Einstein’s equations: good for more than just gravity

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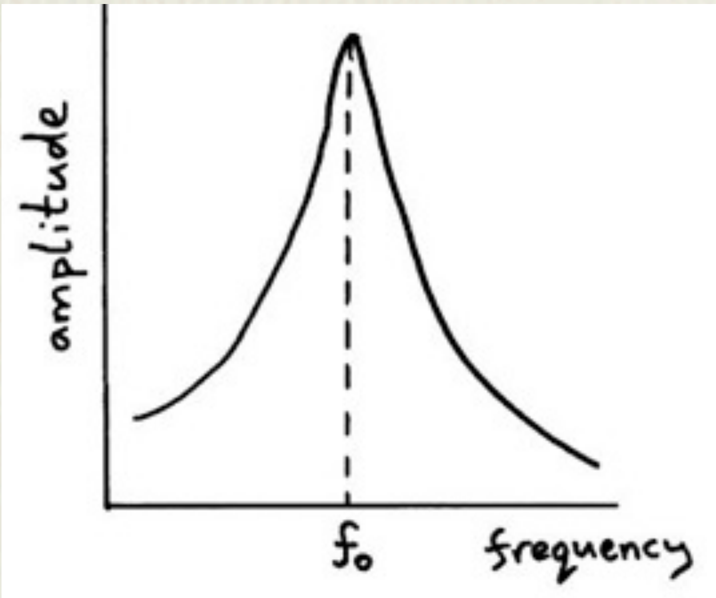
quasiparticles = elementary excitations

Ex: phonons, photons, dressed electrons, plasmons, Z-bosons, pions, polarons, magnons, ...



Quasiparticles =

long-lived (lifetime $\tau \gg \hbar$/energy),

weakly-interacting ($\lambda_{\text{m.f.p.}} \gg \lambda_{\text{de Broglie}}$)

resonances.

Good quasiparticle description $\Rightarrow$

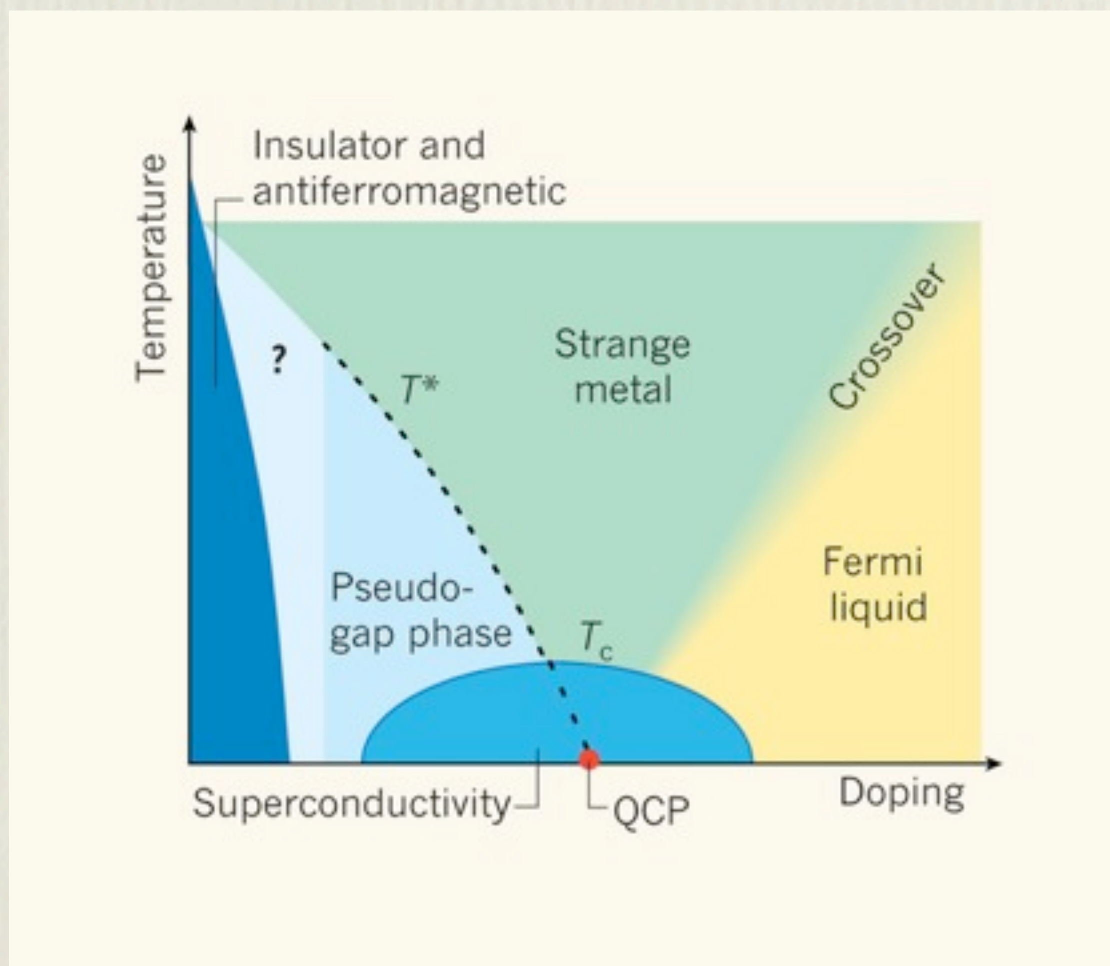
weakly coupled/weakly correlated system

Quasiparticles provide good description of dynamics in vast range of systems ... but not all.

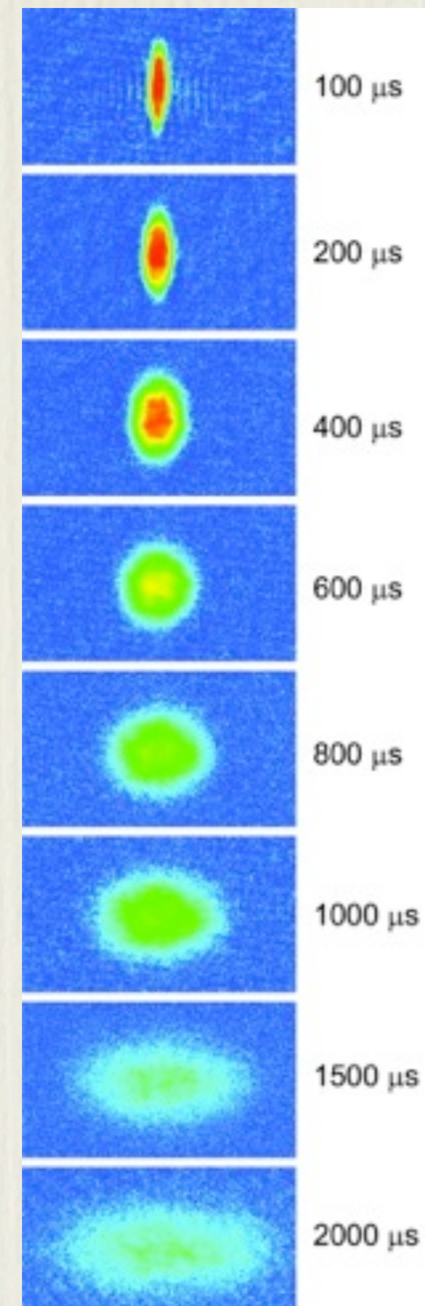
strongly coupled/strongly correlated systems

quark-gluon plasma

high T_c superconductors



unitary fermi gas



strongly coupled/strongly correlated systems

- no quasiparticle description
- no weak coupling approximation
- no kinetic theory approximation to dynamics
- very limited theoretical methods
- certain systems amenable to quantitative analysis using gauge/gravity duality

gauge/gravity duality

- a.k.a. "AdS/CFT duality," "gauge/string duality," "holography"

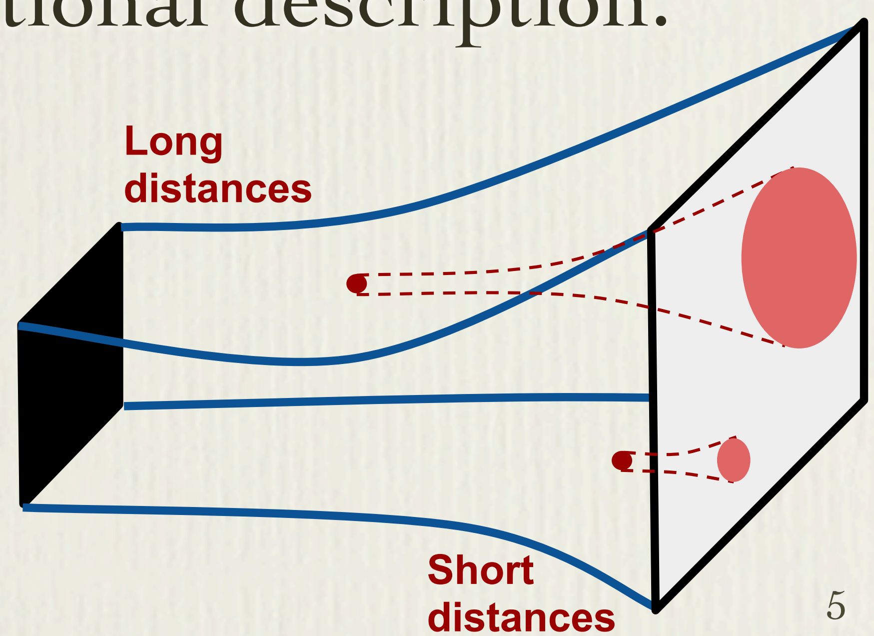
- Some non-Abelian gauge theories have exact reformulation as higher dimensional gravitational (or string) theories.

  Simplest case: maximally supersymmetric SU(N_c) Yang-Mills ($\mathcal{N}$=4 SYM)

  = string theory on $AdS_5 \times S^5$. More complicated generalizations to less supersymmetric, non-conformal theories.

- Strong coupling (and large N_c) limit of quantum field theory given by classical dynamics in dual gravitational description.

- Holographic description gives geometric representation of renormalization flow:



applications of holography

- Equilibrium properties of strongly coupled theories
- Near-equilibrium dynamics
- Far-from-equilibrium dynamics

work with Paul Chesler: arXiv:0812.2053, 0906.4426, 1011.3562

equilibrium properties

Equilibrium ⇔ static asymptotically anti-de Sitter geometries

non-Abelian plasma (≈ quark-gluon plasma) ⇔ AdS_5 black hole

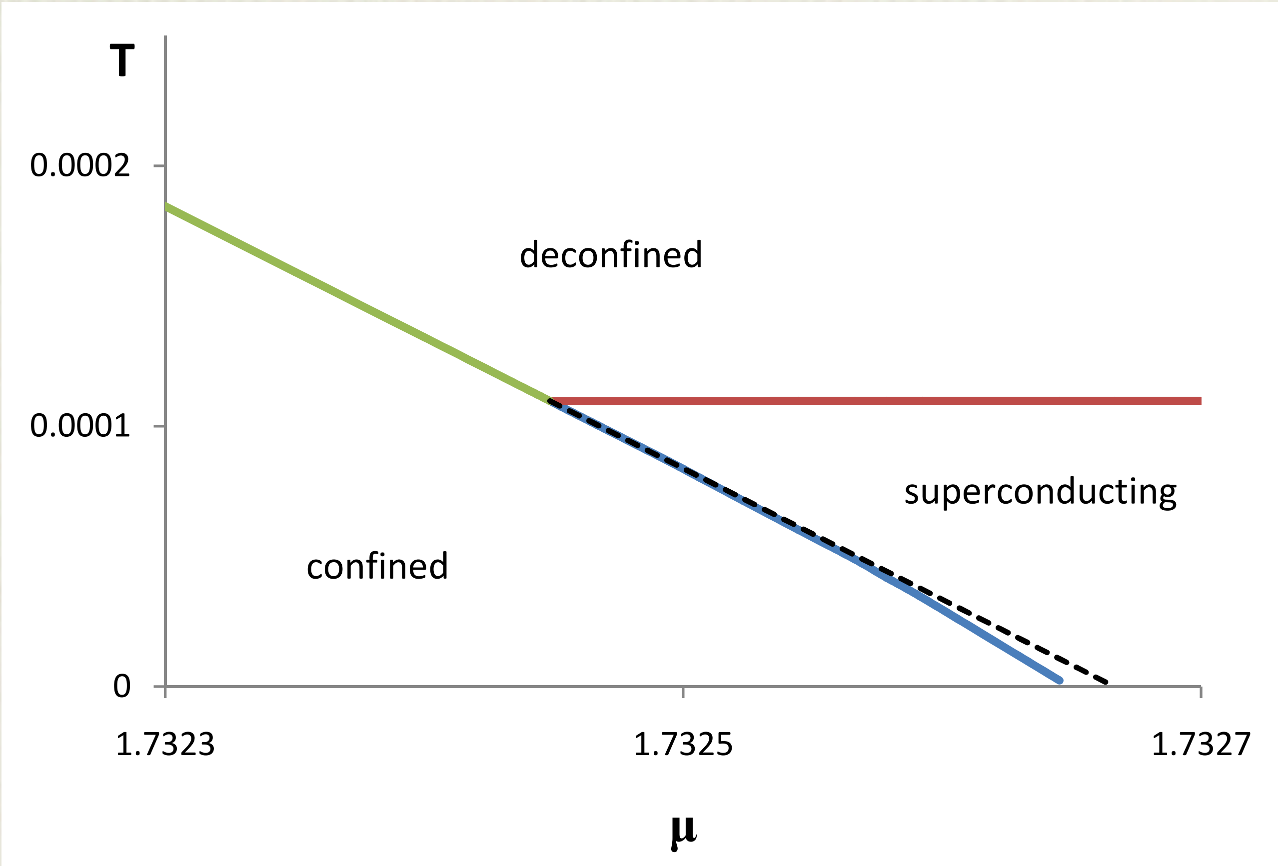
holographic superfluids, non-Fermi liquids ⇔ charged black holes, ...

thermodynamics

phase diagrams

phase transitions

screening lengths



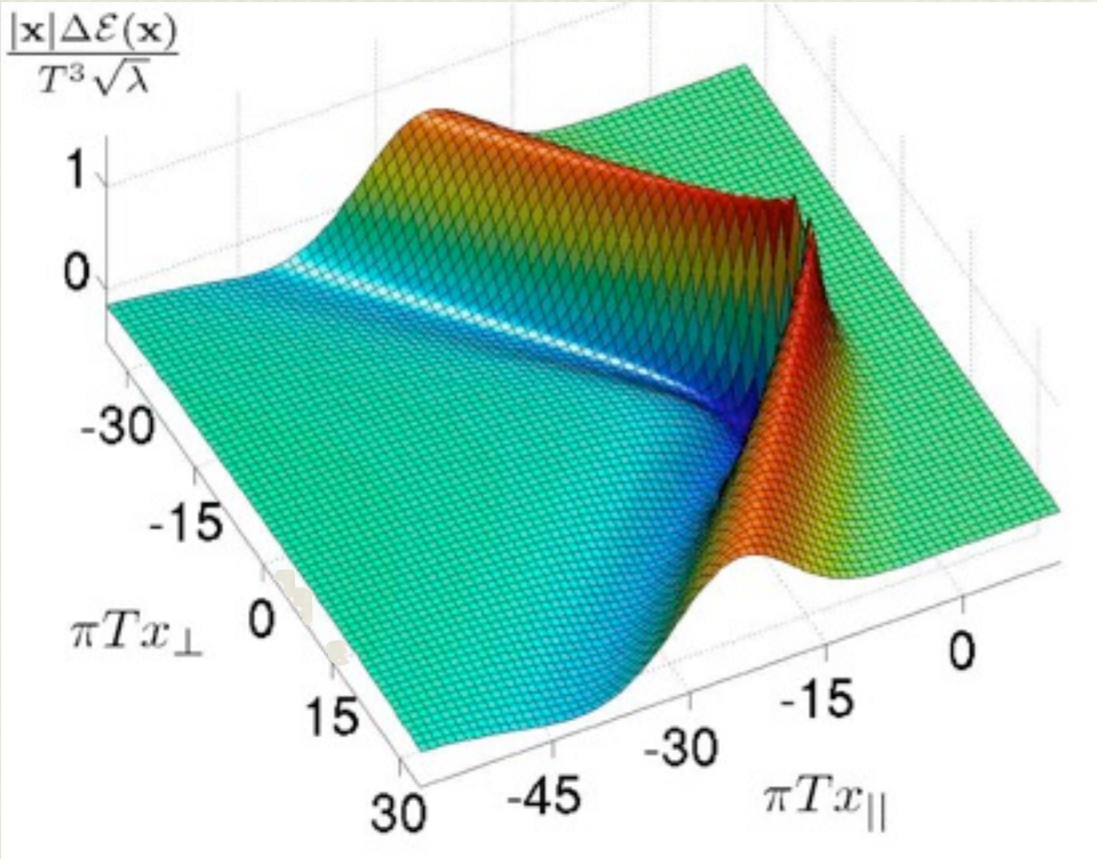
near-equilibrium dynamics

small fluctuations ⇒ linear response, spectral densities

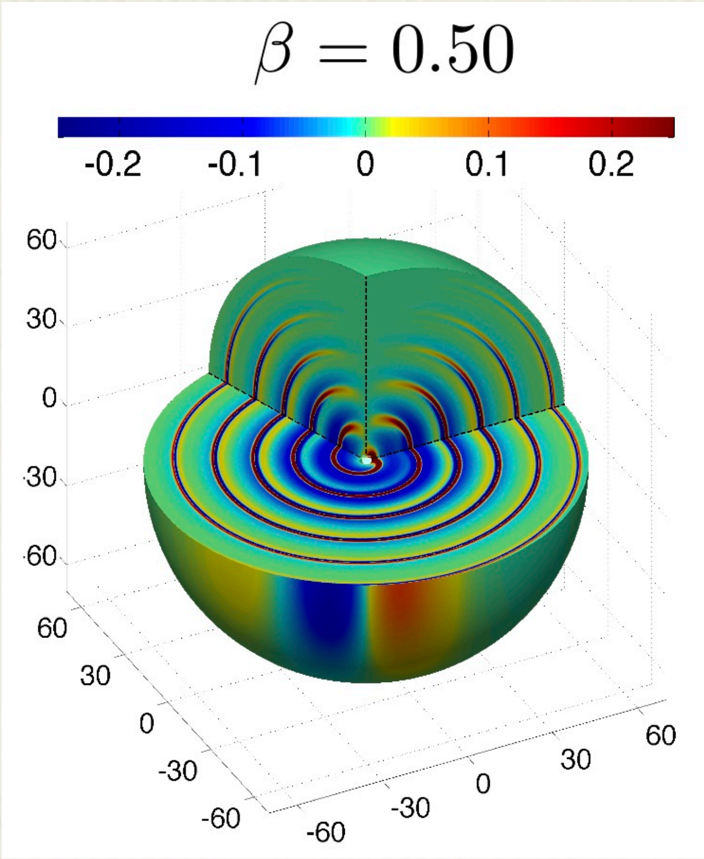
- transport coefficients: viscosity, diffusion, conductivity
- quasi-normal modes, late time equilibration
- photo-emission spectrum

external probes

- friction
- drag
- wakes



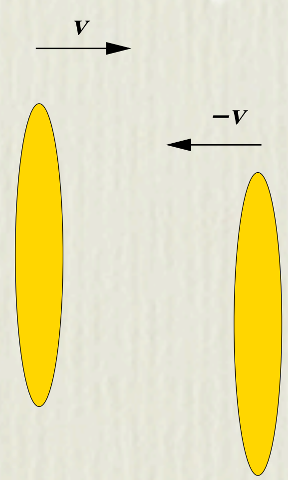
linear motion

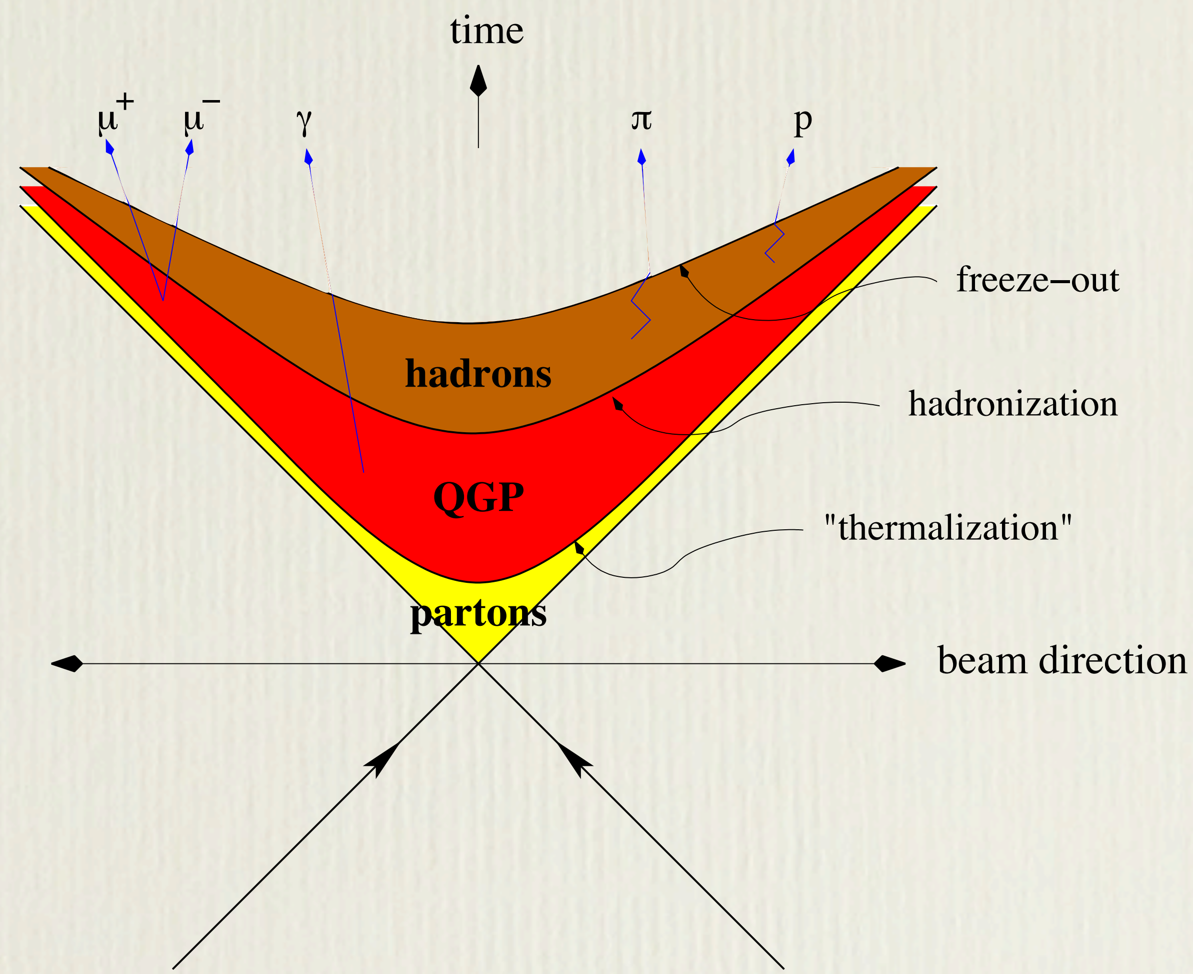


circular motion

far-from-equilibrium dynamics

Motivation: heavy ion collisions





how fast do produced partons isotropize?

when/where is hydrodynamics valid?

signatures of strongly coupled dynamics?

Idealizations:

SU(3) gauge field + quarks ➡ SU(N_c) gauge field + adjoint matter

strongly coupled QCD ➡ strongly coupled $\mathcal{N}$=4 SYM

Large, highly Lorentz contracted nuclei ➡ infinite planar null shocks

initial value problems

- Issues:
  - choice of problem

    isotropization, boost invariant expansion, shock wave collisions

  - choice of initial state

    time-dependent external fields, scattering

  - calculating time evolution

    numerical relativity: coordinate choice, integration scheme, stability

  - measuring observables

    thermalization time, agreement w. hydro, entropy, ...

non-equilibrium initial states

- Specify complete quantum statistical density matrix ρ ? Ugh!

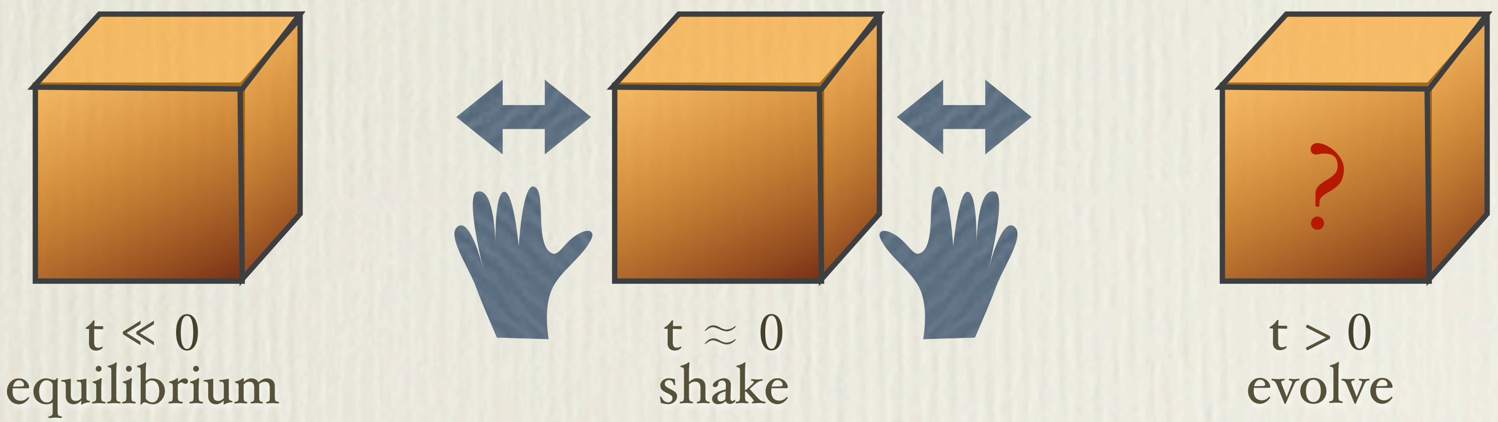
    Pick geometry on initial Cauchy surface ? Ugh!

    but see Heller, Janik, Witaszczyk arXiv:1103.3452

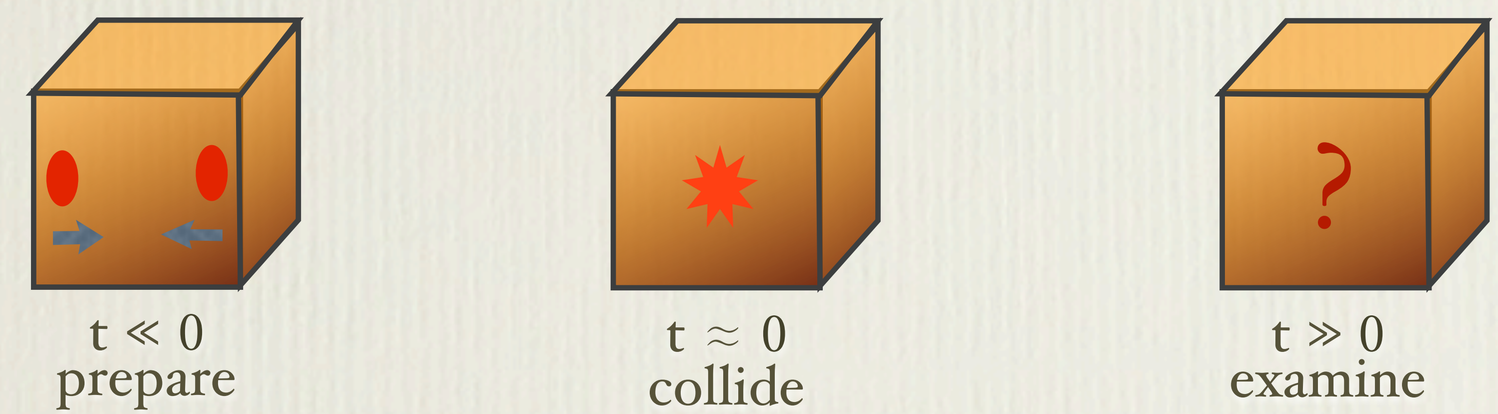
- Want operational description:

    1. Use time-dependent external fields:

time-dependent dynamics ⬌ external work done on system

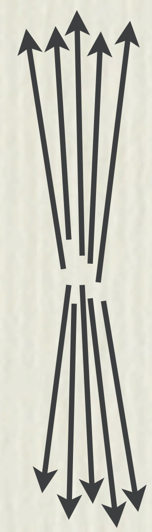


2. Do scattering experiment:



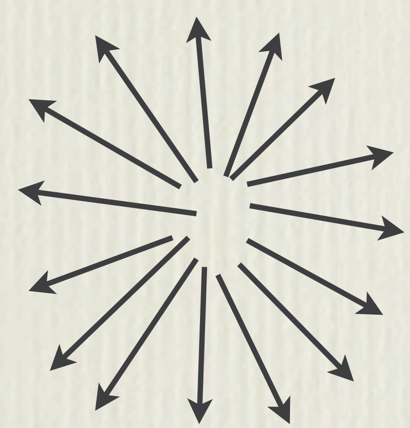
isotropization at RHIC

$t = 0$



mid-rapidity momentum distribution: highly anisotropic (oblate), $T_{xx} = T_{yy} \gg T_{zz}$

$t \approx \text{few fm}/c$



expanding fluid in approx. local equilibrium, $T_{ij} \propto \delta_{ij}$ in local fluid frame

Time scale? Relevant dynamics?

anisotropy dynamics

arXiv:0812.2053

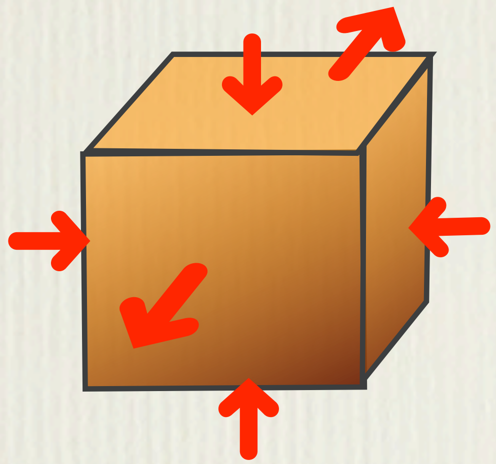
- Metric $g^{\mu\nu}$ = external field coupling to stress-energy $T^{\mu\nu}$

  ∴ time-dependent geometry ➡ non-equilibrium $\langle T^{\mu\nu} \rangle$

- "Simple" case: perfect spatial homogeneity, arbitrary anisotropy

$$ds^2 = -dt^2 + e^{f(t)}(dx^2 + dy^2) + e^{-2f(t)}\,dz^2$$

$$\Rightarrow \quad \langle T^{\mu\nu}(t, \mathbf{x}) \rangle = \begin{bmatrix} \varepsilon(t) & & & \\ & p_\perp(t) & & \\ & & p_\perp(t) & \\ & & & p_{\|}(t) \end{bmatrix}$$



- Choose time dependence, e.g., $f(t) = \frac{1}{2}\, c\, [\, 1 - \tanh(t/\tau)\,]$

gravitational description

- Solve *5D* Einstein equations with time-dependent boundary condition and anti-de Sitter initial condition.

- Coordinate choice:

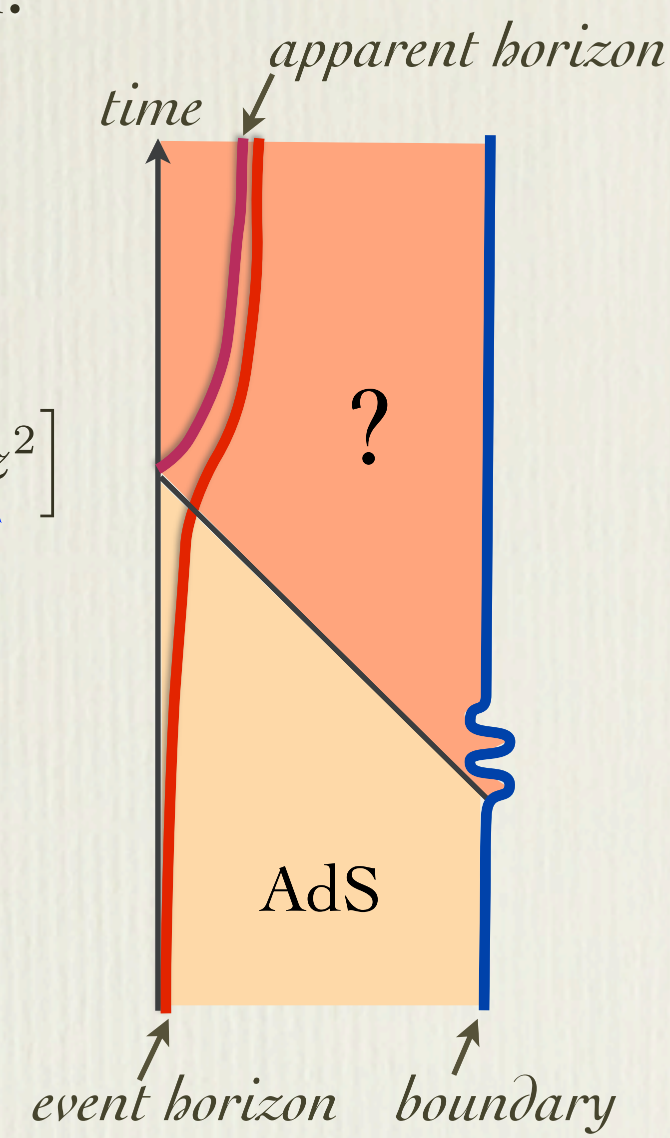
  ✓ Good: Incoming Eddington-Finklestein

$$ds^2 = -A(v,r)\,\underbrace{dv^2}_{\text{time coord.}} + 2\,dv\,\underbrace{dr}_{\text{5D radial coord.}} + \Sigma(v,r)^2\left[e^{B(v,r)}(dx^2+dy^2) + e^{-2B(v,r)}\underbrace{dz^2}_{\text{longitudinal direction}}\right]$$

  v = const. on incoming (radial) null geodesics

- Boundary conditions as $r \to \infty$: $A \to r^2$, $\Sigma \to r$, $B \to f(v)$

- Extract $\langle T^{\mu\nu}\rangle$ from sub-leading near-boundary asymptotics



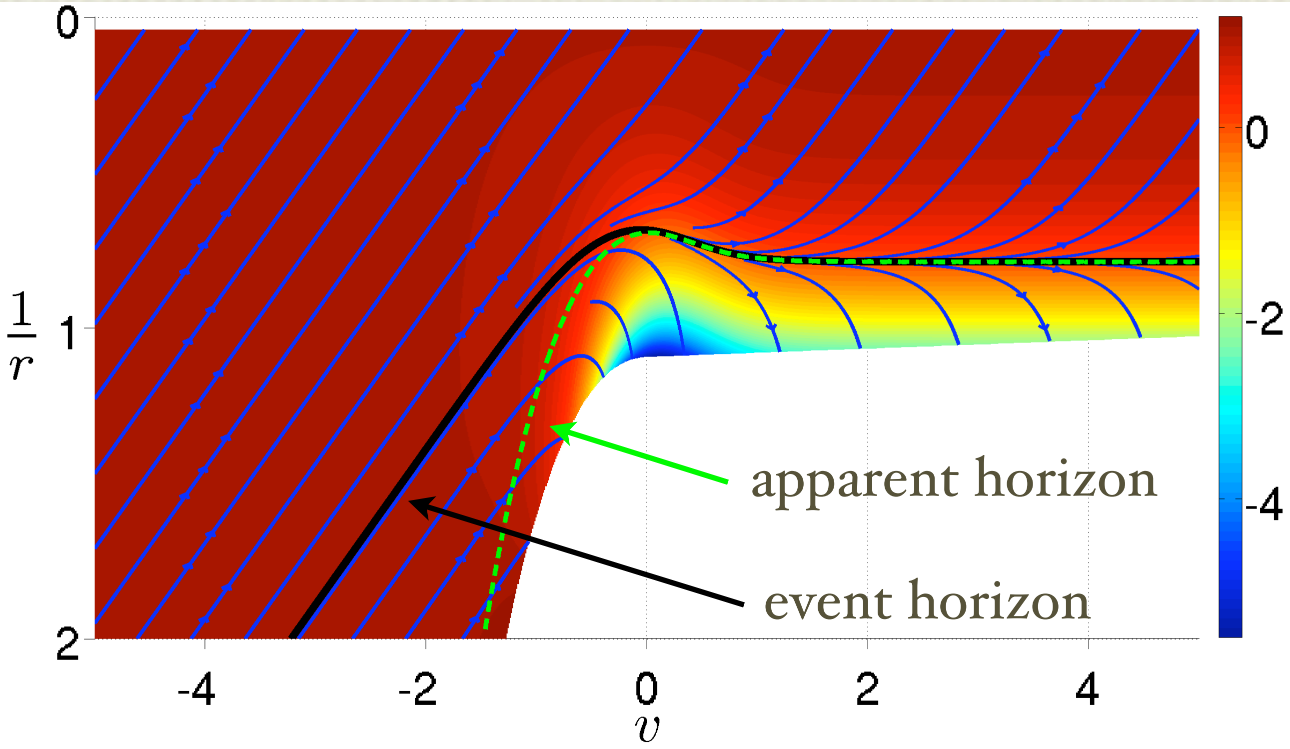
Einstein equations

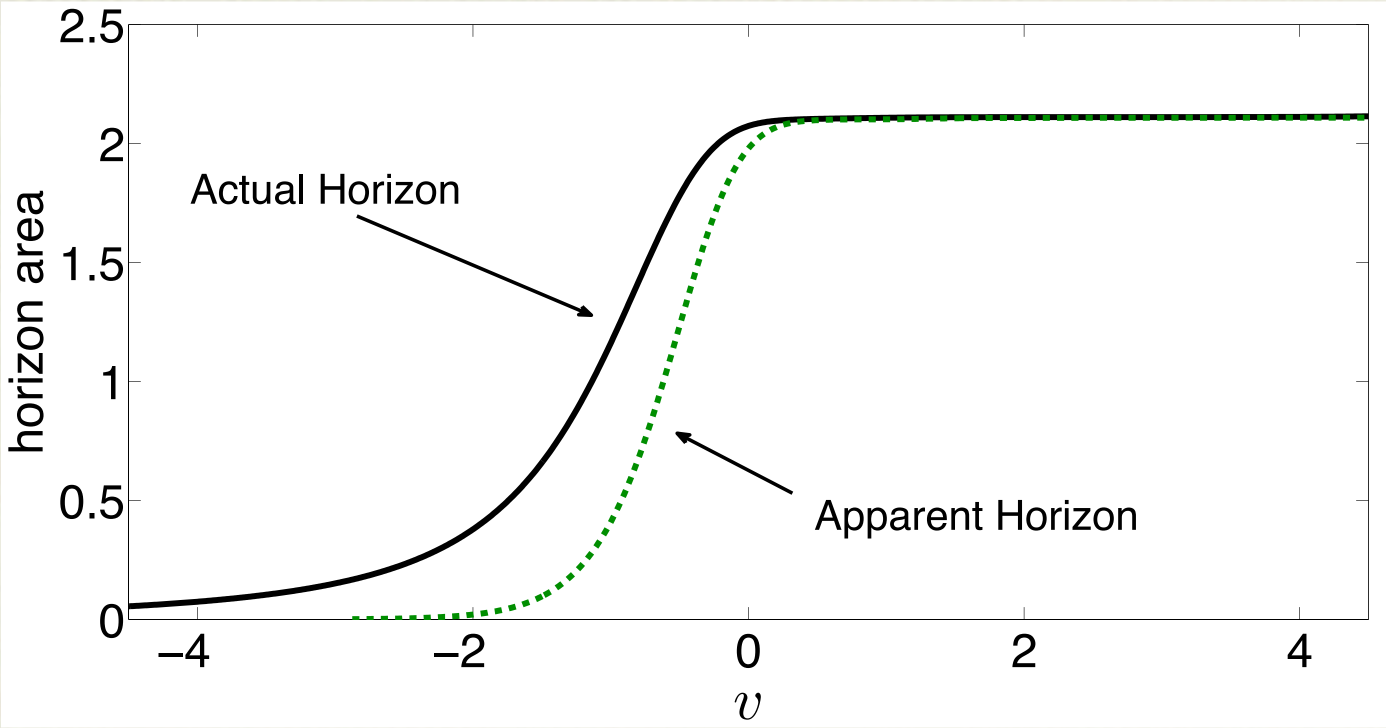
- $R_{MN} - \frac{1}{2} G_{MN}(R + 2\Lambda) = 0$
- 5 non-trivial components: *vv, rr, vr, zz, xx+yy*
  - ➡ 5 equations, 3 unknown functions (A,B,Σ)

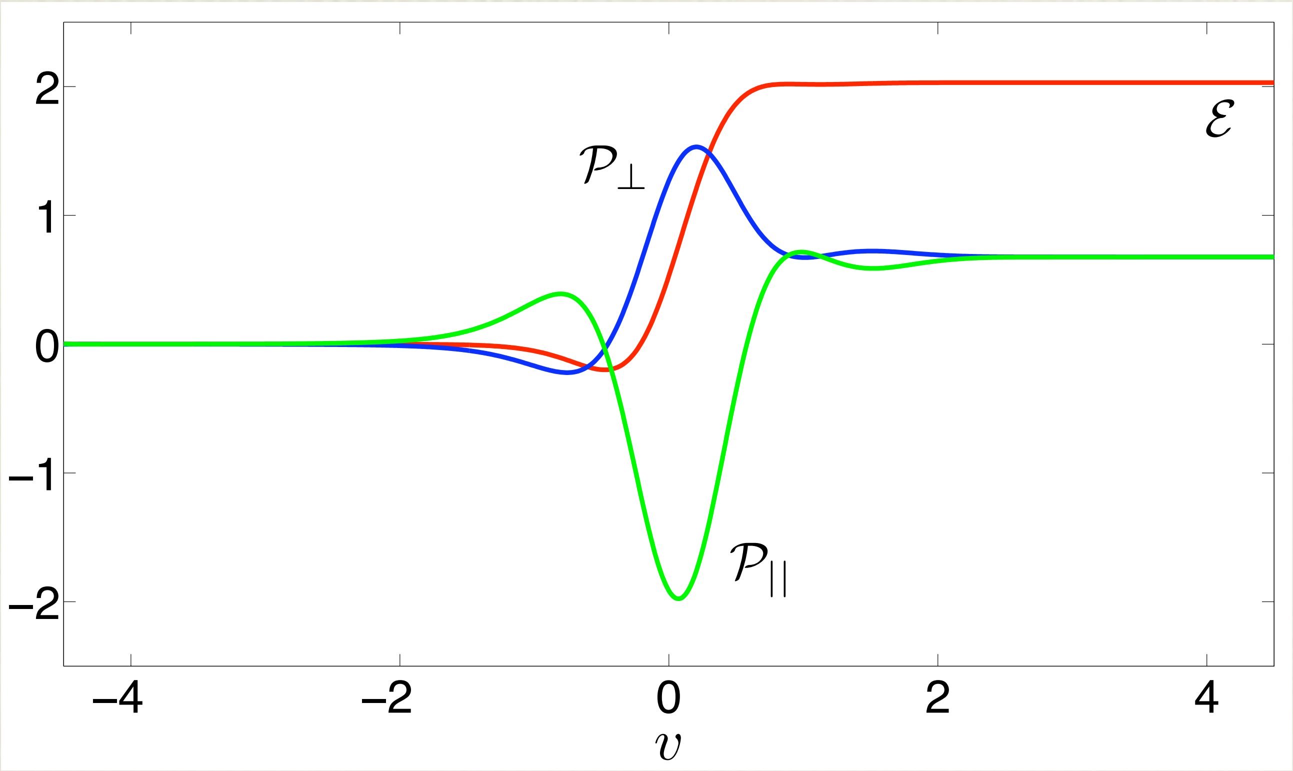
$$
\begin{aligned}
0 &= \Sigma\,(\dot{\Sigma})' + 2\,\Sigma'\,\dot{\Sigma} - 2\,\Sigma^2 \\
0 &= \Sigma\,(\dot{B})' + \tfrac{3}{2}\big(\Sigma'\,\dot{B} - B'\,\dot{\Sigma}\big) \\
0 &= A'' + 3\,B'\,\dot{B} - 12\,\Sigma'\,\dot{\Sigma}/\Sigma^2 + 4 \\
0 &= \ddot{\Sigma} + \tfrac{1}{2}\,(\dot{B})^2\,\Sigma - \tfrac{1}{2}\,A'\,\dot{\Sigma} \qquad \longleftarrow \text{boundary value constraint} \\
0 &= \Sigma'' + \tfrac{1}{2}\,(B')^2\,\Sigma \qquad \longleftarrow \text{initial value constraint}
\end{aligned}
$$

- $g' \equiv \partial_r g$ = directional derivative along incoming null geodesics
- $\dot{g} \equiv \partial_v g + \frac{1}{2} A\, \partial_r g$ = directional derivative along outgoing null geodesics
- N.B.: A = non-dynamical auxiliary field
- Each time step: solve nested linear ODEs to find Σ,B time derivatives - Easy!

isotropization: results







isotropization: results

- stable evolution

- rapid relaxation toward equilibrium

  $\tau_{\mathrm{iso}} \approx 0.7/T \Rightarrow \tau_{\mathrm{iso}} \approx 0.5$ fm/c for $T \approx 350$ MeV

- recent work [Chesler & Teaney arXiv:1112.6196] computes violations of fluctuation-dissipation theorem to probe thermalization of specific momentum modes

  slowest to thermalize: high momentum, lightlike

boost invariant expansion

arXiv:0906.4426

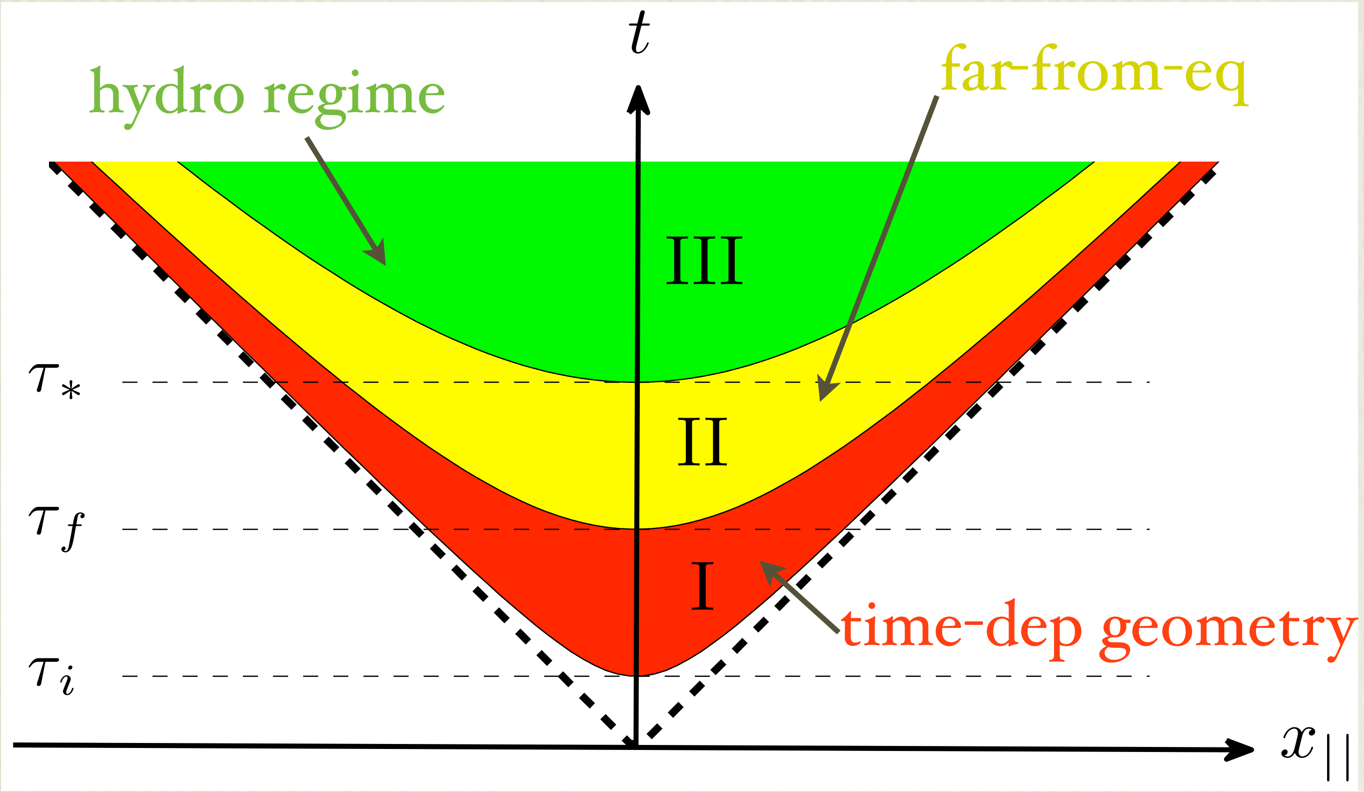
- boost invariance: dynamics only depends on proper time $\boldsymbol{\tau}$, independent of rapidity $\boldsymbol{\eta}$

  crude but instructive caricature of infinitely boosted nuclei, colliding at single spacetime event   Bjorken, 1983

- mimic creation with time dependent geometry:

$$ds^2 = -d\tau^2 + e^{\gamma(\tau)}\, d\boldsymbol{x}_\perp^2 + \tau^2\, e^{-2\gamma(\tau)}\, dy^2$$

proper time   rapidity



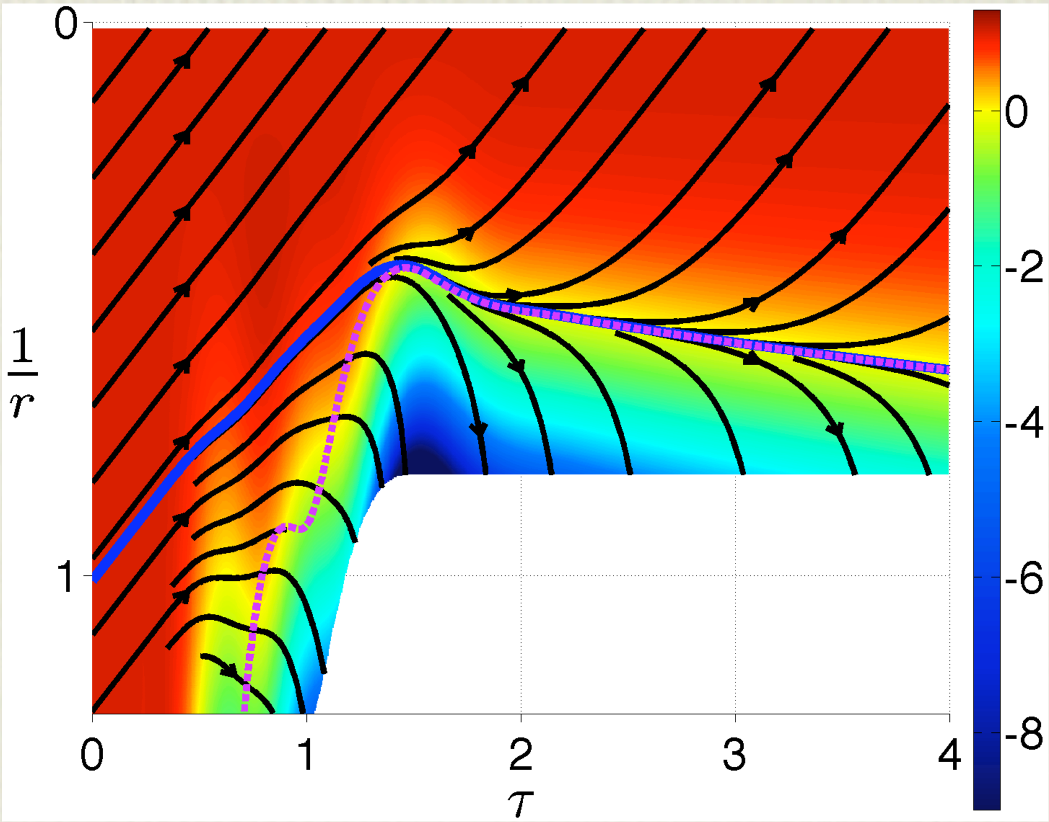
choose forcing function with compact support:

$$\gamma(\tau) = c\, h(\tau-\tau_0)^6\, e^{-1/h(\tau-\tau_0)}$$

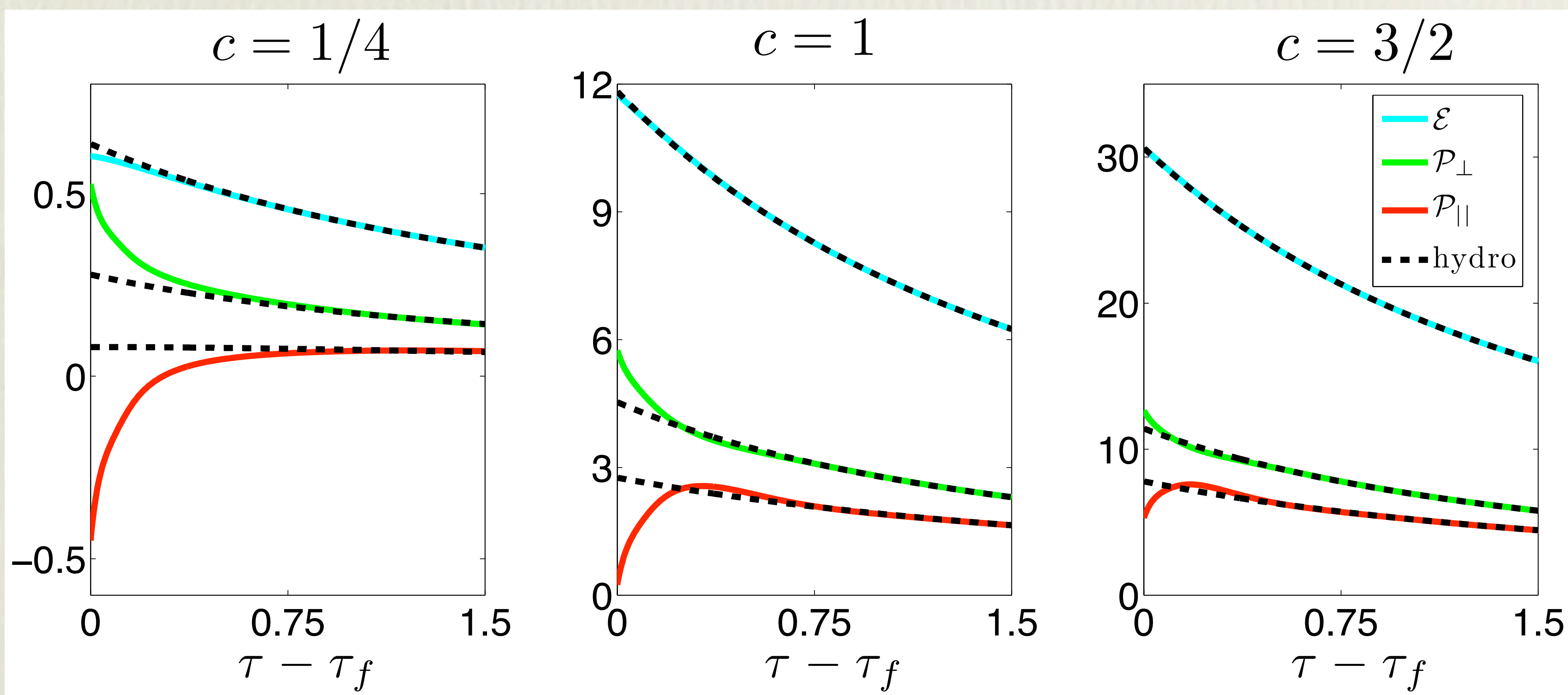
$$h(\delta\tau) = 1 - (\delta\tau)^2/\Delta^2$$

Wednesday, February 1, 12

boost invariant expansion: results



Limit of validity of hydro controlled by relaxation of non-hydro modes, not by growth of higher-order viscous terms



colliding planar shocks

arXiv:1011.3562

- Metric ansatz: $$ds^2 = -A\,dv^2 + \Sigma^2\left[e^B d\boldsymbol{x}_\perp^2 + e^{-2B} dz^2\right] + 2dv\,(dr + F dz)$$

  time coord. collision axis 5D radial coord.

  4 unknown functions of (v,r,z)

  v = const.: null hypersurface

  r = affine parameter along infalling null geodesics

  $r = \infty$: holographic boundary

- Residual diffeomorphism freedom: $r \to r + \xi(v,z)$

- Boundary asymptotics: $$A = r^2\left[1 + \frac{2\xi}{r} + \frac{\xi^2 - 2\partial_v \xi}{r^2} + \frac{a_4}{r^4} + O(r^{-5})\right],$$

$$B = \frac{b_4}{r^4} + O(r^{-5}), \quad \Sigma = r + \xi + O(r^{-7}), \quad F = \partial_z \xi + \frac{f_2}{r^2} + O(r^{-3})$$

- Holographic mapping: $$\mathcal{E} \equiv \frac{2\pi^2}{N_c^2}\,T^{00} = -\tfrac{3}{4}a_4\,, \quad \mathcal{P}_\parallel \equiv \frac{2\pi^2}{N_c^2}\,T^{zz} = -\tfrac{1}{4}a_4 - 2b_4\,,$$

$$\mathcal{S} \equiv \frac{2\pi^2}{N_c^2}\,T^{0z} = -f_2\,, \quad \mathcal{P}_\perp \equiv \frac{2\pi^2}{N_c^2}\,T^{\perp\perp} = -\tfrac{1}{4}a_4 + b_4$$

colliding shocks: initial conditions

- Single shock: analytic solution in Fefferman-Graham coordinates

$$ds^2 = r^2[-dx_+dx_- + d\boldsymbol{x}_\perp^2] + \frac{1}{r^2}\,[dr^2 + h(x_\pm)\,dx_\pm^2]$$

Janik & Peschanski

- Choose Gaussian profile with width w, surface energy density μ^3:

$$h(x_\pm) \equiv \mu^3\,(2\pi w^2)^{-1/2}\,e^{-\frac{1}{2}x_\pm^2/w^2}$$

- Single shock, our coordinates: must solve for diffeomorphism numerically

- Superpose single shocks to generate incoming two-shock initial data

Einstein equations

$$
\begin{aligned}
0 &= \Sigma'' + (B')^2\,\Sigma & (1)\\
0 &= \Sigma^2\left[F'' - 2(d_3B)' - 3B'd_3B\right] + 4\Sigma' d_3\Sigma - \Sigma\left[3\Sigma' F' + 4(d_3\Sigma)' + 6B'd_3\Sigma\right], & (2)\\
0 &= 6\Sigma^3(d_+\Sigma)' + 12\Sigma^2(\Sigma' d_+\Sigma - \Sigma^2) - e^{2B}\left\{2(d_3\Sigma)^2\right. & \\
&\quad \left. + \Sigma^2\left[\tfrac{1}{2}(F')^2 + (d_3F)' + 2F'd_3B - \tfrac{7}{2}(d_3B)^2 - 2d_3^2B\right] + \Sigma\left[(F' - 8d_3B)\,d_3\Sigma - 4d_3^2\Sigma\right]\right\}. & (3)\\
0 &= 6\Sigma^4(d_+B)' + 9\Sigma^3(\Sigma' d_+B + B'd_+\Sigma) + e^{2B}\left\{\Sigma^2[(F')^2 + 2(d_3F)' + F'd_3B - (d_3B)^2 - d_3^2B]\right. & \\
&\quad \left. + 4(d_3\Sigma)^2 - \Sigma\left[(4F' + d_3B)\,d_3\Sigma + 2d_3^2\Sigma\right]\right\}, & (4)\\
0 &= \Sigma^4\left[A'' + 3B'd_+B + 4\right] - 12\Sigma^2\Sigma' d_+\Sigma & \\
&\quad + e^{2B}\left\{\Sigma^2\left[(F')^2 - \tfrac{7}{2}(d_3B)^2 - 2d_3^2B\right] + 2(d_3\Sigma)^2 - 4\Sigma\left[2(d_3B)d_3\Sigma + d_3^2\Sigma\right]\right\}, & (5)\\
0 &= 6\Sigma^2 d_+^2\Sigma - 3\Sigma^2 A' d_+\Sigma + 3\Sigma^3(d_+B)^2 - e^{2B}\left\{(d_3\Sigma + 2\Sigma d_3B)(2d_+F + d_3A)\right. & \\
&\quad \left. + \Sigma\left[2d_3(d_+F) + d_3^2A\right]\right\}, & \\
0 &= \Sigma\left[2d_+(d_3\Sigma) + 2d_3(d_+\Sigma) + 3F'd_+\Sigma\right] + \Sigma^2\left[d_+(F') + d_3(A') + 4d_3(d_+B) - 2d_+(d_3B)\right] & \\
&\quad + 3\Sigma\left(\Sigma d_3B + 2d_3\Sigma\right) d_+B - 4(d_3\Sigma)d_+\Sigma &
\end{aligned}
$$

$$h' \equiv \partial_r h\,, \quad d_+h \equiv \partial_v h + A\,\partial_r h\,, \quad d_3h \equiv \partial_z h - F\,\partial_r h$$

Nested linear radial ODEs! ➡ simple time evolution procedure:

Given $B(v,z,r)$ at time v_0: (1) $\to \Sigma$, (2) $\to F$, (3) $\to \partial_+\Sigma$, (4) $\to \partial_+B$, (5) $\to A$ ➡ $\partial_v B$ ➡ $B(v_0+\delta v,z,r)$

numerical issues

- Computational domain:

  Impose periodic boundary conditions in z

  Excise geometry inside apparent horizon, $r < r_{\mathrm{h}}(v,z)$

- Residual diffeomorphism freedom:

  Fix apparent horizon at $r = r_{\mathrm{h}}$ = const. ➡ $0 = 3\Sigma^2\, d_+\Sigma - \partial_z(F\,\Sigma\, e^{2B}) + \frac{3}{2}F^2\,\Sigma' e^{2B}$

- Singular point at $r = \infty$:

  Use (pseudo)spectral methods: Fourier (z) & Chebyshev (r) basis expansion

- Precision loss due to very rapid growth of A, F deep in bulk:

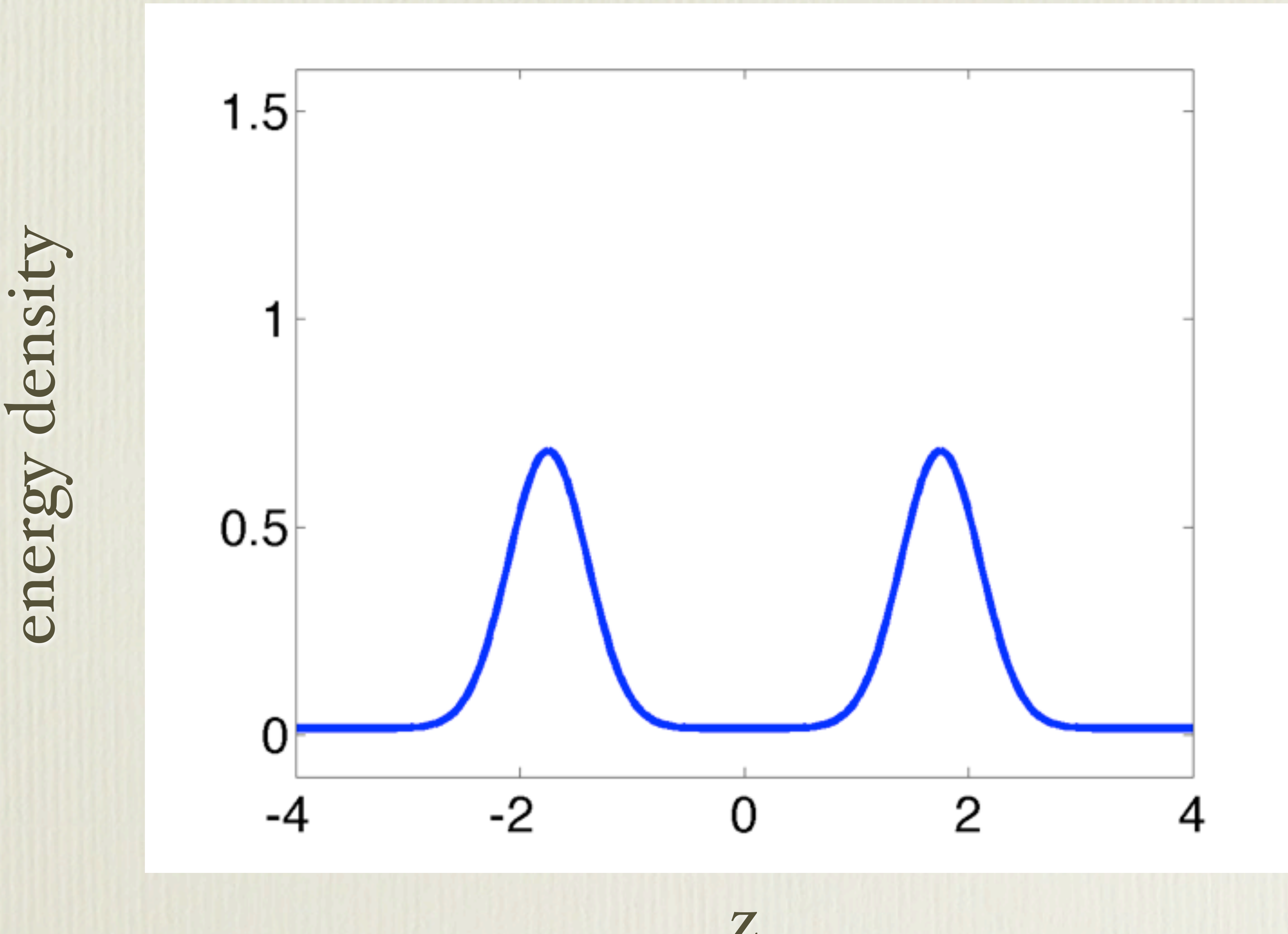
  Add small background energy density ➡ $a_4 \to a_4 - \delta$

- Short wavelength instabilities induced by discretization:

  Introduce tiny numerical viscosity

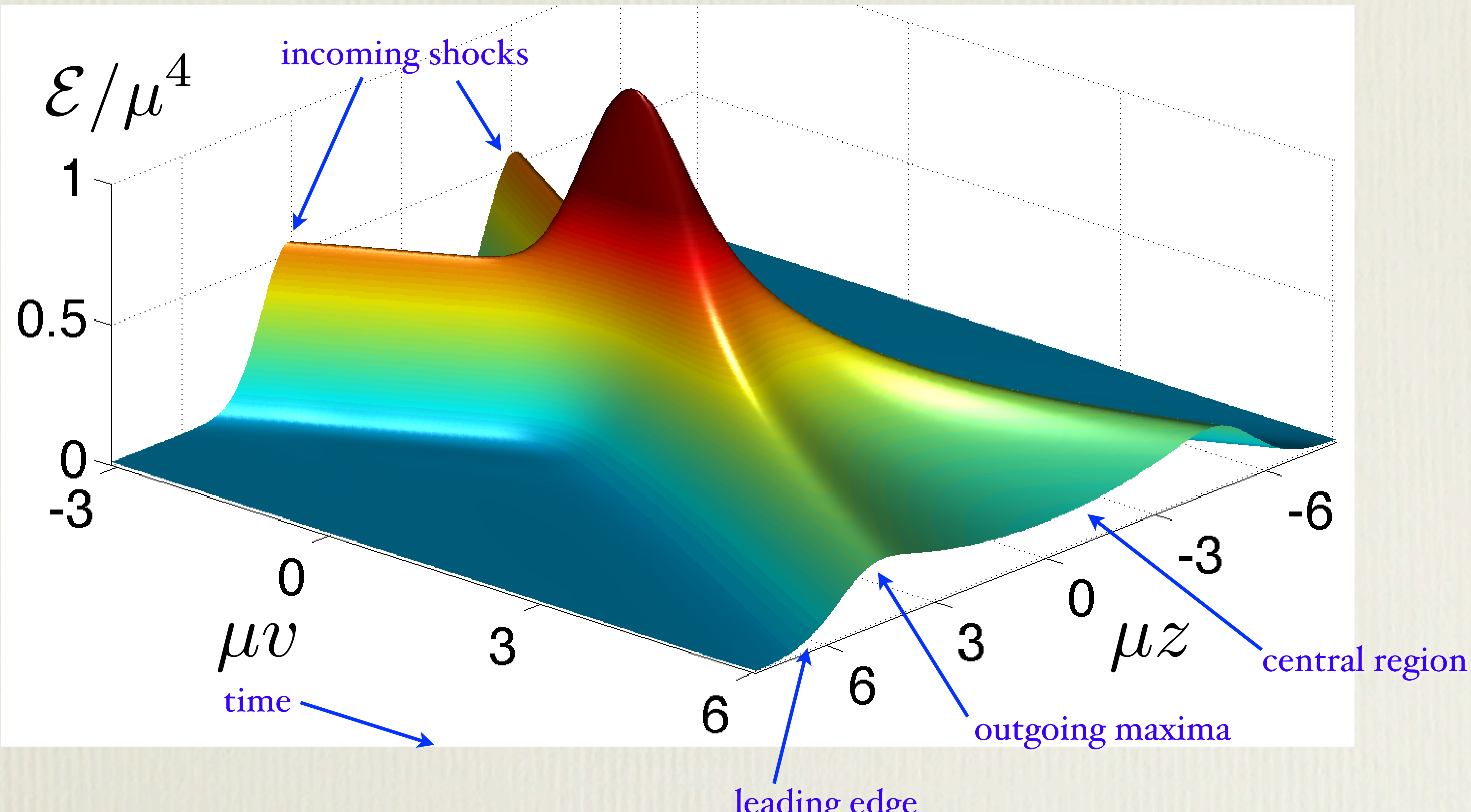
Can achieve stable evolution

colliding shocks: results

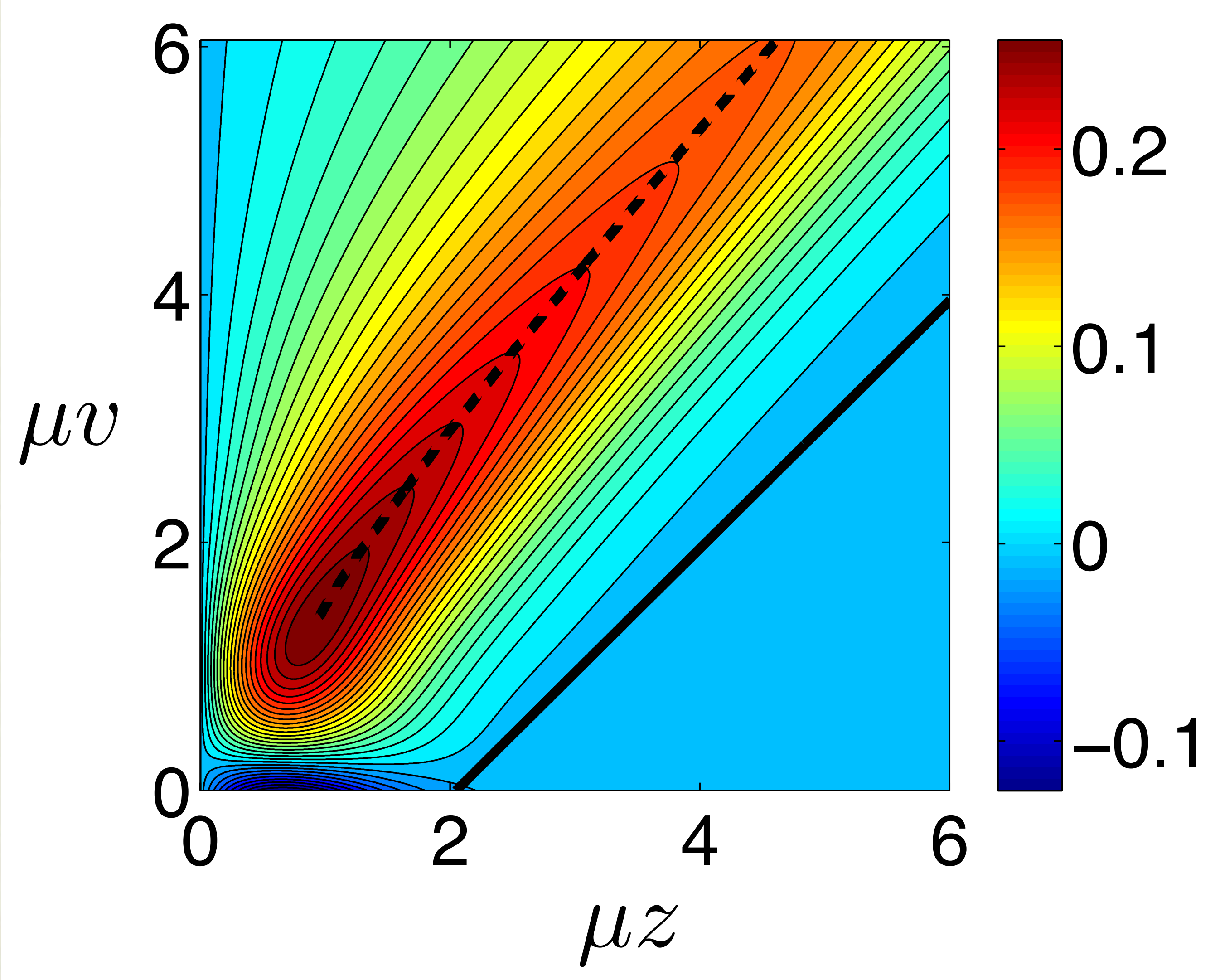


$w = 0.75/\mu,\ \delta = 0.014\ \mu^4$

colliding shocks: energy density



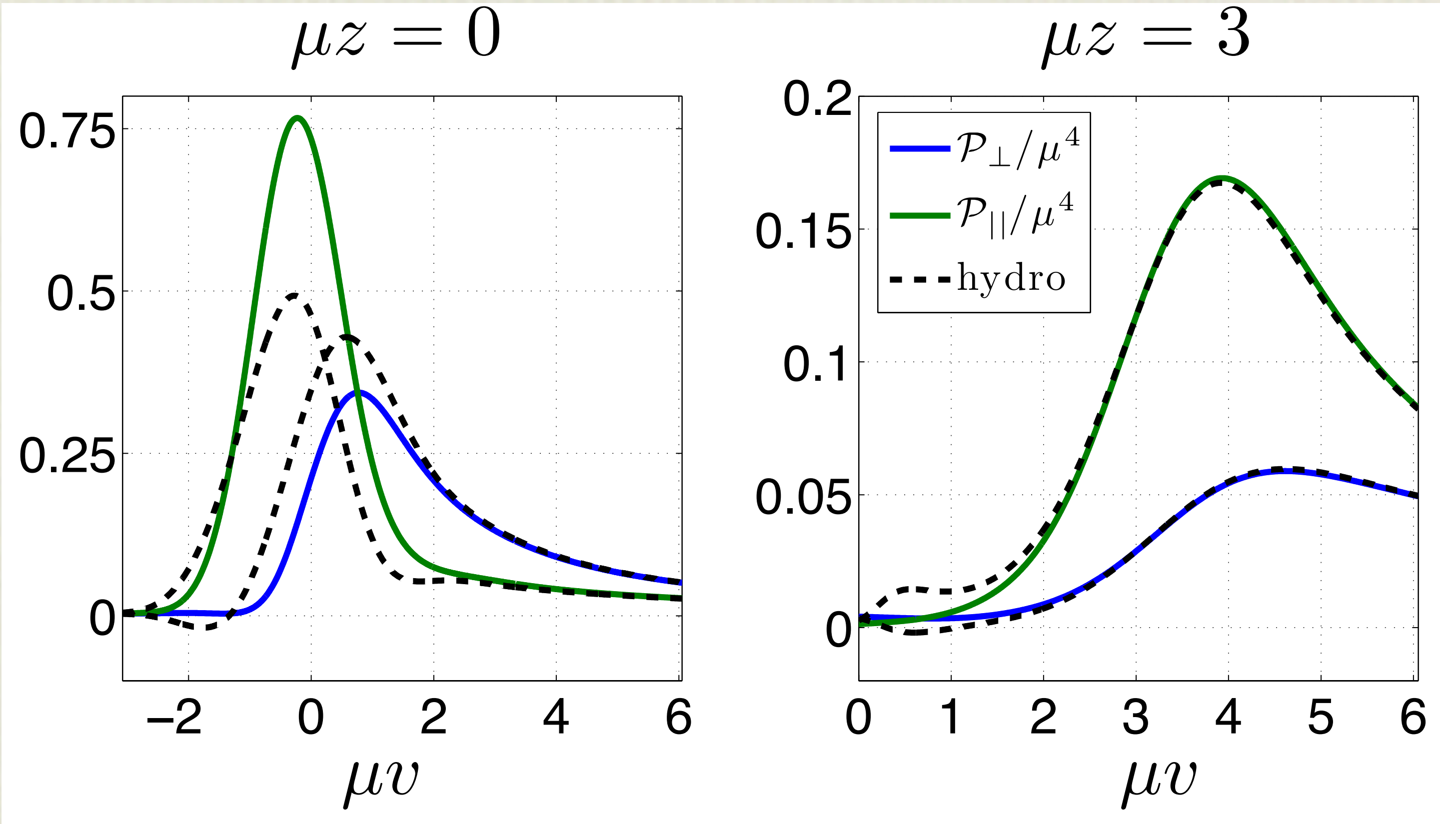
colliding shocks: energy flux



Outgoing maxima move at speed $v \approx 0.86$ c

Not an artifact of background energy density

colliding shocks: hydro validity



- Early times: large anisotropy, far from local equilibrium
- Late times: accurate agreement with hydro constitutive relations
- Central region: onset of hydro validity ≈ 4/μ after initial interaction

  μ ≈ 2.3 GeV for modeling RHIC ➡ $\tau_{\text{hydro}} \approx 0.35$ fm/c

- Near outgoing maxima & leading edges: fortuitous agreement with 1st order hydro: big difference between 1st and 2nd order hydro

remarks

- using gauge/gravity duality to study strongly coupled far-from-equilibrium dynamics works for interesting variety of problems
  - good coordinates, adapted to gravitational infall ➡ remarkably simple equations allowing efficient integration
  - can achieve stable evolution
  - 1+1D, 2+1D problems: computationally "easy" (Matlab code running on laptop)
  - even GR amateurs can make progress!

remarks

- work to date has only scratched the surface; many interesting generalizations await:
  - dependence on shock profile
  - asymmetric shocks
  - shocks with non-zero charge density (Einstein-Maxwell)
  - shocks with finite transverse extent (3+1 PDEs)
  - dynamics in non-conformal theories with (more complicated) dual gravitational descriptions