

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}; \quad d\tau = dx dy dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin\theta d\phi \hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin\theta dr d\theta d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta v_\theta) + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi}$

Curl: $\nabla \times \mathbf{v} = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta} (\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}}$
 $+ \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}}$

Laplacian: $\nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}; \quad d\tau = s ds d\phi dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

FUNDAMENTAL CONSTANTS

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|---|------------------------------|
| $\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2$ | (permittivity of free space) |
| $\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$ | (permeability of free space) |
| $c = 3.00 \times 10^8 \text{ m/s}$ | (speed of light) |
| $e = 1.60 \times 10^{-19} \text{ C}$ | (charge of the electron) |
| $m = 9.11 \times 10^{-31} \text{ kg}$ | (mass of the electron) |

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \quad \begin{cases} \hat{\mathbf{x}} = \sin \theta \cos \phi \hat{\mathbf{r}} + \cos \theta \cos \phi \hat{\boldsymbol{\theta}} - \sin \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin \theta \sin \phi \hat{\mathbf{r}} + \cos \theta \sin \phi \hat{\boldsymbol{\theta}} + \cos \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \quad \begin{cases} \hat{\mathbf{r}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s \cos \phi \\ y = s \sin \phi \\ z = z \end{cases} \quad \begin{cases} \hat{\mathbf{x}} = \cos \phi \hat{\mathbf{s}} - \sin \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin \phi \hat{\mathbf{s}} + \cos \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \quad \begin{cases} \hat{\mathbf{s}} = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{\mathbf{r}} \quad \text{Coulomb's Law} \quad \mathbf{F} = Q\mathbf{E} + Q\mathbf{V} \times \mathbf{B} \quad \text{Lorentz force}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} \, d\tau' \quad \text{electric field from continuous charge distribution}$$

$$\oiint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{encl}} \text{ Gauss' Law (integral)} \quad \oint \mathbf{E} \cdot d\mathbf{l} = 0 \text{ (statics)}$$

$$V(\mathbf{r}) = -\int_{\phi}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \text{ Electrostatic potential, and } \mathbf{E} = -\nabla V \text{ (statics)}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \text{ Poisson's equation, and } \nabla^2 V = 0 \text{ Laplace's equation (regions of no charge)}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \text{ and } V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')}{r} d\tau' \text{ (setting reference point at infinity)}$$

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} \text{ and } \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = \frac{1}{\epsilon_0} \sigma \text{ boundary conditions}$$

$$D_{\text{above}}^{\perp} - D_{\text{below}}^{\perp} = \sigma_f$$

$$V(\mathbf{b}) - V(\mathbf{a}) = \frac{W}{Q} \quad W = \frac{1}{2} \sum_{i=1}^n q_i V(\mathbf{r}_i) \text{ and } W = \frac{1}{2} \iiint \rho V d\tau$$

$$Q = CV \quad W = \frac{1}{2} CV^2 \quad W = \frac{1}{2} Q^2 / C \text{ capacitors}$$

$$\nabla^2 V = -\frac{1}{\epsilon_0} \rho \text{ Poisson's equation} \quad \nabla^2 V = 0 \text{ Laplace's equation}$$

$$V(x, y) = \sum_{n=0}^{\infty} (A_n e^{+kx} + B_n e^{-kx})(C_n \sin ky + D_n \cos ky) \text{ solution to Laplace's equation in Cartesian coordinates in two dimensions}$$

$$\int_0^a \sin(n\pi \frac{y}{a}) \sin(n'\pi \frac{y}{a}) dy = \begin{cases} 0 & n \neq n' \\ \frac{a}{2} & n = n' \end{cases} \text{ orthogonality of sines}$$

$$\int_0^a \sin(n\pi \frac{y}{a}) dy = \begin{cases} 0 & n \text{ even} \\ \frac{2a}{n\pi} & n \text{ odd} \end{cases} \quad e^{+kx} + e^{-kx} = 2 \cosh kx$$

$$\int_0^\pi \cos^2 \theta \sin \theta d\theta = 2/3$$

| |
|------------------------------------|
| $P_0(x) = 1$ |
| $P_1(x) = x$ |
| $P_2(x) = (3x^2 - 1)/2$ |
| $P_3(x) = (5x^3 - 3x)/2$ |
| $P_4(x) = (35x^4 - 30x^2 + 3)/8$ |
| $P_5(x) = (63x^5 - 70x^3 + 15x)/8$ |

TABLE 3.1 Legendre Polynomials.

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta)$$

solution to Laplace's equation in spherical coordinates and azimuthal symmetry

$$\int_0^\pi P_\ell(\cos \theta) P_{\ell'}(\cos \theta) d\cos \theta = \begin{cases} 0 & \ell \neq \ell' \\ \frac{2}{2\ell+1} & \ell = \ell' \end{cases} \quad \text{orthogonality of Legendre polynomials}$$

$V(r, \theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos \theta$ potential outside of neutral conducting sphere in uniform E-field

$$\frac{1}{\mathfrak{R}} = \frac{1}{r} \sum_0^\infty \left(\frac{r'}{r} \right)^n P_n(\cos \alpha) \quad 1/r \text{ expansion in Legendre polynomials}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum_0^\infty \frac{1}{r^{n+1}} \iiint (r')^n P_n(\cos \alpha) \rho(\mathbf{r}') d\tau' \quad \text{multipole expansion}$$

$$\vec{p} = \iiint \vec{r}' \rho(\vec{r}') d\tau' \quad \vec{p} = \sum_1^n q_i \vec{r}'_i \quad V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2} \quad \text{dipole moment}$$

$$\vec{E}_{\text{dip}} = \frac{p}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}) \vec{p}$$

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, & \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, & \mathbf{H} = \frac{1}{\mu} \mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy:
$$U = \frac{1}{2} \int (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) d\tau$$

Momentum:
$$\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B}) d\tau$$

Poynting vector:
$$\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})$$