Towards a numerical solution of large-N QCD

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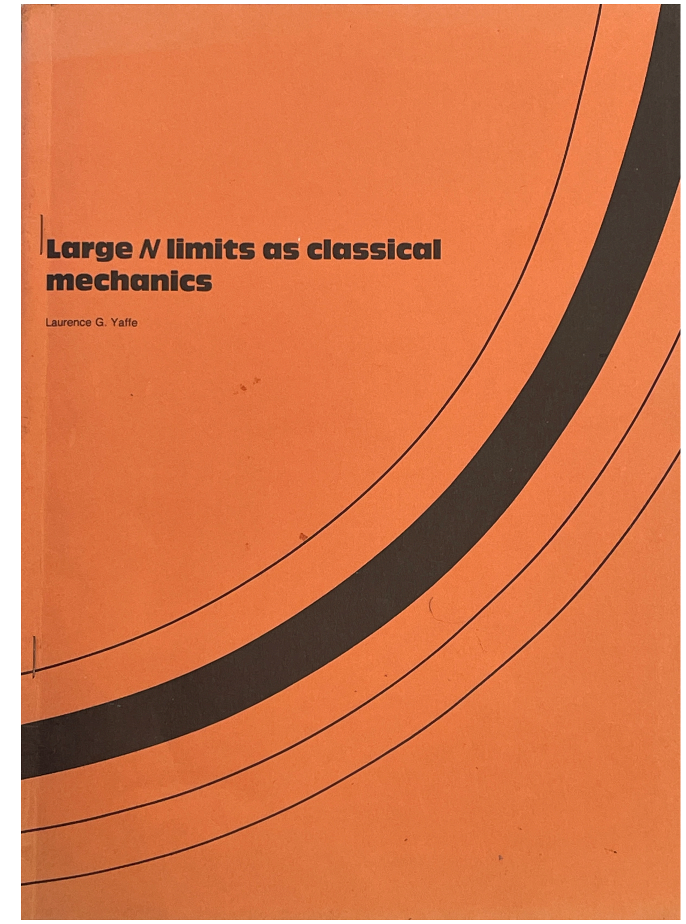
Large-N QCD

- Known for 40+ years: $N \to \infty$ limit is a classical limit
  - quantum dynamics → classical dynamics
  - classical phase space $\sim \{\text{coherent states}\,|u\rangle\}$
    - = coadjoint orbit of infinite dimensional Lie group
    - = over-complete basis for gauge-invariant Hilbert space
  - coherence algebra generators $\sim \{W_\Gamma, \partial_t W_\Gamma\}$
  - classical Hamiltonian $h_{\rm cl}(u) \equiv \lim_{N\to\infty} N^{-2} \langle u|\hat{H}|u\rangle$
  - coherent state expectation values → phase space functions
  - factorization: $\langle AB\rangle \to \langle A\rangle\langle B\rangle$ for suitable observables
  - commutators → Poisson brackets
  - fundamental rep fermions → nested $O(N^2)$ & $O(N)$ classical dynamics



Rev. Mod. Phys. 54, 407 (1982)

Large-N QCD

- Solving large-N QCD $\Leftrightarrow$ minimizing $h_{\mathrm{cl}}(u)$ on classical phase space
  - minimum value $\Rightarrow$ $O(N^2)$ vacuum energy [+ $O(N)$ fermion contribution]
  - Taylor expand around minimum:
    - small oscillation frequencies $\Rightarrow$ glueball, meson spectra
    - cubic terms $\Rightarrow$ $1 \to 2$ particle decay widths
    - quartic terms $\Rightarrow$ $2 \to 2$ particle scattering amplitudes
- Merely a classical numerical minimization problem!
  - Need well-defined (UV finite) theory $\Rightarrow$ lattice gauge theory
  - Minimization (beyond one plaquette) is computationally hard (exponentially)

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No stochastic simulation!
No sign problem!
No DMRG, DLCQ, ...
No quantum computing!

Solving large-N QCD: key decision points

- Worth the trouble?
- Euclidean or Hamiltonian formulation?
- State representation?
- Variational parameters?
- Truncation strategy?
- Implementation?

Worth the trouble?

- Large-N QCD is hard numerical problem, major investment of effort
- Lattice simulations today can yield fully controlled calculations of growing number of QCD observables: thermodynamics, light & heavy quark spectrum, weak matrix elements, ...
- Holographic theories provide classical description of $N = \infty$ dynamics and can mimic many aspects of large-N QCD physics
- But:
    - Holographic theories $\neq$ real QCD
    - Despite limited progress, still very difficult to extract real-time physics (scattering amplitudes) from Euclidean lattice simulations

State representation

- Need some coordinatization on phase space ⇒ encoding of properties of specific large-N coherent state
  - Assume unbroken lattice symmetries (translations, cubic, charge conjugation)
  - Can work directly in infinite volume
- Possibilities:
  - Wilson loops expectations & conjugates $\{W_\Gamma, \pi_\Gamma\}$
  - Master field link matrices $\{u_k\}$, $k = 1, \cdots, D$, D = spatial dimension
  - Expectations of Wilson loops $\{W_\Gamma\}$ + single & double-E inserted loops
- Issues:
  - Truncation error
  - Consistency conditions, inequalities ⇒ coordinate boundaries

Variational parameters

- Variational parameters = basis for local deformations in phase space
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  - Wilson loop expectations
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  - Riemann normal coordinates on phase space

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$$|u\rangle = \exp\left[\Lambda(\{a_\Gamma, b_{\alpha,\Gamma}\})\right]|0\rangle \equiv \exp\left[iN\left(\sum_\Gamma a_\Gamma \operatorname{tr} U_\Gamma + \sum_{\alpha,\Gamma} b_{\alpha,\Gamma} \operatorname{tr}(E_\alpha U_\Gamma) - \text{h.c.}\right)\right]|0\rangle$$

coherence algebra element

$$\delta|u\rangle = \Lambda(\{\delta a_\Gamma, \delta b_{\alpha,\Gamma}\})|u\rangle$$

local Riemann normal coordinates = coherence algebra coefficients

Truncation & minimization strategy I

- Keep finite set of Wilson loops $\{W_\Gamma \,|\, \Gamma \in S\}$, neglect loops $\Gamma' \notin S$. Directly vary selected $\{W_\Gamma\}$. Issues:
  - Selection of loop set S, resulting truncation error
    - Underlying assumption: correlation length limits size of important loops
  - Consistency conditions, inequalities $\Rightarrow$ coordinate boundaries, $|W_\Gamma| \leq 1$, etc.
    - But see recent work: Koch, Jevicki, Liu, Mathaba & Rodrigues, Kazakov & Zheng, Lin, Anderson & Kruczenski, …
  - Kinetic energy evaluation: $\langle u |\sum_\ell tr(E_\ell^2) | u \rangle = \sum_{\Gamma,\Gamma'} \omega_\Gamma \, (\Omega^{-1})_{\Gamma\Gamma'} \, \omega_{\Gamma'} + \pi_\Gamma \Omega_{\Gamma\Gamma'} \pi_{\Gamma'}$
    - loop joining matrix $\Omega_{\Gamma\Gamma'} \equiv \langle \sum_\ell [E_\ell, W_\Gamma], [E_\ell, W_{\Gamma'}] \rangle$,
    - splitting vector $\omega_\Gamma \equiv \langle \sum_\ell [[E_\ell, [E_\ell, W_\Gamma]] \rangle$
    - insufficient diagonal dominance, $(\mathrm{trunc}\,\Omega)^{-1} \not\approx \mathrm{trunc}\,(\Omega^{-1})$

Truncation & minimization strategy II

- Finite dimensional approximation to master field, $u_k = M \times M$ unitary. Evaluate $\{W_\Gamma \,|\, \Gamma \in S\}$. Vary u_k matrix elements. Issues:
  - Selection of set S, kinetic energy evaluation, boundaries
  - Bad truncation error even at infinite coupling
    - $\beta_L = 0$ $(g^2 = \infty) \Longrightarrow$ all $W_\Gamma = 0$ (except identity).
    - Impossible to reproduce with finite dimensional master field.
    - Increasing lattice dimension $\Rightarrow$ worse scaling, $O(M^{-1/D})$

Truncation & minimization strategy III

- Keep finite set of Wilson loops $\{W_\Gamma \,|\, \Gamma \in S\}$, plus associated finite set of loops with two electric field insertions. Vary finite set of Riemann normal coordinates $\Longrightarrow$ geodesic equations for variation of observables in phase space. Issues:
  - Neglect loops $\Gamma' \notin S$ (1980's[†]) or approximate $W_{\Gamma'} = f(\{W_\Gamma\})$ for $\Gamma' \notin S, \Gamma \in S$ (new proposal)
    - N.B.: most loops self-intersect. $\langle \frac{1}{N} \mathrm{tr}\, U_{\Gamma_1\Gamma_2} \rangle \neq \langle \frac{1}{N} \mathrm{tr}\, U_{\Gamma_1} \rangle \langle \frac{1}{N} \mathrm{tr}\, U_{\Gamma_2} \rangle$, but $\langle \frac{1}{N} \mathrm{tr}\, U_{\Gamma_1\Gamma_2} \rangle \approx \langle \frac{1}{N} \mathrm{tr}\, U_{\Gamma_1} \rangle \langle \frac{1}{N} \mathrm{tr}\, U_{\Gamma_2} \rangle$ in both strong and weak coupling regimes.
  - Truncation of normal coordinate set
    - little group $\Rightarrow$ degenerate variations
  - Complexity of implementation

† F. Brown & L.Y., Nucl.Phys.B 271 (1986) 267-332; T. Dickens, U. Lindqwister, W. Somsky, L.Y., Nucl.Phys.B 309 (1988) 1-119

Implementation issues

- Strong coupling ⇒ short correlation length ⇒ computationally easy
- Weak coupling ⇒ long correlation length ⇒ computationally hard
  - Feasibility always limited by max practical correlation length
  - # loops of characteristic size = exponential in size

➡ Efficient implementation still matters, despite machine improvements

  - Programming, debugging, testing = very serious investment of effort

Choices

- Worth doing: yes, with Hamiltonian approach
- State representation: list of expectations of Wilson loops $\{W_\Gamma\}$ + double-E inserted loops
- Variational parameters: Riemann normal coordinates
- Truncation: finite set of loops, approximating non-retained loops
- Implementation: C++, in progress

Status report

- Since July, finished (or nearly so):
  - ✓ Basic encoding of loops (& fermion bilinears), $D = 2,3,4,\ \ n_F = 0,1,2,$ Hamiltonian or Euclidean
  - ✓ Symmetry transformations & lattice irreps
  - ✓ Observable canonicalization = unique representative of equivalent symmetry transforms & loop starting points
  - ✓ Commutators of coherence algebra generators and observables
    - ➡ geodesic equations for Riemann normal variations of observables, gradient & curvature of Hamiltonian
- Still to-do:
  - ✓ Implement new truncation approximation
  - ✓ ODE solver, selection of linear algebra package
  - ✓ Testing on soluble models (Euclidean YM_2), comparison with old programs
  - ✓ Applications to YM_3, QCD_3, YM_4, QCD_4
- Results: "soon"

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