Phys. 513
Mid-term Exam
Fall 2018

Problem 1 Solution V1.0.

Inside the sphere (r < R) there's Poisson's equation for the "inside" potential $\Phi_<$:

$$\nabla^2 \Phi_<(r, \theta) = -\frac{\rho_0(r)}{\varepsilon_0}.$$  

Outside the sphere (r > R) there's Laplace's equation for the "outside" potential $\Phi_>$:

$$\nabla^2 \Phi_>(r, \theta) = 0.$$  

Solutions having azimuthal symmetry have form

$$\Phi_< = \frac{q_0}{r} + \sum A_k r^k P_k(\cos \theta)$$

(The $1/r^2$ solutions are excluded in considering the $\Phi_<(r \to \infty)$ limit).

$$\Phi_> = \sum \frac{B_k}{r} r^{k+1} P_k(\cos \theta).$$

Apply boundary conditions

$$\Phi_<(r=R, \theta) = \Phi_>(r=R, \theta) = \Phi_0 \cos \theta.$$
Given \( n \neq 0 \):
- \( A_n = -\frac{Q}{4\pi \varepsilon_0 R} \)
- \( A_1 = \frac{Q}{R} \)
- \( A_n (n \neq 0, n \neq 1) = 0 \)
- \( B_1 = \frac{Q R^2}{2} \)
- \( B_n (n \neq 1) = 0 \), so

\[
\Phi_\infty = \frac{Q}{4\pi \varepsilon_0} \left( \frac{1}{R} - \frac{1}{R^2} \right) + \Phi_0 \frac{R}{R^2} \cos \theta,
\]

\[
\Phi_\infty = \Phi_0 \frac{R^2}{R^2} \cos \theta.
\]

**Check:**

a) These are by construction and examination solutions to Laplace's equation (outside the charge);

b) The \( r = R \) boundary condition is satisfied, so outside the charge the solution is unique.

c) Near the charge \( \nabla^2 \Phi \)
satisfies Poisson's equation.

So, the solution is sensible.
PROBLEM 2 SOLUTION V1.0

1. Start with $\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0$, then insert the expression for $\vec{E}$, giving

$$\vec{\nabla} \cdot \vec{E} = \frac{-k \rho}{r^2}$$

Refer to Jackson "front cover" identity $\vec{\nabla} \cdot \vec{A} = \vec{A} \cdot \vec{\nabla} + \nabla \cdot \vec{A}$, so

$$\vec{\nabla} \cdot \vec{E} = \frac{-k \rho}{r^2} + \vec{\nabla} \cdot \vec{E} = \frac{-k \rho}{r^2}$$

Thinking of the potential $1/r$ from a point charge, we replace $1/r$ with $-\vec{\nabla} 1/r$. The second term is then Poisson's equation.

The gradient in the first term is evaluated directly:

We then have

$$-\frac{M}{r^2} \hat{r} - \frac{k \rho}{r^2} \hat{r} + 4\pi \varepsilon_0 \hat{r} = \frac{\rho}{\varepsilon_0}$$

or $\rho = -\varepsilon_0 \frac{M}{r^2} e^{-kr} + 4\pi \varepsilon_0 \hat{r} \delta(r)$. 
b. The second term in \((a)\) is a charge of magnitude \(4\pi \varepsilon_0 m\) placed at \(r = 0\). The first term is a spherically-symmetric opposite-signed charge distribution that rises in distance. Graphically, this is

\[\rho(r)\]

Charge at \(r = 0\):
Magnitude \(4\pi \varepsilon_0 m\)

Spherically-symmetric, opposite sign (6a) magnitude of density

\[\varepsilon_0 \frac{4\pi}{r^2} e^{-kr}\]

c. Apply Gauss's law to the spherical surface at \(r \to \infty\):

\[\frac{Q_{\text{total}}}{\varepsilon_0} = \iiint E \cdot \hat{n} \, dA\]

\[= \int_0^\infty \int_0^{2\pi} \int_0^\infty E \cdot \hat{n} \, r \, dr \, d\theta \, d\phi\]

\[= 4\pi \frac{mc}{r^2} r^2 \]

\[\to 0 \text{ total charge}\]
Problem 3 Solution v1.0

You are free to choose different coordinates)

a. The 2D geometry is

Here, the problem is translation invariant in \( z \), \( I \) is the intersection point of the plates, and \( \phi_0 \) the small angle.

In this figure any plane (which appears as a straight line in the figure) passing through \( I \) is an equipotential (via symmetry).

Hence, the potential depends only on \( \phi \) (ignoring fringe fields).

b. The potential obeys Laplace's equation (in 2D cylindrical coordinates)
\[ \nabla^2 \Phi = \frac{1}{r^2} \frac{d}{d\phi} r^2 \frac{d}{d\phi} \Phi, \quad \text{where it's a total derivative as there's no } r \text{- dependence.} \]

This has solution \( \Phi(\phi) = C_0 + C_1 \phi \)

with \( C_0, C_1 \) constants.

Apply boundary conditions:

\[ \left. \Phi(\phi=0) = 0, \quad \Phi'(\phi=\phi_0) = \Phi_0 \right. \]

\[ C_0 = 0, \quad C_1 = \Phi_0 / \phi_0 \]

Now, from the geometry,

\[ \tan \phi_0 = \frac{8}{L} \quad \text{and} \quad \tan \phi = \frac{y}{x + 8w/\phi}, \quad \text{hence} \]

\[ \phi(x,y) = \Phi_0 \phi_0 = \Phi_0 \arctan \frac{y}{x + 8w/\phi} \]

C. Put charge \( +Q \) on the lower plate,

Put zero charge on the upper plate.

The (no-fringe) electric field is purely along \( \Phi \):

\[ E = -\nabla \Phi = -\frac{1}{r} \frac{d}{d\phi} r^2 \Phi = -\frac{1}{r} \Phi_0 \phi' \]
On the lower plate, $\mathbf{E}$ is in the $y$ direction (normal to the plate) and the surface charge $\sigma$ is

$$\sigma / \varepsilon_0 = |E| = \frac{\Phi_0}{\Phi_0} \frac{1}{xw/\varepsilon + x}$$

where we used $r = xw/\varepsilon + x$ for a field point on the lower plate (see figure in (a)).

Now find the total charge per length on the lower plate

$$\sigma / \text{length} = \int_0^L \sigma \, dx$$

$$= -\varepsilon_0 \int_0^L \frac{\Phi_0}{\Phi_0} \frac{1}{xw/\varepsilon + x} \, dx$$

$$= -\varepsilon_0 \left[ \frac{\Phi_0}{\arctan \sqrt{L}} \right]_0^L \frac{1}{xw/\varepsilon + x} \, dx$$

$$= -\varepsilon_0 \left[ \frac{\Phi_0}{\arctan \sqrt{L}} \right]_0^L \frac{\ln \left( \frac{w + \varepsilon}{\varepsilon} \right)}{w}$$

The capacitance is the ratio of the charge to potential:

$$C = \frac{\varepsilon_0 / \text{length}}{\text{length} / \arctan \sqrt{L}} \ln \frac{w + \varepsilon}{\varepsilon}$$