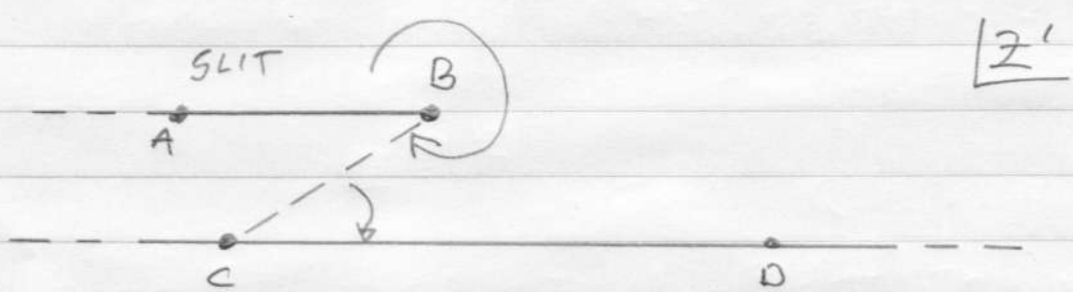
2. CHALLENGE PROBLEM (SMYTHE)
WHAT'S THE EXACT FRINGE FIELD OF A SEMI-INFINITE PARALLEL-PLATE CAPACITOR?

USE CONFORMAL MAPPING.

START WITH A COMPLEX PLANE WITH A SLIT ALONG A HALF-LINE PARALLEL TO THE REAL AXIS:



NOTICE THERE'S A POLYGON ABCD. WE TAKE A LIMIT $C \to -\infty$ ALONG THE REAL AXIS; ALSO A, B, D ARE AT ∞, AND THE ANGLES AROUND B AND C ARE 2π AND 0.

WE NOW MAP A, B, C, D TO $\infty, -1, 0, \infty$.

FROM COMPLEX ANALYSIS

$$dz'/dz = \gamma (z-a_1)^{\alpha_1 - 1} (z-a_2)^{\alpha_2 - 1} \cdots$$

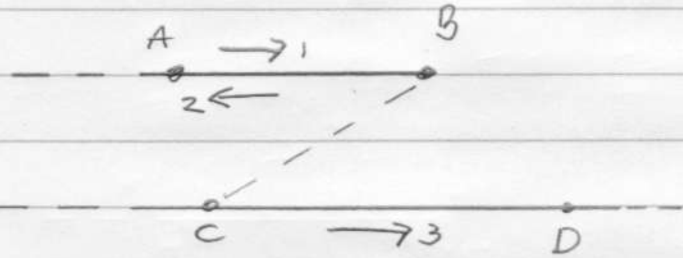
TO MAP A POLYGON ONTO A HALF-PLANE.

WITH a_n THE VERTEX POINT, AND
$\pi\alpha_n$ THE BEND ANGLE.

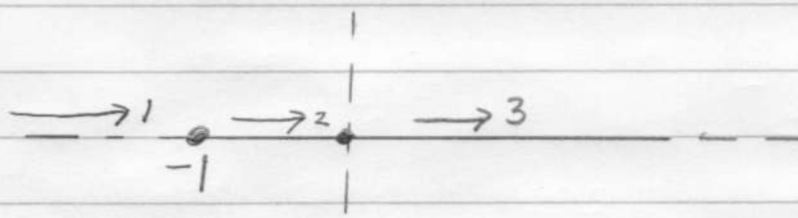
FOR $\gamma = a > 0$,

$$dZ'/dZ = \frac{a(z+1)}{z} = a\left(1 + 1/z\right) \quad \mathrm{I}$$

THE MAPPED SYSTEMS LOOK LIKE



BECOMES



dZ'/dZ HAS SOLUTION $Z' = a(z + \log z)$ II

NOTICE EQUATION I HAS dZ'/dZ REAL WHEN X IS REAL, HENCE dZ' IS PARALLEL TO dZ AS X VARIES OVER

$$-\infty < x < -1, \quad -1 < x < 0, \quad 0 < x < +\infty$$

WITH $\mathrm{Im}(\log z)$ OVER THESE INTERVALS

$$i\pi, \quad i\pi, \quad 0$$

WITH II, WE ALSO HAVE IN THESE INTERVALS

$$dz'/dx > 0, \quad dz'/dx < 0, \quad dz'/dx > 0$$

NOTICE FOR $x = -1$, $z' = a(\pi i - 1)$ (POINT B).

WE HAVE MAPPED THE BOUNDARY OF THE z' PLANE TO THE REAL AXIS OF THE z PLANE.

THERE IS A SUBLETY THAT CONFUSES ME: IS THE MAPPING ONE-TO-ONE? I THINK SO, AND I THINK THE CUT ENSURES THIS, BUT ...

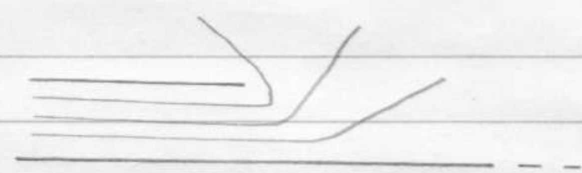
THERE'S A THEOREM IN COMPLEX ANALYSIS: f & g ARE ANALYTIC IN D. $f(z) = g(z)$ NEAR α IN D. THEN $f = g$ IN D.

HENCE THE MAPPING OF THE INTERIORS IS LIKEWISE ONE-TO-ONE.

NOW WE CAN FIND THE EQUIPOTENTIAL LINES ("STREAMLINES"):
THESE ARE LINES OF CONSTANT $\text{Im} \log z$ (OR $\theta = \text{CONSTANT} = \eta$) FROM A SOURCE AT THE ORIGIN. THESE STREAMLINES MAP ONTO

$$\Phi \propto z' = re^{i\eta} + \log r + i\eta \qquad 0 < r < \infty$$

THESE ARE EXACT EQUIPOTENTIALS AND LOOK LIKE



REFLECT THIS ABOUT THE REAL AXIS TO FIND EQUIPOTENTIALS FOR THE BOTTOM HALF OF THE CAPACITOR.