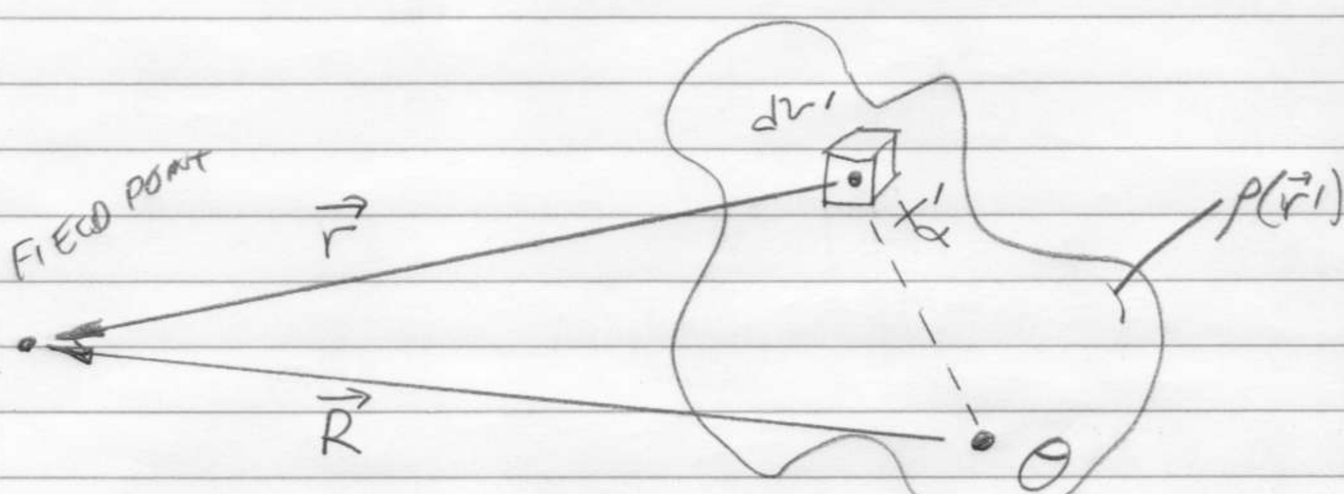
1. MULTIPOLES. TAYLOR EXPANSION

WE'RE LOOKING AT A LOCALIZED CHARGE DISTRIBUTION $\rho(\vec{r}\,')$. THE CARTESIAN-COORDINATE ORIGIN IS IN OR NEAR THE CHARGE DISTRIBUTION RELATIVE TO THE FIELD POINT;



TAYLOR-EXPAND $1/r$ IN POWERS OF x'_α ($\alpha = x, y, z$) ABOUT O:

$$\frac{1}{r} = \frac{1}{R} + x'_\alpha \left[\frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right)\right]_{r=R} + \frac{1}{2!}\, x'_\alpha x'_\beta \left[\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta}\left(\frac{1}{r}\right)\right]_{r=R} + \cdots$$

WHERE WE USE "SUMMATION NOTION", e.g.

$$\vec{A}\cdot\vec{B} = \sum_{\alpha=x,y,z} A_\alpha B_\alpha = A_\alpha B_\alpha .$$

THE $1/r$ EXPANSION APPLIED TO THE FORM OF THE POTENTIAL

$$\Phi(\vec{R}) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(x_\alpha')}{r}\, dv'$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{1}{R} \iiint \rho\, dv'$$

$$+ \frac{1}{4\pi\varepsilon_0} \left[\frac{d}{dx_\alpha'}\left(\frac{1}{r}\right)\right]_{r=R} \iiint x_\alpha' \rho\, dv'$$

$$+ \frac{1}{4\pi\varepsilon_0} \frac{1}{2!} \left[\frac{d^2}{dx_\alpha' dx_\beta'}\left(\frac{1}{r}\right)\right]_{r=R} \iiint x_\alpha' x_\beta' \rho\, dv'$$

$$+ \ldots$$

THE VOLUME INTEGRALS ARE "MOMENTS" OF THE CHARGE DISTRIBUTION, e.g.,

$\iiint \rho\, dv'$ IS THE TOTAL CHARGE;

$\iiint x_\alpha' \rho\, dv'$ IS THE α^{th} COMPONENT OF THE DIPOLE MOMENT;

$\iiint x_\alpha' x_\beta' \rho\, dv' \equiv Q_{\alpha\beta}$ IS THE SYMMETRIC QUADRUPOLE TENSOR.

IN PRINCIPLE, A SYMMETRIC TENSOR HAS 6 INDEPENDENT NUMBERS. IF WE DIAGONALIZE $Q_{\alpha\beta}$ (THAT IS, ROTATE $Q_{\alpha\beta}$ INTO ITS PRINCIPAL AXES, IT SEEMS $Q_{\alpha\beta}$ HAS 3 INDEPENDENT NUMBERS; BUT THIS IS NOT SO.

ISOLATING THE QUADRUPOLE TERM IN THE POTENTIAL:

$$\Phi^{(4)}(\vec{R}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{2!}\left[\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta}\left(\frac{1}{r}\right)\right]_{r=R} Q_{\alpha\beta}$$

THIS CAN BE WRITTEN AS

$$\Phi^{(4)}(\vec{R}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{2!\cdot 3}\left[\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta}\left(\frac{1}{r}\right)\right]_{r=R}$$

$$\times \left(3Q_{\alpha\beta} - \delta_{\alpha\beta} Q_{\gamma\gamma}\right)$$

(THIS NEEDS TO BE SHOWN).
FIRSTLY, RECALL $\delta_{\alpha\beta} = 0$ FOR $\alpha \neq \beta$ AND $\delta_{\alpha\beta} = 1$ FOR $\alpha = \beta$.

SECONDLY, THE TERM IN $\Phi^{(4)}$ CONTAINING $\delta_{\alpha\beta}$ DOESN'T AFFECT $\Phi^{(4)}$ BECAUSE

$$\left[\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta}\left(\frac{1}{r}\right)\right]_{r=R} \delta_{\alpha\beta} Q_{\gamma\gamma}$$

$$= \frac{\partial^2}{\partial x'^2_\gamma}\left(\frac{1}{r}\right) Q_{\gamma\gamma}$$

$$= \nabla^2\left(\frac{1}{r}\right) Q_{\gamma\gamma} = 0.$$

SO, INDEED THE TERM CONTAINING $\delta_{\alpha\beta}$ DOES NOT CONTRIBUTE TO $\Phi^{(4)}$. BY WHY DO THIS?

SINCE THE $\delta_{\alpha\beta}$ TERM DOESNT CONTRIBUTE, WE CAN REPLACE $Q_{\alpha\beta}$ WITH $D_{\alpha\beta}/3$ WHERE

$$D_{\alpha\beta} = 3Q_{\alpha\beta} - \delta_{\alpha\beta} Q_{\gamma\gamma}.$$

SO, IN TERMS OF $\Phi^{(4)}$, $Q_{\alpha\beta}$ AND $D_{\alpha\beta}/3$ ARE EQUIVALENT. IN THE PRINCIPAL-AXIS REPRESENTATION $D_{\alpha\beta}$ IS DIAGONAL, BUT ALSO $D_{\gamma\gamma} = 0$ (THIS LATER IS EASY TO CHECK). SO, 3 DIAGONAL NUMBERS LESS THE $Q_{\gamma\gamma} = 0$ CONSTRAINT YIELDS TWO INDEPENDENT NUMBERS.