

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}; \quad d\tau = dx dy dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin \theta dr d\theta d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$

Curl: $\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}}$
 $+ \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}}$

Laplacian: $\nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}; \quad d\tau = s ds d\phi dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$	(permeability of free space)
$c = 3.00 \times 10^8 \text{ m/s}$	(speed of light)
$e = 1.60 \times 10^{-19} \text{ C}$	(charge of the electron)
$m = 9.11 \times 10^{-31} \text{ kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \quad \begin{cases} \hat{\mathbf{x}} = \sin \theta \cos \phi \hat{\mathbf{r}} + \cos \theta \cos \phi \hat{\boldsymbol{\theta}} - \sin \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin \theta \sin \phi \hat{\mathbf{r}} + \cos \theta \sin \phi \hat{\boldsymbol{\theta}} + \cos \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \quad \begin{cases} \hat{\mathbf{r}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s \cos \phi \\ y = s \sin \phi \\ z = z \end{cases} \quad \begin{cases} \hat{\mathbf{x}} = \cos \phi \hat{\mathbf{s}} - \sin \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin \phi \hat{\mathbf{s}} + \cos \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \quad \begin{cases} \hat{\mathbf{s}} = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{\mathbf{r}} \quad \text{Coulomb's Law} \quad \mathbf{F} = Q\mathbf{E} + Q\mathbf{v} \times \mathbf{B} \quad \text{Lorentz force}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} \, d\tau' \quad \text{electric field from continuous charge distribution}$$

$$\oiint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{encl}} \quad \text{Gauss' Law (integral)} \quad \oint \mathbf{E} \cdot d\boldsymbol{\ell} = 0 \quad \text{(statics)}$$

$$V(\mathbf{r}) = -\int_{\varnothing}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \quad \text{Electrostatic potential, and } \mathbf{E} = -\nabla V \text{ (statics)}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad \text{and} \quad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')}{r} d\tau' \quad \text{(setting reference point at infinity)}$$

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} \quad \text{and} \quad \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = \frac{1}{\epsilon_0} \sigma \quad \text{boundary conditions}$$

$$D_{\text{above}}^{\perp} - D_{\text{below}}^{\perp} = \sigma_f$$

$$V(\mathbf{b}) - V(\mathbf{a}) = \frac{W}{Q} \quad W = \frac{1}{2} \sum_{i=1}^n q_i V(\mathbf{r}_i) \quad \text{and} \quad W = \frac{1}{2} \iiint \rho V d\tau$$

$$Q = CV \quad W = \frac{1}{2} CV^2 \quad W = \frac{1}{2} Q^2 / C \quad \text{capacitors}$$

$$\nabla^2 V = -\frac{1}{\epsilon_0} \rho \quad \text{Poisson's equation} \quad \nabla^2 V = 0 \quad \text{Laplace's equation}$$

$$V(x, y) = \sum_{n=0}^{\infty} (A_n e^{+kx} + B_n e^{-kx})(C_n \sin ky + D_n \cos ky) \quad \text{solution to Laplace's equation in Cartesian coordinates in two dimensions}$$

$$\int_0^a \sin(n\pi \frac{y}{a}) \sin(n'\pi \frac{y}{a}) dy = \begin{cases} 0 & n \neq n' \\ \frac{a}{2} & n = n' \end{cases} \quad \text{orthogonality of sines}$$

$$\int_0^a \sin(n\pi \frac{y}{a}) dy = \begin{cases} 0 & n \text{ even} \\ \frac{2a}{n\pi} & n \text{ odd} \end{cases} \quad e^{+kx} + e^{-kx} = 2 \cosh kx$$

$$\int_0^{\pi} \cos^2 \theta \sin \theta d\theta = 2/3$$

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = (3x^2 - 1)/2$
$P_3(x) = (5x^3 - 3x)/2$
$P_4(x) = (35x^4 - 30x^2 + 3)/8$
$P_5(x) = (63x^5 - 70x^3 + 15x)/8$

TABLE 3.1 Legendre Polynomials.

$$V(r, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}}) P_{\ell}(\cos \theta)$$

solution to Laplace's equation in spherical coordinates with azimuthal symmetry

$$\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)d\cos\theta = \begin{cases} 0 & \ell \neq \ell' \\ \frac{2}{2\ell+1} & \ell = \ell' \end{cases} \quad \text{orthogonality of Legendre polynomials}$$

$$V(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta \quad \text{potential outside a neutral conducting sphere in uniform field}$$

$$\frac{1}{\mathfrak{R}} = \frac{1}{r} \sum_0^\infty \left(\frac{r'}{r} \right)^n P_n(\cos\alpha) \quad 1/r \text{ expansion in Legendre polynomials}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum_0^\infty \frac{1}{r^{n+1}} \iiint (r')^n P_n(\cos\alpha) \rho(\mathbf{r}') d\tau' \quad \text{multipole expansion}$$

$$\vec{p} = \iiint \vec{r}' \rho(\vec{r}') d\tau' \quad \vec{p} = \sum_1^n q_i \vec{r}'_i \quad V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2} \quad \text{dipole moment}$$

$$\vec{E}_{\text{dip}} = \frac{P}{4\pi\epsilon_0 r^3} (2 \cos\theta \hat{r} + \sin\theta \hat{\theta})$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad \mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \quad \mathbf{D} = \epsilon \mathbf{E}$$

$$\mathbf{F}_{\text{mag}} = \int I d\mathbf{l} \times \mathbf{B}$$

$$\mathbf{K} = d\mathbf{I} / d\ell_\perp \quad \mathbf{J} = d\mathbf{I} / da_\perp \quad \text{surface and volume currents}$$

$$\nabla \cdot \mathbf{J} + \frac{d\rho}{dt} = 0 \quad \text{conserved current}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{r}}{|\mathbf{r} - \mathbf{r}'|^2} \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K} \times \hat{r}}{|\mathbf{r} - \mathbf{r}'|^2} da' \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{\mathbf{J} \times \hat{r}}{|\mathbf{r} - \mathbf{r}'|^2} d\tau'$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

$$\nabla \cdot \mathbf{A} = 0 \quad \text{“Coulomb gauge” convention} \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad (\text{for Coulomb gauge})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} da' \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{I}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{l}'$$

$$\oint \mathbf{A} \cdot d\mathbf{l} = \Phi_m$$

$$\mathbf{A}_{\text{above}} - \mathbf{A}_{\text{below}} = 0 \quad \frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$$

$$\mathbf{m} = I \iint \hat{n} da = I \mathbf{a}$$

$$\mathbf{m} = \frac{1}{2} \oint \mathbf{r} \times I d\ell \quad \mathbf{m} = \frac{1}{2} \iint \mathbf{r} \times \mathbf{K} da \quad \mathbf{m} = \frac{1}{2} \iiint \mathbf{r} \times \mathbf{J} d\tau$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \quad \mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \quad U = -\mathbf{m} \cdot \mathbf{B}$$

$$\mathbf{J}_b = \nabla \times \mathbf{M} \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f \quad \oint \mathbf{H} \cdot d\mathbf{l} = I_{f,\text{encl}}$$

$$B_{\text{above}}^\perp - B_{\text{below}}^\perp = 0 \quad B_{\text{above}}^\parallel - B_{\text{below}}^\parallel = \mu_0 K \quad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}}$$

$$H_{\text{above}}^\perp - H_{\text{below}}^\perp = -(M_{\text{above}}^\perp - M_{\text{below}}^\perp) \quad \mathbf{H}_{\text{above}}^\parallel - \mathbf{H}_{\text{below}}^\parallel = \mathbf{K}_f \times \hat{\mathbf{n}}$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad \mathbf{M} = \chi_m \mathbf{H} \quad \mathbf{B} = \mu \mathbf{H}$$

$$U = \frac{1}{2} \iiint \mathbf{H} \cdot \mathbf{B} \, dv \quad U = \frac{1}{2} \iiint \mathbf{D} \cdot \mathbf{E} \, dv$$

$$\Phi_b = M_{ba} I_a \quad M_{ba} = \frac{\mu_0}{4\pi} \oint \oint \frac{dl_a \cdot dl_b}{r} \quad \mathcal{E}_b = -M_{ba} \frac{dI_a}{dt} \quad \Phi = LI \quad \mathcal{E} = -L \frac{dI}{dt}$$

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, & \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, & \mathbf{H} = \frac{1}{\mu} \mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy:
$$U = \frac{1}{2} \int (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) d\tau$$

Momentum:
$$\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B}) d\tau$$

Poynting vector:
$$\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})$$