The Construction of Attention Functions for Phase Regulation

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Abstract

In this report we describe a method for building an *attention function* on the two dimensional torus. Such a function is used as an analytic switch by an underactuated robotic system to change its attention from one controlled cyclic process to another. A point in the torus represents two phase variables which each describe the progress of a cyclic system around a circle, such as that of a bouncing ball or a hopping leg. The attention function is designed so that under a certain flow on the torus, attention is paid to the phase which will next become zero.

1 Introduction

In [2] we introduced a method for regulating the phases of two cyclic processes and applied it to the task of juggling two balls with a robotic paddle. To juggle one ball, the paddle uses a *mirror law* [1] which takes the form of a reference trajectory $\mu(t)$ that the paddle must follow. The reference trajectory $\mu(t)$ guides the paddle to go down when the ball goes up and up when the ball goes down, hence the name "mirror law", and is such that the collisions between ball and paddle take the ball to an asymptotically stable orbit. To juggle two balls, we introduce two mirror laws μ_1 and μ_2 between which the paddle must alternate its attention. Controlling the paddle's attention so that it tracks the appropriate mirror law is the subject of this report. The *attention function*, denoted by s, will be designed so that the paddle "attends" to ball one ($s = 1$) for some time before and after ball one collisions, attends to ball two ($s = 0$) near ball two collisions, and smoothly switches between the two reference trajectories μ_i as needed. We also want that s does not change the attention of the paddle between collisions with the same ball (e.g. in a 3:2 juggle, the paddle will hit the balls in the order ball 1, ball 1, ball 0, ball 1, ball 0 — over and over). In combination with the mirror laws the attention function results in the conglomerate reference trajectory

$$\mu = s\mu_1 + (1 - s)\mu_2.$$

To describe the bouncing balls as cyclic processes [2], we introduce phase variables $\phi_i \in S^1$, for $i = 1, 2$, and consider points of the form $(\phi_1, \phi_2) \in \mathbb{T}^2$. We so arrange the definition of ϕ_i so that the points where $\phi_i = 0$ correspond to collisions between the paddle and ball i. Furthermore we construct mirror laws so that the system, in the limit, behaves according to

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = \kappa_1 \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{and} \quad A\phi_2 = B\phi_1 + \frac{1}{2} \pmod 1 \tag{1}$$

where A and B are integers. For example with $A\!:\!B = 1\!:\!1$, (1) encodes the task of juggling two balls so that one is at its apex while the other is colliding with the paddle.

The following *reference field*, also introduced in [2], has (1) as a stable limit cycle:

$$\mathcal{R}(\phi_1, \phi_2)^T = \kappa_1 \begin{pmatrix} A \\ B \end{pmatrix} - \kappa_2 \nabla V(\phi_1, \phi_2). \tag{2}$$

The collisions are controlled so that they "service" this reference dynamic system. Thus, the attention function we construct in this report presupposes that points on the torus flow according to this system. By careful design, therefore, we have reduced the problem of designing the attention function to that of determining which points in $\mathbb{T}^2$ will flow to which collisions next under the action of (2). The limit cycle for a particular case is illustrated in Figure 1 and the associated vector field is shown in Figure 2.

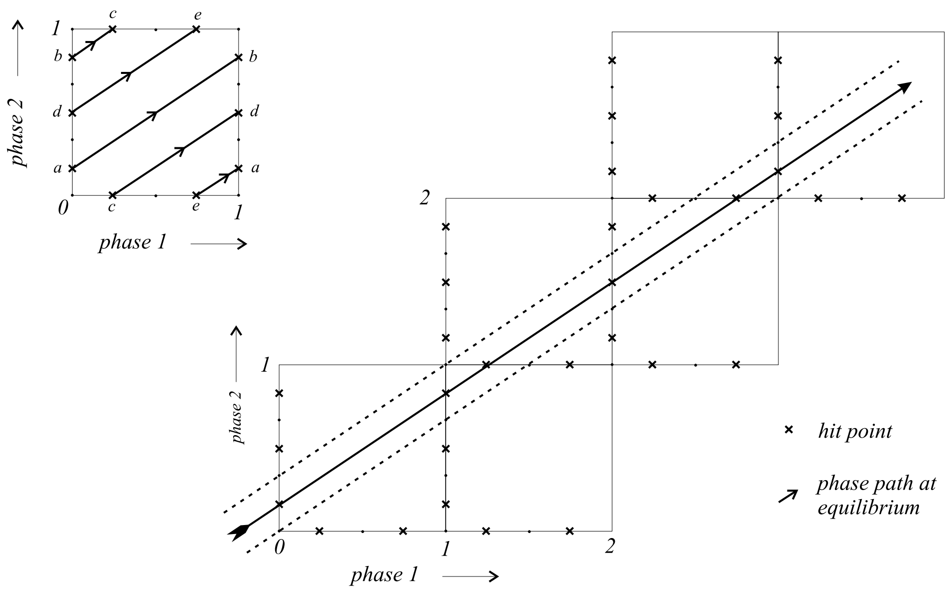

Figure 1: The task of phase regulation characterized as a limit cycle (shown on the unit square and *unwrapped* on the plane). The limit cycle can be parameterized by $t \mapsto (At, Bt + \frac{1}{2A})$. Note that the y-intercept is $\frac{1}{2A}$.

2 The Construction

Let $\mathbb{T}^2$ denote the two dimensional torus $S^1 \times S^1$, where S^1 is the circle obtained via the quotient topology on $S^1/\mathbb{Z}$. A point in this representation of $\mathbb{T}^2$ is therefore of the form $(x \pmod 1), y \pmod 1))$ for some $x, y \in \mathbb{R}$. In this paper we will generally work in the covering space $\mathbb{R}^2$ or restrict our attention to $[0,1]^2$ — thinking of the torus as the square $[0,1]^2$ with opposite edges identified. The attention function we define is a two times differentiable function $s : [0,1]^2 \to [0,1]$, as required by PID control, that remains C^2 under the identification of opposite edges.

The construction of s takes several steps. First, we consider just the attracting limit cycle of the reference field $\mathcal{R}$ (2) as encoding the essential behavior of the system. Collisions occur where this cycle crosses either of the two sections $\phi_1 = 0$ or $\phi_2 = 0$ of $\mathbb{T}^2$, thereby defining a characteristic sequence of collisions in the limiting behavior. We define a function s_1 from the limit cycle to the interval $[0,1]$ where $s_1 = 1$ near $\phi_1 = 0$ and $s_1 = 0$ near $\phi_2 = 0$ and varies smoothly between these extremes at all other points. Next, we extend s_1 to a switch function on the rectangle $[0,1] \times [-1,1]$ via an operation that is like a reverse deformation retraction. The result is a switch function on a 2-dimensional strip built around a copy of the limit cycle. The rectangle is then stretched, rotated and offset to lie between the lines $Ay = Bx$ and $Ay = Bx + 1$. Finally, this strip is wrapped around

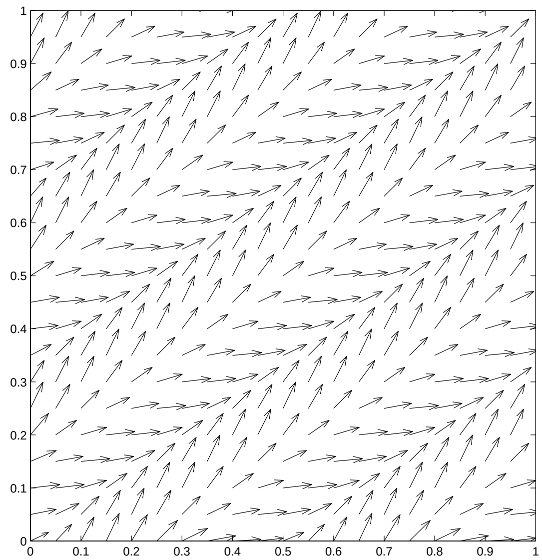

Figure 2: The reference field $\mathcal{R}(\phi_1, \phi_2)$ for $A\!:\!B = 3\!:\!2$.

$\mathbb{T}^2$ to complete the construction. These steps are illustrated in Figure 3. We describe the details next.

2.1 The Attention Function on the Limit Cycle

To construct s_1, we first examine some basic properties of the limit cycle and in particular the sequence of collisions that a given $A\!:\!B$ generate. To this end, parameterize the limit cycle in (2) by $t \mapsto (At, Bt + \frac{1}{2A})$ (taken modulo 1), $t \in \mathbb{R}$.

Definition 2.1 *A* **hit point** *is a pair* (ϕ_1, ϕ_2) *where either* $\phi_1 \equiv 0 \pmod 1$ *or* $\phi_2 \equiv 0 \pmod 1$.

Hit points occur along the limit cycle when $At \equiv 0 \pmod 1$ or $Bt + \frac{1}{2A} \equiv 0 \pmod 1$ or equivalently when there is some $j \in \mathbb{Z}$ such that

$$t = \frac{j}{A} \text{ or } t = \frac{1}{B}(j - \frac{1}{2A}). \tag{3}$$

Property 2.1 *On the limit cycle, points of the form $(k, k) \in \mathbb{Z}^2$ are not hit points. That is, hits do not occur simultaneously.*

Proof: Suppose $(k, k) \in \mathbb{Z}^2$ is a hit point on the limit cycle. Then for some t, $At = k$ and $Bt + \frac{1}{2A} = k$. Thus, substituting k/A for t in the second equation gives

$$B\frac{t}{k} + \frac{1}{2A} = k$$

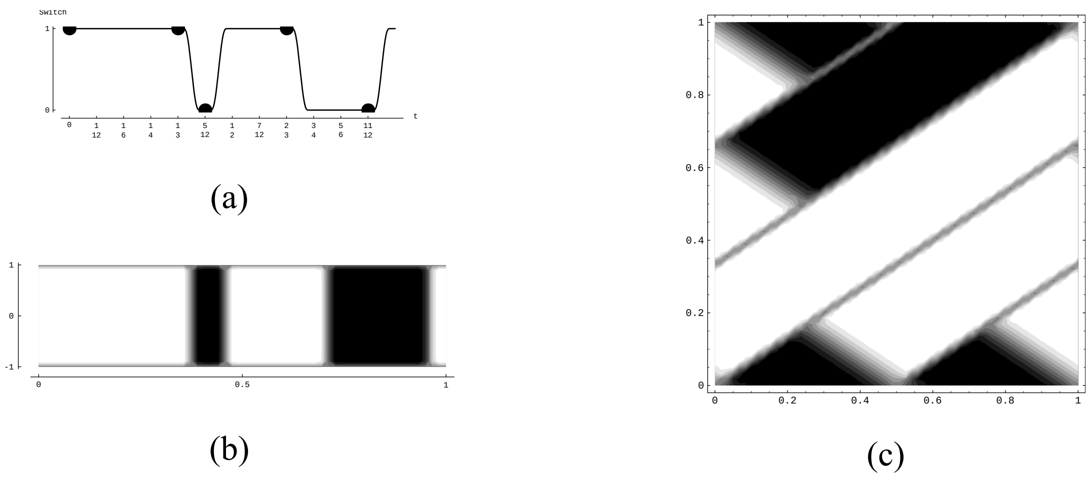

Figure 3: The construction of an attention function for the case $A:B = 3:2$. (a) The attention function on an ideal version of the limit cycle of (2) with dots added to indicate when hits occur. (b) The limit cycle extended to $[0,1]\times[-1,1]$. (c) The strip in (b) wrapped around $\mathbb{T}^2$.

or equivalently

$$Bk + \frac{1}{2} = k,$$

which is a contradiction. The term on the left hand side is not an integer, while the term on the right hand side is. $\Box$

It is convenient to define $\sigma = 2AB$, for it allows us an easy characterization of hit points, as the following property demonstrates.

Property 2.2 *All hit points are of the form* $(At, Bt + \frac{1}{2A})$ *where* $t = \frac{n}{\sigma}$ *for some* $n \in \mathbb{Z}$.

Proof: Suppose $(At, Bt + \frac{1}{2A})$ is a hit point. Then either $At = k$ or $Bt + \frac{1}{2A} = k$ for some integer k. In the first case, $t = 2Bk/\sigma$ so that $n = 2Bk$. In the second case $t = (2Ak-1)/\sigma$ so that $n = 2Ak - 1$. $\Box$

We define the partial function $h : \mathbb{Z} \to \{0, 1, \perp\}$ to determine which kind of hit point $\frac{n}{\sigma}$ is, if it is a hit point at all. We let 1 correspond to a ball one hit, and 0 to a ball two hit. We correspond the special object $\perp$ (to mean *undefined*) to the case that neither ball is hit. Then we have

Proposition 2.1 *Let* $t = \frac{n}{\sigma}$ *and* $P = (At, Bt + \frac{1}{2A})$. *Define the function* $h : \mathbb{Z} \to \{0, 1, \perp\}$ *by* $h(n) = 1$ *when* P *is a ball 1 hit point (i.e.* $At \equiv 0$*),* $h(n) = 0$ *when* P *is a ball 2 hit point (i.e.*

$Bt + \frac{1}{2A} \equiv 0$) *and* $h(n) = \perp$ *when* P *is not a hit point. Then*

$$h(n) = \begin{cases} 1 \text{ if } \frac{nA}{\sigma} \in \mathbb{Z} \\ 0 \text{ if } \frac{(n+1)B}{\sigma} \in \mathbb{Z} \\ \perp \text{ otherwise } . \end{cases} \tag{4}$$

Note by Property 2.1 that h is well defined. The proof is a simple rewriting of the conditions that $(At, Bt + \frac{1}{2A})$ be a hit point.

For those n such that $h(n) = \perp$, we have to decide which ball to attend to. An obvious choice is to have the paddle attend to the ball that will be hit next. So we define a new function $\tilde{h}$ recursively:

$$\tilde{h}(n) = \begin{cases} h(n) \text{ if } h(n) \neq \perp \\ \tilde{h}(n+1) \text{ otherwise.} \end{cases} \tag{5}$$

The function $\tilde{h}$ is the basis for the switch function on $\mathbb{T}^2$, which we are now prepared to build. Notice that $\tilde{h}$ is defined on the integers, but is described completely by its values on the set $\{0, 1, 2, ..., \sigma - 1\}$:

Property 2.3 $\tilde{h}(n + k\sigma) = \tilde{h}(n)$ *for all integers* k.

Proof: First we show that $\frac{nA}{\sigma} \in \mathbb{Z}$ is equivalent to $\frac{(n+k\sigma)A}{\sigma} \in \mathbb{Z}$. Accordingly, if $\frac{nA}{\sigma} = j$ for some $j \in \mathbb{Z}$ then

$$\frac{(n + k\sigma)A}{\sigma} = j + kA \in \mathbb{Z}$$

since kA is an integer. Conversely, if $\frac{(n+k\sigma)A}{\sigma} = j$ for some $j \in \mathbb{Z}$ then

$$\frac{nA}{\sigma} = j - kA \in \mathbb{Z}.$$

By a similar argument, $\frac{(n+1)B}{\sigma} \in \mathbb{Z}$ is equivalent to $\frac{(n+k\sigma+1)B}{\sigma} \in \mathbb{Z}$. The desired result follows. □

Next, we define a function s_1 on the interval $[0, 1]$ by dividing $[0, 1]$ into σ subintervals $[\frac{n}{\sigma}, \frac{n+1}{\sigma}]$, $0 \leq n < \sigma$. The function $\tilde{h}$ tells us what to do at the endpoints of these intervals. Thus, $s_1(t)$ agrees with $\tilde{h}(\sigma t)$ when σt is an integer, or equivalently, when $\lfloor \sigma t \rfloor = \lceil \sigma t \rceil$. We just need to fill in the rest of the intervals. We will need C^2 step functions $up : \mathbb{R}^3 \to \mathbb{R}$ and $down : \mathbb{R}^3 \to \mathbb{R}$. The function $up(t, a, b)$ is used to fill in an interval $[a, b]$ wherein $\tilde{h}(a) = 0$ and $\tilde{h}(b) = 1$. It is thus 0 when $t \leq a$, between 0 and 1 when $a < t < b$ and 1 otherwise. Such a function can be constructed with polynomial splines of degree 6. *down* is defined similarly. The function *down* is used, for example, between the second and third hit points in Figure 3(a). The smoothed function s_1 is then

$$s_1(t) = \begin{cases} 0 \text{ if } \tilde{h}(\lfloor \sigma t \rfloor) = \tilde{h}(\lceil \sigma t \rceil) = 0 \\ 1 \text{ if } \tilde{h}(\lfloor \sigma t \rfloor) = \tilde{h}(\lceil \sigma t \rceil) = 1 \\ up(t, \frac{\lfloor \sigma t \rfloor}{\sigma}, \frac{\lceil \sigma t \rceil}{\sigma}) \text{ if } \tilde{h}(\lfloor \sigma t \rfloor) = 0 \wedge \tilde{h}(\lceil \sigma t \rceil) = 1 \\ down(t, \frac{\lfloor \sigma t \rfloor}{\sigma}, \frac{\lceil \sigma t \rceil}{\sigma}) \text{ if } \tilde{h}(\lfloor \sigma t \rfloor) = 1 \wedge \tilde{h}(\lceil \sigma t \rceil) = 0. \end{cases} \tag{6}$$

Notice that $s_1(0) = s_1(1)$ by Property 2.3. This function is shown for the case $A : B = 3 : 2$ in Figure 3(a).

2.2 Extending the Attention Function to $\mathbb{T}^2$

We further extend s_1 to the function $s_2 : [0, 1] \times [-1, 1] \to [0, 1]$ defined by

$$s_2(x, y) = b(y)s_1(x) + \frac{1}{2}(1 - b(y))$$

where

$$b(y) = up(y, 0, \epsilon) \cdot down(y, 1 - \epsilon, 1).$$

Here, ϵ is some small number that defines the width of the "borders" along $y = 1$ and $y = -1$. This function is shown is Figure 3(b) as a contour plot. Note that if we form a cylinder or Möbius strip from the domain of s_2 by identifying the segment $x = 0$ with the segment $x = 1$, then s_2 is C^2 along the identification line. This follows from the cyclic nature of $\tilde{h}$ noted above.

We next take the domain of s_2, distort it and wrap it around $\mathbb{T}^2$ to complete the attention function. We wish for the line $y = 0$ in the domain of s_2 to be mapped to the limit cycle on T^2. Thus we define a map $f : [0, 1] \times [-1, 1] \to R^2$ by

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} A & -\frac{1}{4B} \\ B & \frac{1}{4A} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2A} \end{pmatrix}. \tag{7}$$

As desired, $f(t, 0)^T = (At, Bt + \frac{1}{2A})^T$.

The final step is to collapse the image of f down to $[0, 1]^2$. For this we simply set $w(x, y) = (x \pmod 1, y \pmod 1)$ and obtain the desired switch function

$$s(\phi_1, \phi_2) = s_2 \circ f^{-1} \circ w^{-1}(\phi_1, \phi_2). \tag{8}$$

Since we designed s_2 so that for all x, $s_2(x, \pm 1) = \frac{1}{2}$, we are assured that along the "seams" of the wrapped s_2, s is still C^2. Notice that the paddle pays attention to neither ball along the seam. This is certainly a choice which we could have made differently. A contour plot of the case $A : B = 3 : 2$ is shown if Figure 3(c).

Presently, although we know w^{-1} exists, we have only been able to find it for specific cases of A and B and a general formula has yet to be found. For practical purposes, however, it is not likely that arbitrary attention functions are useful. Rather, an attention function will be worked out for each of the basic cases 1:1, 1:2, 2:3 and 2:5 for example.

3 Concluding Remarks

The class of attention functions we have constructed is probably not the only useful class. Simpler constructions may be found. We have simply demonstrated that a useful place to start the construction is on the limit cycle, describing the sequence of collisions a stable juggle admits and building up attention mechanisms from there.

It remains to be seen whether these functions are practicable in hardware implementations or if they require further tuning, to accommodate follow through for example, as the work on spatial juggling [3] suggests they may. We are also not certain that attention functions, or similar such functions, will find general utility in robotics or if they are limited in use to the juggling domain. One promising direction, initiated in [2], is to turn juggling *upside down* into walking, running and hopping. The purpose of the attention function (or some extension of the idea to more than two phases) is then to direct the attention paid by a walking robot by its legs to the ground.

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