of the geometry are

\[ \begin{align*}
\mathbf{H} & \approx \frac{1}{2\pi} \frac{I}{d^2 + y^2} \mathbf{\hat{y}} + \frac{1}{2\pi} \frac{I'}{d^2 + y^2} \mathbf{\hat{y}} \\
\mathbf{H}_x & \approx \frac{1}{2\pi} \frac{I}{d^2 + y^2} - \frac{d}{d^2 + y^2} \\
\mathbf{H}_y & \approx \frac{1}{2\pi} \frac{I'}{d^2 + y^2}
\end{align*} \]

Let's place an image current at \( x = -d \) flowing anti-parallel to the true current.

Since the field \( \mathbf{H} \) from a single wire is \( \mathbf{H} = \frac{1}{2\pi} \frac{I}{d^2 + y^2} \mathbf{\hat{y}} \), we have image-charge solutions:

\[ \begin{align*}
\mathbf{H}_x (\text{vacuum}) & = \frac{1}{2\pi} \frac{I}{d^2 + y^2} - \frac{d}{d^2 + y^2} \\
\mathbf{H}_y (\text{vacuum}) & = \frac{1}{2\pi} \frac{I'}{d^2 + y^2}
\end{align*} \]

And \( \mathbf{H}_y (\text{media}) = \frac{1}{2\pi} \frac{I''}{d^2 + y^2} \\
\mathbf{H}_x (\text{media}) = \frac{1}{2\pi} \frac{I'}{d^2 + y^2} \)

Here \( I' \) is the image current to find \( \mathbf{B} \) in vacuum, and
$I''$ is the image current to find $B$ in the media. We guessed the image current is at $-d$ (why?). Now apply boundary conditions:

$B_x$ is continuous (from $\vec{\nabla} \cdot \vec{B} = 0$)

and $H_y$ is continuous (from $\vec{\nabla} \times \vec{H} = 0$ on the surface.)

So $I - I' = I''$ and $\mu_0 (I + I') = \mu I''$

and $I' = \frac{M - \mu_0 I}{M + \mu_0}$ (we made a good choice for the position of images.)

Recall the force between two conductors force/length $= F/L = \frac{I_1 I_2}{2\pi d}$

Hence $F/L = \frac{\mu_0 I^2}{4\pi d} \frac{M - \mu_0}{M + \mu_0}$

Since the fields are antiparallel, the wire is repulsed from the medium.

b. With the loop on the surface, the image current is as well on the surface. What does the "shape" of the $B$-field look like?

$\vec{B}$ is sourced by true currents and $\vec{\nabla} \times \vec{M}$. But for this medium $\vec{M} \times \vec{H}$, so $\vec{B}$ is sourced by
something proportional to $\bar{J}^{\text{true}}$ (since $\bar{J}^{\text{true}} \propto \nabla \times \bar{J}$).

the source of $\bar{M}$ looks like

the true current loop superimposed on the magnetization current loop $\nabla \times \bar{J}$, which is at the same position as the true current loop.

since the image current is antiparallel to the true current, the shape of the $\bar{B}$ field is the same as if the loop were in free space; but the magnitude of the $\bar{B}$ field is reduced from that in free space.

we have $\int_{\text{true}} \bar{B} \cdot d\bar{r} = \int_{\text{true}} I_{\text{true}} \cdot d\bar{r}$ around any loop. since lines of $\bar{B}$ "curl around" we can choose an integration path of constant $\bar{B}$.

in free space, $\bar{B} = \mu_0 \bar{H}$.

in media, $\bar{B} = \mu \bar{H}$.

when we evaluate $\int_{\text{loop}} \bar{H} \cdot d\bar{r}$ for the loop on the surface,
\[ I = \Phi \vec{H}_s \cdot d\vec{l} = \int_{\text{vacuum}} \frac{\vec{B}_s}{\mu_0} \cdot d\vec{l} + \int_{\text{slab}} \frac{\vec{B}_s}{\mu} \cdot d\vec{l} \]

\[ = \left( \frac{1}{\mu_0} + \frac{1}{\mu} \right) \int_{\text{vacuum}} \vec{B}_s \cdot d\vec{l} \]

Now consider the loop in free space:

\[ I = \Phi \vec{H}_c \cdot d\vec{l} = \int_{\text{vacuum}} \frac{\vec{B}_c}{\mu_0} \cdot d\vec{l} + \int_{\text{vacuum}} \frac{\vec{B}_c}{\mu_0} \cdot d\vec{l} = 2 \int_{\text{vacuum}} \frac{\vec{B}_c}{\mu_0} \cdot d\vec{l} \]

Notice above we decomposed the closed line integral into a line integral inside the slab and a line integral in vacuum; for the case where the loop is in free space, the path is the same and the path is all in vacuum.

Hence \[ \frac{2}{\mu_0} \vec{B}_c = \left( \frac{1}{\mu_0} + \frac{1}{\mu} \right) \vec{B}_s \] in general.

From the expression for magnetic flux \[ \Phi_M = LI \]

\[ \frac{\Phi_s}{\Phi_V} = \frac{Ls}{Lv} \frac{I}{I} = \frac{Ls}{Lv} = \frac{\int \vec{B}_s \cdot \hat{n} dA}{\int \vec{B}_V \cdot \hat{n} dA} = \frac{2}{1 + \mu_0 / \mu} \]