SPECIAL LECTURE
VOLUME FORCES IN DIELECTRICS
ADAPTED FROM STRATTON "ELECTROMAGNETIC THEORY": CHAPT II.

RECALL, THE (FREE) ENERGY IN THE PRESENCE OF DIELECTRICS IS

$$U = \frac{1}{2}\iiint \vec{E}\cdot\vec{D}\, dV \quad \text{(JACKSON EQN 4.89)}$$

WE FOUND THIS IN LECTURE BY STUDYING A "VIRTUAL" PROCESS IN WHICH FREE CHARGES WE SLOWLY ADDED TO A SYSTEM, NOT ALLOWING BOUNDARIES TO MOVE (THEREBY ENSURING NO MECHANICAL WORK IS DONE BY MECHANICAL MOTIONS).

WE'LL NOW CONSIDER A DIFFERENT "VIRTUAL" PROCESS: THE POSITIONS OF CHARGES AND DIELECTRIC MATERIAL ARE "VIRTUALLY" DISPLACED A DISTANCE $\delta\vec{r}$ AT EVERY POINT IN THE SYSTEM, BUT WE DONT ADD FREE CHARGES.

A SUBTLETY: IN THIS LATER PROCESS, IS $U = \frac{1}{2}\iiint \vec{E}\cdot\vec{D}\, dV$ STILL VALID? A LONG DISCUSSION LEADS TO "YES", BASED ON THE FIELDS BEING CONSERVATIVE.

IN THIS "VIRTUAL" DISPLACEMENT, THE (FREE) ENERGY CHANGES BY

$$\delta U = -\iiint \vec{F} \cdot \delta\vec{r} \, dv,$$

WHERE $\vec{F}$ IS THE "VOLUME FORCE" $\frac{dF}{dv}$

A SUBLETY: THE "VIRTUAL VELOCITIES" $\vec{v}$ DUE TO THE VIRTUAL DISPLACEMENTS ARE SO LOW THAT THE DISPLACEMENT PROCESS IS REVERSIBLE AND ISOTHERMAL. HENCE, THE CHANGE IN (FREE) ENERGY IS THE MECHANICAL WORK.

IT IMMEDIATELY FOLLOWS THAT

$$\frac{dU}{dt} = -\iiint \vec{F} \cdot \vec{v} \, dv$$

THE VIRTUAL DISPLACEMENTS CAN CHANGE HOW CHARGE $\rho$ IS DISTRIBUTED (LEADING TO $\delta\rho$) AND A CHANGE IN THE DIELECTRIC CONSTANT $\varepsilon$ (MATERIAL MOVES). THE RESULTING (FREE) ENERGY CHANGE IS

$$\begin{aligned} \delta U &= \delta \frac{1}{2} \iiint \vec{E} \cdot \vec{D} \, dv \\ &= \frac{1}{2} \iiint D^2 \left(\frac{1}{\varepsilon}\right) dv + \iiint \vec{E} \cdot \delta\vec{D} \, dv \\ &= -\frac{1}{2} \iiint E^2 \, \delta\varepsilon \, dv + \iiint \vec{E} \cdot \delta\vec{D} \, dv \end{aligned}$$

THE FIRST TERM IS THE (FREE) ENERGY CHANGE DUE TO THE CHANGE IN THE DIELECTRIC CONSTANT INDUCED BY THE VIRTUAL DISPLACEMENT.

RECALL THE (INTEGRATION BY PARTS) IDENTITY $\vec{\nabla}\cdot(\phi\vec{A}) = \phi\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}\phi$, APPLIED TO THE SECOND TERM:

$$\iiint \vec{E}\cdot\delta\vec{D}\,dv = -\iiint \vec{\nabla}\Phi\cdot\delta\vec{D} = \iiint \Phi\,\vec{\nabla}\cdot\delta\vec{D}\,dv$$

(WHERE GAUSS'S LAW ALLOWS US TO IGNORE THE TOTAL DIVERGENCE). SO

$$\iiint \vec{E}\cdot\delta\vec{D}\,dv = \iiint \Phi\,\delta\rho\,dv$$

WE THUS HAVE

$$\frac{dU}{dt} = \iiint \left\{ \Phi\frac{d\rho}{dt} - \frac{1}{2}E^2\frac{d\epsilon}{dt} \right\} dv$$

WE'D LIKE TO SOMEHOW EXPRESS THIS IN TERMS OF $\vec{F}$. TO THIS END, WE RECALL CONSERVATION FOR CHARGE $\rho$ AND MASS $\rho_m$:

$$\vec{\nabla}\cdot\rho\vec{v} + \frac{d\rho}{dt} = 0; \quad \vec{\nabla}\cdot\rho_m\vec{v} + \frac{d\rho_m}{dt} = 0.$$

THIS IS A GOOD PLACE TO INTRODUCE THE "CONVECTIVE DERIVATIVE". THIS IS THE TOTAL DERIVATIVE OF A CERTAIN THING EVALUATED SO THE OBSERVATION POINT FOR THE DERIVATIVE IS CO-MOVING WITH THE VELOCITY FLOW. WE WRITE THIS AS $\frac{D}{Dt}$.

FOR EXAMPLE, THE CONVECTIVE DERIVATIVE OF THE DIELECTRIC CONSTANT IS

$$\frac{D\varepsilon}{Dt} = \frac{\partial \varepsilon}{\partial t} + \frac{\partial \varepsilon}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \varepsilon}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \varepsilon}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{\partial \varepsilon}{\partial t} + \vec{V}\cdot\vec{\nabla}\varepsilon$$

FOR CHARGE $\rho$ AND DIELECTRIC CONSTANT $\varepsilon$, WE THEN HAVE

$$\frac{\partial \rho_m}{\partial t} = \frac{D\rho_m}{Dt} - \vec{V}\cdot\vec{\nabla}\rho_m$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{D\varepsilon}{\partial t} - \vec{V}\cdot\vec{\nabla}\varepsilon$$

SUBLETY: WE ASSUME THE DEPENDENCE OF THE DIELECTRIC CONSTANT $\varepsilon$ ON MASS DENSITY $\rho_m$ IS KNOWN, EITHER THROUGH THEORY OR EMPIRICALLY.

THAT IS, WE HAVE $\frac{D\varepsilon}{Dt} = \frac{d\varepsilon}{d\rho_m}\frac{D\rho_m}{Dt}$,
WITH $d\varepsilon/d\rho_m$ KNOWN.
BY THE WAY, THE CLAUSIUS-MOSSOTTI FROM SPECIAL LECTURE I IS A GOOD ESTIMATE OF $d\varepsilon/d\rho_m$.

SUBTLETY: ASSUMING $\varepsilon$ HAS NO TEMPERATURE DEPENDENCE WORKS IN GENERAL ONLY IF THE VIRTUAL PROCESS IS ISOTHERMAL.

INSERTING THE CONSERVATION EQUATIONS INTO THE EXPRESSION FOR $D\varepsilon/dt$ ABOVE GIVES

$$\frac{D\varepsilon}{Dt} = \frac{d\varepsilon}{d\rho_m}\left\{\frac{d\rho_m}{dt} + \vec{V}\cdot\vec{\nabla}\rho_m\right\}$$

$$= \frac{d\varepsilon}{d\rho_m}\left\{\vec{V}\cdot\vec{\nabla}\rho_m - \vec{\nabla}\cdot\rho_m\vec{V}\right\}$$

$$= \frac{d\varepsilon}{d\rho_m}\rho_m\vec{\nabla}\cdot\vec{V}$$

WE THUS HAVE

$$\frac{d\varepsilon}{dt} = -\frac{d\varepsilon}{d\rho_m}\rho_m\vec{\nabla}\cdot\vec{V} - \vec{V}\cdot\vec{\nabla}\varepsilon$$

WE THE HAVE THE EXPRESSION FOR $dU/dt$ ON PAGE 3 IN THE FORM

$$\frac{dU}{dt} = \iiint \Big\{ -\underset{(1)}{\Phi\, \vec{\nabla}\cdot\rho\vec{v}} + \underset{(2)}{\tfrac{1}{2} E^2 \frac{d\varepsilon}{d\rho_m}\, \vec{\nabla}\cdot\vec{v}} + \vec{v}\cdot\big(\underset{(3)}{\tfrac{1}{2}E^2\vec{\nabla}\varepsilon}\big) \Big\}$$

WE'RE ALMOST DONE. TO FIND THE VOLUME FORCE $\vec{F}$, WE NEED TO WRITE THE INTEGRAND AS A DOT PRODUCT OF "SOMETHING" $\cdot\, \vec{v}$. WE CAN THEN IDENTIFY "SOMETHING" AS THE VOLUME FORCE.

TERM 1 HAS APPLIED THE IDENTITY $\vec{\nabla}\cdot(\phi\vec{A}) = \phi\,\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}\phi$, GAUSS'S LAW, WITH A SURFACE COMPLETELY OUTSIDE THE SYSTEM. THIS GIVES

$$\iiint \Phi\, \vec{\nabla}\cdot(\rho\vec{v})\, dv = -\iiint \underbrace{\rho\,\vec{\nabla}\Phi}\cdot\vec{v}\, dv$$

TERM 2 IS SIMILAR, AGAIN ELIMINATING THE SURFACE TERM:

$$\frac{1}{2}\iiint E^2 \rho_m \frac{d\varepsilon}{d\rho_m}\,\vec{\nabla}\cdot\vec{v}\, dv$$

$$= -\frac{1}{2}\iiint \underbrace{\vec{\nabla}\Big(E^2\frac{d\varepsilon}{d\rho_m}\rho_m\Big)}\cdot\vec{v}$$

COMBINED, THESE GIVE

$$\frac{dU}{dt} = \iiint \left\{ -\rho \vec{E} + \frac{1}{2} E^2 \vec{\nabla} \varepsilon - \frac{1}{2} \vec{\nabla} \left( E^2 \frac{d\varepsilon}{d\rho_m} \rho_m \right) \right\} \cdot \vec{v} \, dV$$

THE "SOMETHING" IN BRACKETS IS THE VOLUME FORCE:

$$\vec{F} = \underset{①}{\rho \vec{E}} - \underset{②}{\frac{1}{2} E^2 \vec{\nabla} \varepsilon} + \underset{③}{\frac{1}{2} \vec{\nabla} \left( E^2 \frac{d\varepsilon}{d\rho_m} \rho_m \right)}$$

TERM 1 IS THE USUAL VOLUME FORCE $\vec{F} = \rho \vec{E}$ $\left\{ = (\vec{\nabla} \cdot \vec{D}) \vec{E} \right\}$

TERM 2 ARISES FROM AN INHOMOGENEOUS DIELECTRIC $(\vec{\nabla} \varepsilon \neq 0)$ IN AN ELECTRIC FIELD.

TERM 3 IS USUALLY IGNORED, THE EFFECT IS CALLED "ELECTROSTRICTION" AN IT ARISES FROM A DENSITY-DEPENDENT DIELECTRIC IN AN INHOMOGENEOUS ELECTRIC FIELD. NOTICE THIS TERM IS A PURE GRADIENT, SO IF THE VOLUME IS LARGE ENOUGH (TO BE OUTSIDE THE FIELD) ITS INTEGRAL GIVES NO NET FORCE. THAT'S WHY THIS TERM IS OFTEN IGNORED.

SUBTLETY: IGNORING THE ELECTROSTRICTION IN A LARGE ENOUGH VOLUME INDEED GIVES THE CORRECT TOTAL BULK FORCE, BUT THE INCORRECT PRESSURE AT POINTS WITHIN THE DIELECTRIC