C2.1 ①

1. CHALLENGE PROBLEM (SMYTHE JACKSON, ZANGWILL) "WEDGE" GEOMETRY (CLASSIC PROBLEM DONE IN A NUMBER OF TEXTS).

THIS IS AN ELABORATION OF JACKSON SECTION 2.11 "FIELDS & CHARGE DENSITIES IN 2D CORNERS & ALONG EDGES".

THAT SECTION HAD SOLUTIONS OF

$$V(\rho,\phi) = \mathbb{R}(\rho)\Phi(\phi) \quad \text{OF}$$

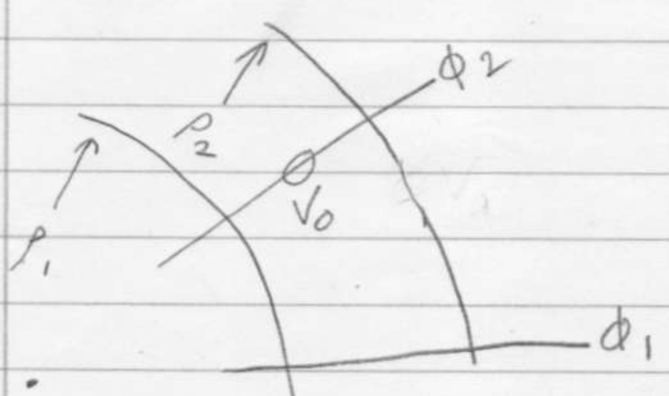
$$\left.\begin{aligned} \mathbb{R}(\rho) &= a_0 + b_0 \ln \rho \\ \Phi(\phi) &= A_0 + B_0 \phi \end{aligned}\right\} \nu = 0$$

$$\mathbb{R}(\rho) = a\rho^{\nu} + b\rho^{-\nu}$$

$$\Phi(\phi) = A\cos(\nu\phi) + B\sin(\nu\phi)$$

AND $\pm\nu^2$ THE SEPARATION CONSTANTS.

WE APPLY THIS TO A DIFFERENT PROBLEM, THE GEOMETRY IS 2D:



WITH TRANSLATIONAL INVARIANCE INTO AND OUT OF THE PAGE. THE VOLUME IS BOUNDED BY PLANES $\phi = \phi_1$ AND $\phi = \phi_2$, AND ARCS $\rho = \rho_1$ AND $\rho = \rho_2$. THE FIELD POINT IS INSIDE THE VOLUME. ALL SURFACES ARE AT $V = 0$ EXCEPT THE $\phi = \phi_2$ PLANE $V(\phi = \phi_2) = V_0$.

THE $\rho = \rho_1$ AND $\rho = \rho_2$ BOUNDARY CONDITIONS IMPLY $a_0 = b_0 = A_0 = B_0 = 0$.

HOW DO YOU ENFORCE THE $\rho = \rho_1$ AND $\rho = \rho_2$ BOUNDARY CONDITIONS WITH TERMS LIKE ρ^{ν} AND $\rho^{-\nu}$?

TRICK: WRITE $\eta = i\nu$ AND

$$\rho^{\pm\nu} = e^{\pm\nu \ln \rho}$$

THIS HAS THE EFFECT OF TURNING THE POWER LAW TERM INTO HARMONIC FUNCTIONS, AND WE KNOW HOW TO ENFORCE $V = 0$ BOUNDARY CONDITIONS WITH HARMONIC FUNCTIONS.

WE THEN HAVE

$$V(\rho,\phi) = \sum_{m \neq 0} \left(A_m e^{+im \ln \rho} + B_m e^{-im \ln \rho}\right) \times$$

$$\left(a_m \sinh m\phi + b_m \cosh m\phi\right).$$

WITH $e^{\pm im \ln \rho} = \cos m \ln \rho \pm i \sin m \ln \rho$
AND HAVING $V(\rho=\rho_2, \phi) = 0$, AND
CHOOSING THE $\sin m \ln \rho$ TERM
TO GET ZERO AT $\rho = \rho_1$ AND $\rho = \rho_2$,
WE HAVE $A_m = -B_m$, AND
$C_m = A_m - B_m$. REQUIRING
BOTH $V(\rho_1, \phi) = V(\rho_2, \phi) = 0$ REQUIRES
THE USUAL HARMONIC CONDITION
$m = n\pi / \ln \rho_1/\rho_2$. THIS LEAVES
US WITH

$$V(\rho,\phi) = \sum_n C_m \sin\left\{ n\pi \frac{\ln \rho/\rho_2}{\ln \rho_1/\rho_2} \right\}$$

$$\times \sinh\left\{ \frac{m\pi(\phi - (\phi_2 - \phi_1))}{\ln \rho_1/\rho_2} \right\}$$

WE NEED TO FIND COEFFICIENTS C_m.
ORTHOGONALITY IS TRICKY.
TRICK: MULTIPLY BY
$\sin m' \pi \ln \rho/\rho_2$
AND INTEGRATE OVER THE DOMAIN
$\pi \ln \rho/\rho_2 : 0, 2\pi$
THIS IS GOING TO BE COMPLICATED!