Chapter 1

# AUTOMATICALLY SYNTHESIZED CONTROLLERS FOR DISTRIBUTED ASSEMBLY

*Partial Correctness*

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**Abstract** We consider the task of assembling a large number of self controlled parts (or robots) into copies of a prescribed assembly (or formation). In particular, we describe a computationally tractable way to synthesize, from a specification of the desired assembly, local controllers to be used by each part, which when taken together, have the global effect of assembling the parts. We then prove that the controlled discrete dynamics of the system are correct with respect to a simplified model of the dynamics— meaning that a maximal number of parts are correctly assembled into copies of the desired assembly.

**Keywords:** Controller synthesis, distributed control, self-assembly.

## 1. INTRODUCTION

We consider the problem of controlling hundreds or thousands of robots to perform a task in concert. This problem presents many fundamental issues to robotics, control theory and computer science. With a great number of robots, decentralization is critical due to the cost of communication and the need for fault tolerance. In decentralized control, each robot should act based only on information local to it. It then becomes difficult, however, to guarantee or even derive the behavior of the entire system given the behaviors of the individual

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components. In this paper we address this difficulty in a novel way: We begin with a specification of an assembly and develop methods that allow us to automatically *synthesize* individual behaviors so that they are guaranteed to produce the desired global behavior.

Specifically, we consider the task of assembling many disk-shaped parts in the plane into copies of a prescribed assembly (formation), which is specified by a graph with n vertices. We do not allow the parts to collide, making the task more difficult due to the non-trivial topology of the resulting $2n$ dimensional configuration space. As shown in Figure 1.1 we suppose that each part can move itself and can play any role in an assembly, which makes the task particularly rich. We first demonstrate a means of synthesizing from the specified assembly, a set of identical controllers for the parts to run which have the net effect of moving the parts to form copies of the specified assembly without colliding. The idea is that parts should join together into subassemblies which should in turn join together to make larger assemblies and so on. To achieve this, some theory is developed along with algorithms that compile a specified assembly into a list of allowable subassemblies. Next we show how to produce a lookup table from the list which can be used as a discrete event controller (Figure 1.2) that guides parts through a "soup" of other parts and subassemblies. Then we add a continuous motion controller based on the assembly rules represented by the lookup table and a (provably correct) method for deadlock avoidance.

Finally, we show formally that the discrete dynamics given by the lookup table and the deadlock avoidance mechanism (and employed in the control of each part) are correct. The proof assumes a certain logical model of the dynamics which accounts for the discrete interactions between parts (forming neighbor relationships) but neglects the continuous dynamics. A formal analysis of the complete, highly nonlinear hybrid dynamics is not yet forthcoming.

## 1.1. RELATED RESEARCH

We are most strongly inspired by the work of Whitesides and his group ( Bowden et al., 1999; Breen et al., 1999) in meso-scale self-assembly. In this work, small, regular plastic tiles with hydrophobic or hydrophylic edges are placed on the surface of some liquid and gently shaken. Tiles with hydrophobic edges are attracted along those edges while hydrophylic edges repel. Striking "crystals" emerge as larger structures self assemble. By using different shapes and edge types, different gross structures can be created. A similar idea is used on a much smaller scale in (Mirkin, 2000) where strands of DNA are attached to tiny gold balls in solution. Complementary strands attract and a gross structure is revealed. By choosing which strands go where, the "programmer" has some control over the resulting emergent structure. At least two next steps are apparent. First, these and similar (Bonabeau et al., 2000) methods generally produce arrays or lattices of parts, meaning that there is no general way to *terminate* a regular pattern at, say, a $5 \times 5$ array of parts (There has been

work on changing the function of parts as they combine (Saitou, 1999). Second, there is no known formal method of starting with a *specification* of the desired emergent structure and devising the structure of the individual parts. In this paper we address both of these issues by supposing that each part can run a *program* that tells it when to join with another part, and when to repel it, based on some state information. Of course, this is a far way away from the reality of small plastic parts or gold balls, but our ideas could easily be implemented with teams of robots and may even, when developed further, present the chemist with new tools.

The motivation for considering disk shaped parts in the plane and for the potential field construction in Section 4 comes from the work of Koditschek and others (Koditschek and Bozma, 2000; Karagoz et al., 2000) in assembly. There, a global artificial potential function over the configuration space of $n$ disk shaped parts is used to guide the parts to their assembled state, corresponding to the unique minimum of the potential function. The approach is not distributed, however, because it requires that each part knows the full state of the system to act. Other work has applied similar ideas, in a distributed fashion (Reif and Wang, 1995; Balch and Hybinette, 2000), although without a means of synthesizing the desired behavior. Still other approaches to the control of a group of robots (Desai et al., 1999) assume a leader. In contrast, the present paper commits to the synthesis idea and to a strong degree of decentralization, using decentralized potential fields merely as a *primitive* in a more sophisticated hybrid control scheme.

The ideas in this paper also grow from our own work in controller synthesis in manufacturing systems (Klavins and Koditschek, 2000; Klavins, 2000). Our approach to manufacturing has been to synthesize a decentralized automated factory description from a description of a product. The description includes the layout of the factory and the control programs the robots should run to produce the product. In that sense, the present work is an extension of the idea, although it assumes fewer constraints on the topology of the workspace.

## 2. THE PROBLEM

We consider a simple form of assembly process by assuming that parts are programmable and able to sense the position and state of other nearby parts. We start with $m$ disk-shaped parts (of radius $r$) confined to move in $\mathbb{R}^2$. Denote the position of part $i$ by the vector $x_i$. We desire that each part move smoothly, without colliding with other parts, so that all parts eventually take some role in an *assembly* or *formation*. This is shown graphically in Figure 1.1. For simplicity, we assume that the dynamics of each disk are given by $\ddot{x}_i = u_i$. We believe that control of parts with more complicated dynamics can be based on the control algorithms we develop for this simple situation. In this section we describe the goal of assuming a role in a formation formally.

Let $G = (V, E)$ be a finite undirected, acyclic graph. Thus, $V$ is a finite set of nodes (in this paper, $V = \{1, ..., n\}$) and $E$ is a collection of edges of

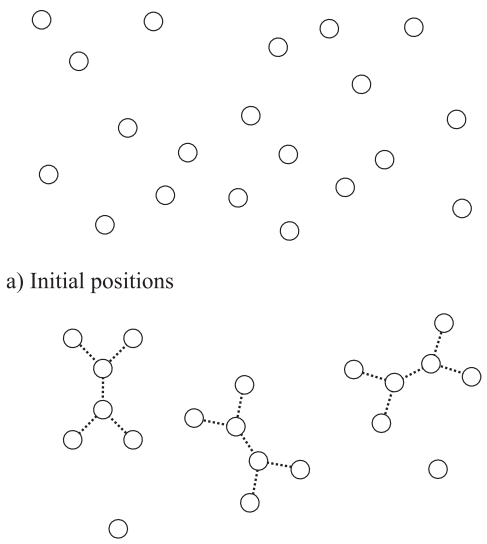


*Figure 1.1.* The goal of the assembly problem. Each disk shaped part must move from its initial position (a) to a position in a a copy of the specified assembly (b). Dashed lines show the resulting adjacency relationship $E$. There may be leftover parts.

the form $\{a, b\}$ with $a, b \in V$ and $a \neq b$. In this paper, we will call such a graph an *assembly* and only consider the case where $G$ is a tree (i.e., contains no cycles). There are technical details, which are solvable but not addressed in this paper except briefly, that prevent the direct application of the methods in this paper to general graphs.

Given an assembly $G = (V, E)$ with $|V| = n$, consider the case where $m = n$. The problem is to produce a control algorithm to be used by each part that will control the $m$ parts to move, without colliding, from arbitrary initial conditions to positions such that there exists a permutation $h$ of $\{1, ..., m\}$ such that

1 If $\{h(i), h(j)\} \in E$ then $d_{nbr} - \epsilon < ||x_i - x_j|| < d_{nbr} + \epsilon$;

2 If $\{h(i), h(j)\} \notin E$ then $||x_i - x_j|| > d_{nbr}$.

Here $d_{nbr} > 0$ and $\epsilon > 0$ are parameters. The image $h(i)$ of $i$ is called the *role* of $i$ in the assembly. We furthermore require that these assemblies be stable to disturbances in the sense that the set of points $x_1, ..., x_m$ satisfying the above conditions is an attractor of the closed loop dynamics we will construct. If $m = kn$ for some $k \in \mathbb{Z}$ then we still require the above except now with respect to a disjoint union of $k$ copies of $G$. And of course, if $m$ is not a

multiple of $n$, then we require that as many parts as possible form assemblies in the obvious way.

We note that not all trees can be embedded in the plane in such a way that neighbors are distance $d_{nbr}$ apart and non-neighbors are distance greater than $d_{nbr}$ apart. For simplicity in what follows, we restrict the assemblies we specify to those that can be so embedded.

## 2.1. CONTROLLER STRUCTURE

In general we will assume that parts have limited sensing and communication capabilities and we allow them to store a discrete state, $s_i$, along with their control programs. In particular, we assume that part $i$ can sense its own position and the positions and discrete states of other parts within some range $d_{max} > 0$ of $x_i$.

The methods we develop below will, given a description of the desired assembly structure, *synthesize* a hybrid controller $H_i$ of the form shown in Figure 1.2. The goal is that when each part runs a copy of $H_i$ (from different initial conditions), the parts will self assemble.

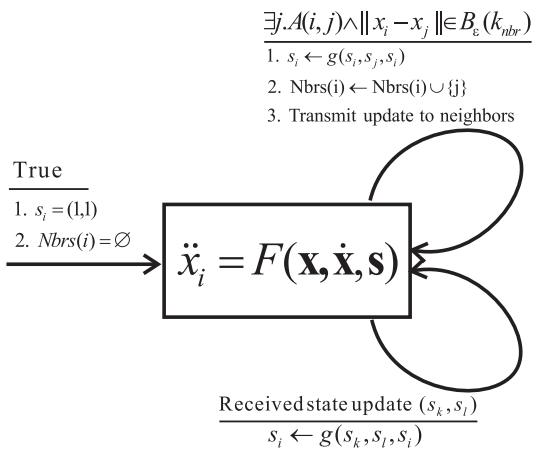


*Figure 1.2.* The structure of the hybrid controller that is constructed by the compilation scheme in this paper. Arcs denote transitions and are labeled by a predicate/action pair. When an arc's predicate becomes true, the action is taken and control transfers from the source of the arc to the target of the arc.

The controller $H_i$ is described by a continuous control law $F_i$, a predicate $\mathcal{A}$ called the *attraction predicate* and a discrete update rule $g$. $F_i$ describes the force that the part should apply to itself. $\mathcal{A}(s_i, s_j) \in \{true, false\}$ determines whether parts $i$ and $j$ with states $s_i$ and $s_j$ should try to become neighbors, thereby forming a larger assembly. The update rule $g(s_i, s_j, s_k)$ determines the new discrete state of part $k$ based on the joining of parts $i$ and $j$. Loosely,

the operation of $H_i$ is as follows. Part $i$ starts with some initial position $x_i(0)$, the initial state $s_i(0) = (1, 1)$ and no neighbors. It then applies the control force $F_i(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{s})$ to itself until either a new neighbor is detected or it receives a state update from a neighbor. Here $\mathbf{x}$, $\dot{\mathbf{x}}$ and $\mathbf{s}$ are $m$ dimensional vectors describing the complete state of the system. However, $F_i$ may only use the states of the parts within distance $d_{max}$ of part $i$. The force $F_i$ is computed based on the position, velocity and discrete state of part $i$ and on the discrete states of the sensed parts.

The task of an automatic synthesis procedure, performed by what we are calling a *compiler*, is to take a description of a desired assembly and produce $H_i$ — in this case, $F_i$, $\mathcal{A}$ and $g$. The construction of $\mathcal{A}$ and $g$ are described next and the construction of $F_i$, which requires $\mathcal{A}$, is discussed after that.

# 3. COMPILATION OF ASSEMBLY RULES FROM SPECIFICATIONS

In this paper, an assembly can be specified simply by listing which roles in the assembly are adjacent — that is, by a graph. As mentioned above, we restrict ourselves to the situation where the adjacency graphs are trees, leaving the detail of arbitrary graphs to future work. In any case, we believe that assembling an arbitrary graph will start with the assembly of a spanning tree of that graph.

The goal of this section is to produce the attraction predicate $\mathcal{A}$ and the update rule $g$ from a specified assembly $G_{spec} = (V_{spec}, E_{spec})$, which we assume is a tree. This requires first generating a set of subassemblies of $G_{spec}$ and then compiling $\mathcal{A}$ and $g$ from the set.

## 3.1. DISCRETE STATE OF A PART

We intend that the parts control themselves to first form subassemblies of $G_{spec}$, and from those subassemblies form larger subassemblies and so on until $G_{spec}$ is finally formed. The discrete state of a part must, therefore, include a reference to the subassembly in which it currently plays a role. To this end, we build a list (in Section 3.2) of the particular (connected) subassemblies we will allow: $\mathcal{G} = \{G_1, ..., G_p\}$. We require that each $G_i \in \mathcal{G}$ is of the form $(V_i, E_i)$ where $V_i = \{1, ..., |V_i|\}$ and $E_i \subseteq V_i \times V_i$. Although this representation of subgraphs in $\mathcal{G}$ is arbitrary, because the vertices in $V_i$ could have been named in other ways, some common scheme is required for a graceful definition of the states of parts.

Now, the discrete state of a part consists of a pair $s_i = (j, k) \in \mathbb{Z}^2$ where $j$ is the index of a subassembly in $\mathcal{G}$ and $k \in V_i$ is a *role* in that subassembly.

## 3.2. GENERATING ASSEMBLY SEQUENCES

Define an operation on assemblies $G_1$ and $G_2$ as follows

**Definition 1** *The* **join** *of $G_1$ and $G_2$ via vertices $u \in V_1$ and $v \in V_2$, denoted $G_1.u \oplus G_2.v$, is defined as $G_1.u \oplus G_2.v = (V, E)$ where*

$$V = \{1, ..., |V_1| + |V_2|\} \text{ and}$$

$$E = E_1 \cup \{\{a + |V_1|, b + |V_1|\} \mid \{a, b\} \in E_2\} \cup \{u, v + |V_1|\}.$$

For example

$$(\{1, 2\}, \{\{1, 2\}\}).2 \oplus (\{1, 2\}, \{\{1, 2\}\}).1$$
$$= (\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}).$$

We will also use the notations $i.j \oplus k.l$ and $(i, j) \oplus (k, l)$ to mean the join of the assemblies with indices $i$ and $k$ in a given $\mathcal{G}$ via the vertices with indices $j$ and $l$.

The set of subassemblies $\mathcal{G}$ must have the following property:

**Property 1** *For all $G \in \mathcal{G}$ there exist $G_1, G_2 \in \mathcal{G}$, $u \in V_1$ and $v \in V_2$ such that $G_1.u \oplus G_2.v \simeq G$ unless $G = \{\{1\}, \emptyset\}$ and there does not exist a $G' \in \mathcal{G} - \{G\}$ with $G \simeq G'$.*

Here "$\simeq$" means isomorphic in the usual sense: $(V_1, E_1) \simeq (V_2, E_2)$ if there exists a function $h : V_1 \to V_2$ such that $(u, v) \in E_1$ if and only if $(h(u), h(v)) \in E_2$. Such an $h$ is called a *witness* of the isomorphism. Witnesses are used in this paper to "translate" the representation of the join of two graphs to the representation of that graph in $\mathcal{G}$. Property 1 assures that any assembly can be constructed from exactly two other assemblies, so that only pairwise interactions between parts need to be considered by the ultimate controller, and that there is only one representation of each subassembly in the list. (We will also require another property, Property 2 in Section 5, later).

The simplest means of automatically constructing $\mathcal{G}$ from $G_{spec}$ is to simply set $\mathcal{G}$ to be all possible connected subgraphs of $G$ up to isomorphism, producing a set of size $\mathrm{O}(2^n)$. This set can be computed using a simple exhaustive search. Since $\mathcal{A}$ and $g$ will be obtained from a table constructed from $\mathcal{G}$, this may be an impracticably large set for large $G_{spec}$, although for small assemblies the set of all subassemblies is quite reasonable and produces good controllers. A $\mathcal{G}$ thus constructed trivially satisfies Property 1.

Another means of constructing $\mathcal{G}$ is to build subtrees of $G_{spec}$ one node at a time, starting at some base node and adding nodes to the leaves of subtrees. This algorithm, which we call $A_1$, requires an assembly $G_{spec}$ and a base node $i$. It produces a set $\mathcal{G}_{A_1,i}$ of size exactly $n$, there being one subassembly for each size 1 to $n$. The set $\mathcal{G}_{A_1,i}$ constructed using $A_1$ satisfies Property 1 easily since each subassembly (except the singleton assembly) can be obtained by joining the next smallest subassembly with $\{\{1\}, \emptyset\}.1$. Richer subassembly sets can be made by calling $A_1$ again, starting with a different base node, and combining it with the first set. In this manner a set of size $\mathrm{O}(cn)$ can be constructed from a set of $c$ nodes $U \subseteq V_{spec}$. Call this set $\mathcal{G}_{A_1,U}$. It satisfies

Property 1 because each of the sets $\mathcal{G}_{A_1,i}$ for $i \in U$ do. The process of combining the sets requires some computation, however, because we must maintain the second part of Property 1. To combine the list $\mathcal{G}_{A_1,i}$ with list $\mathcal{G}_{A_1,j}$ we must compare each element of the first list with each element of the second list to make sure they are not isomorphic. If they are, we keep only one of them for the combined list. Although there is no known polynomial time algorithm for checking the isomorphism of two graphs, checking the isomorphism of two trees of size $n$ takes $\mathrm{O}(n^{3.5})$ steps (Reyner, 1977). Thus, combining two size $n$ lists takes time $\mathrm{O}(n^{5.5})$. The reader can check that the combination of sets satisfying Property 1 also satisfies Property 1.

## 3.3. GENERATING UPDATE RULES

From an assembly set $\mathcal{G}$ satisfying Property 1, we can state the definition of $\mathcal{A}$ simply:

**Definition 2** *Given $\mathcal{G}$ satisfying Property 1, the* **attraction predicate** *$\mathcal{A}$ is defined as*

$$\mathcal{A}(s_i, s_j) = \mathrm{t}rue \Leftrightarrow \exists\, G \in \mathcal{G} \; such \; that \; s_i \oplus s_j \simeq G.$$

We can also define the update rule $g$.

**Definition 3** *Given $\mathcal{G}$ satisfying Property 1 and states $s_i$ and $s_j$ with $\mathcal{A}(s_i, s_j) = \mathrm{t}rue$, the* **update rule** *$g$ is defined as follows. Suppose $G \simeq s_i \oplus s_j$ has index $k$ in $\mathcal{G}$, suppose $h : s_i \oplus s_j \rightarrow G$ witnesses this isomorphism and suppose $s_l = (a, b)$. Then*

$$g(s_i, s_j, s_l) \doteq (k, h(b'))$$

*where $b' \in V(s_i \oplus s_j)$ is the name of vertex $b$ after taking disjoint unions in Definition 1 of the join operation. If $\mathcal{A}(s_i, s_j) = \mathrm{f}alse$ then the update rule is not defined: $g(s_i, s_j, s_l) \doteq \perp$.*

The procedure for determining the values of $\mathcal{A}$ and $g$ require determining tree isomorphisms — which is likely too time consuming to be done online. We can, however, perform all the necessary computations offline by compiling $\mathcal{G}$ into a table. The result is that $H_i$ can make all discrete transitions essentially instantaneously because all that is required is a table look-up. Furthermore, the size of the table is $\mathrm{O}(|\mathcal{G}|^2 n^3)$. As was shown, $|\mathcal{G}|$ can be taken to be $cn$, so that even complicated assemblies require only $\mathrm{O}(n^5)$ storage.

The construction proceeds in two steps. First, we determine a representation of the update function $g$ resulting from a join of $G_i.j$ with $G_k.l$. Second we build a table of all possible joins between all possible pairs of distinct graphs taken from $\mathcal{G} - G_{spec}$. The result is a four dimensional table $T$ where each entry $T_{i,j,k,l}$ is the representation of $G_i.j \oplus G_k.l$.

Given $G_i.j$ and $G_k.l$, let $G = (V, E) = G_i.j \oplus G_k.l$. We must first determine whether there exists a $G' \in \mathcal{G}$ such that $G \simeq G'$ then, we require a

**Algorithm $A_2$:**

Input: $\mathcal{G}$, a list of subgraphs with Property 1
Output: T, a tabular representation of $\mathcal{A}$ and $g$

For $i = 1$ to $|\mathcal{G}| - 1$
  For $k = i$ to $|\mathcal{G}| - 1$
    For $j = 1$ to $|V_i|$
      For $l = 1$ to $|V_k|$
        If $\exists\ G \in \mathcal{G}$ with $i.j \oplus k.l \simeq G$
          Let $h$ be the witness
          $T_{i,j,k,l} = (\ index(G'),\ \langle h \rangle)$
        Else $T_{i,j,k,l} = \perp$

*Figure 1.3.* The procedure for constructing a table of size $O(|\mathcal{G}|^2 n^3)$ from a list of subassemblies $\mathcal{G}$ of a specified tree $G_{spec}$. The predicate $\mathcal{A}$ and the update rule $g$ can be read off the resulting table in constant time.

witness $h$ of this isomorphism because we must have a means of translating the new roles of each part in the new assembly into their representations in $\mathcal{G}$. Suppose such an $h$ exists. Then we represent the table entry $T_{i,j,k,l}$ as a pair

$$(\ index(G'),\ \langle h(1), ..., h(|V_i| + |V_j|) \rangle\ ).$$

Otherwise, set $T_{i,j,k,l} = \perp$. The procedure for constructing $T$ is shown in Figure 1.3, it takes time $O(|\mathcal{G}|^3 n^{6.5})$ because of the added complexity of finding a witness for each join.

To summarize, given $G_{spec}$, constructing $\mathcal{A}$ and $g$, the discrete part of the controller $H_i$, proceeds in two steps. First, a list of subassemblies $\mathcal{G}$ is build from $G_{spec}$ using one of the methods discussed in Section 3.2. Second, using algorithm $A_2$, a table $T$ is built from the $\mathcal{G}$. $\mathcal{A}(s_i, s_j)$ can be computed simply by checking whether $T_{s_i,s_j} \neq \perp$ and $g(s_i, s_j, (a, b))$ can be determined by looking up $T_{s_i,s_j}$ and reading off $h(b)$.

# 4. IMPLEMENTATION OF ASSEMBLY RULES

Completing the controller $H_i$ shown in Figure 1.2 requires a definition of $F_i$ as well as some method by which parts can communicate. In this section we define an $F_i$ and assume a simple communications scheme that works in simulation and about which we have a preliminary analytical understanding.

We suppose that parts can only communicate with their neighbors. The difficulty is then that two parts playing roles in the same subassembly might try to update the state of that subassembly simultaneously. Thus, such an update

requires a means of obtaining consensus among all parts in the subassembly. Consensus can be difficult or even impossible if the processing is asynchronous and there are process failures or link failures (Lynch, 1996), although approximate algorithms exist for these situations (Franceschetti and Bruck, 2001). In what follows, we assume a good consensus algorithm with no process or communication failures. Consideration of the many complications we may add, although important, would take us too far afield of the present topic and are somewhat independent of methods we have so far described.

## 4.1. AN EXAMPLE IMPLEMENTATION

For each part $i$, we can decide, using $\mathcal{A}$, whether part $i$ should move toward $j$ or not. To this end define

$$\begin{aligned} S(i) &= \{j \mid ||x_i - x_j|| < d_{max}\} \\ Attract(i) &= (\{j \mid \mathcal{A}(s_i, s_j)\} \cup Nbrs(i)) \cap S(i) \\ Repel(i) &= (\{j \mid \neg\mathcal{A}(s_i, s_j)\} - Nbrs(i)) \cap S(i). \end{aligned}$$

$S(i)$ is the set of parts that $i$ can sense. Note that these sets are easily computed from a table compiled from a given $G_{spec}$. One way of forming the control law $F_i$ is to sum, for each $j \in Attract(i)$ a vector field $F_{att}$ which has an equilibrium set at distance $d_{nbr}$ from $x_j$ and for each $j \in Repel(i)$ a vector field $F_{rep}$ which has $x_j$ as a repellor. We can construct these fields from the potential functions defined by

$$\begin{aligned} V_{att}(x_i, x_j) &= \left(\frac{||x_i - x_j|| - d_{nbr}}{||x_i - x_j|| - r}\right)^2 \\ V_{rep}(x_i, x_j) &= \left(\frac{1}{||x_i - x_j|| - r}\right)^2. \end{aligned}$$

Recall that $r$ is the radius of the (disk shaped) parts. Then we set

$$\begin{aligned} F_{att}(x_i, x_j) &= -\frac{1}{||x_i - x_j||}\frac{\partial V_{att}}{\partial x_i}(x_i, x_j) \\ F_{rep_1}(x_i, x_j) &= -\frac{1}{||x_i - x_j||}\frac{\partial V_{rep}}{\partial x_i}(x_i, x_j) \\ F_{rep_2}(x_i, x_j) &= \begin{cases} 0 \text{ if } ||x_i - x_j|| > d_{nbr} + \delta \\ F_{rep_1}(x_i, x_j) \text{ otherwise} \end{cases} \end{aligned}$$

where $\delta > 0$ is some small constant. We have scaled the gradients of the potential functions by $||x_i - x_j||^{-1}$ so that the "influences" of parts nearest $i$ are felt most strongly. We have also defined two versions of the repelling field. We use $F_{rep_2}$ because it is only active when parts violate condition (2) from Section 2. We will see the reason for this shortly.

For the complete control law we use

$$\begin{aligned} F_i(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{s}) &= \sum_{j \in Attract(i)} F_{att}(x_i, x_j) \\ &+ \sum_{j \in Repel(i)} F_{rep_2}(x_i, x_j) - b\dot{x}_i \end{aligned}$$

where $b > 0$ is a damping parameter. In practice we assume a maximum actuator force, setting $u_i = max\{u_{max}, F_i(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{s})\}$.

We have built a simulation environment for the above system. We have investigated a variety of initial conditions, with varying numbers of agents (from tens to hundreds), and various specifications of the desired assembly $G_{spec}$. Some simulations can be viewed at http://www.cs.caltech.edu/∼klavins/rda/.

Two deadlock situations arose in our initial simulations. First, $F$ may have spurious stable equilibriums which prevent attracting pairs from moving toward each other. Second, it is possible that the set of currently formed subassemblies admit no joins in $\mathcal{G}$. That is, it may be that at some time there do not exist parts $i$ and $j$ such that $\mathcal{A}(s_i, s_j)$ is true.

To avoid these situations, we employ a simple deadlock avoidance method. For each subassembly $G_k \in \mathcal{G}$ we define a *stale time* $stale(k) \in \mathbb{R}$. Any subassembly that has not changed state within $stale(i)$ seconds of its formation time should (1) break apart, setting the state of each part in it to $(1, 1)$ and (2) have each part "ignore" other parts from that same assembly for $stale(k)$ seconds. If $k_{spec}$ is the index of $G_{spec}$ in $\mathcal{G}$, we set $stale(k_{spec}) = \infty$. The result is a new controller $H_{d,i}$ that checks for staleness and implements (1) and (2) above, but is otherwise similar to $H_i$ in Figure 1.2. We also change the definitions of $Attract(i)$ and $Repel(i)$. Suppose that $Ignore(i)$ is the set of all part indices that part $i$ is presently ignoring due to a staleness break-up. Then

$$\begin{aligned} Attract_d(i) &= Attract(i) - Ignore(i) \\ Repel_d(i) &= Repel(i) - Ignore(i). \end{aligned}$$

$F_i$ is then changed accordingly. Using this deadlock avoidance measure, we have not yet seen a set of initial conditions for any $G_{spec}$ we tried for which our simulation did not converge upon the maximum possible number of parts playing roles in a final assembly, except when partial assemblies were out of sensor range of parts needed to complete them.

## 5. PARTIAL CORRECTNESS OF THE ASSEMBLY PROCESS

In this section, we describe a *discrete* model of the assembly process that allows to prove properties of the rules generating by the compiler, independent of the physical setting in which the assembly process is implemented. This is

only half the story, of course. A complete proof of correctness of the hybrid assembly system is not yet available.

For each part $i \in \{1, ..., m\}$ we define its state at step $k$ by a triple

$$q_i(k) = (s_i(k), N_i(k), \mathfrak{I}_i(k)).$$

Here $s_i(k)$ is the assembly index, role pair as in the description of the compiler, $N_i(k) \in \{1, ..., m\}$ is the set of neighbors of $i$ at step $k$ and $\mathfrak{I}_i(k)$ is the "ignore"set of part $i$ at step $k$. Initially,

$$q_i(0) = (s_i(0), N_i(0), \mathfrak{I}_i(0)) = ((1,1), \emptyset, \emptyset).$$

Let $q(k) = \{q_1(k), ..., q_m(k)\}$.

Define $G(k) = (\{1, ..., m\}, E(k))$ to be the graph induced by the neighbor sets: $\{a, b\} \in E(k)$ if and only if $a \in N_b(k)$. Define $S_i(k)$ to be the connected component of $G_k$ containing $i$. Thus, $S_i(k)$ is the subassembly of agent $i$ at step $k$. Define $R(k)$ to be the binary relation on $\{1, ..., n\}$ defined by

$$\begin{aligned} i\ R(k)\ j \quad &\Leftrightarrow \quad \mathcal{A}(s_i(k), s_j(k)) \\ &\wedge \quad S_i(k) \neq S_j(k) \\ &\wedge \quad (s_i(k) = s_j(k) = (1,1) \Rightarrow i \notin \mathfrak{I}_j(k)). \end{aligned}$$

To advance the system from state $q(k)$ proceed as follows. There are three cases.

1 There exist $i$ and $j$ such that $i\ R(k)\ j$. In this case, set

$$s_l(k+1) = \begin{cases} g(s_i, s_j, s_l) \text{ if } l \in S_i(k) \cup S_j(k) \\ s_l(k) \text{ otherwise} \end{cases}$$

and set

$$N_l(k+1) = \begin{cases} N_l(k) \cup \{j\} \text{ if } l = i \\ N_l(k) \cup \{i\} \text{ if } l = j \\ N_l(k) \text{ otherwise} \end{cases}$$

for all $l \in \{1, ..., n\}$. The ignore sets do not change.

2 $R(k) = \emptyset$ and there are at least two components of $G(k)$ that are not isomorphic to a final subassembly. In this case, we advance the system by breaking up a subassembly. First, choose $X(k)$ to be a minimally sized assembly of $G(k)$. And set

$$s_l(k+1) = \begin{cases} (1,1) \text{ if } l \in X(k) \\ s_l(k) \text{ otherwise} \end{cases}$$

and put

$$N_l(k+1) = \begin{cases} \emptyset \text{ if } l \in X(k) \\ N_l(k) \text{ otherwise} \end{cases}$$

and finally

$$\mathcal{I}_l(k+1) = \begin{cases} X(k) \text{ if} l \in X(k) \\ \emptyset \text{ otherwise} \end{cases}$$

for all $l \in \{1, ..., n\}$.

3 $R(k) = \emptyset$ and there are zero or one components that are not isomorphic to a final assembly.

The system is assembled. Set $s_l(j) = s_l(k)$ and $N_l(j) = N_l(k)$ for all $j > k$.

We prove two properties about the system defined above. The first is a *safety* property, asserting that only subassemblies in $\mathcal{G}$ form during executions of the system. The second is a *progress* property, asserting essentially that the number of components of $G(k)$ decreases as $k$ increases. From this property we can conclude that every run of the system ends with a maximum number of final subassemblies being formed.

**Theorem 1** *For all $k \in \mathbb{N}$, every component of $G(k)$ is isomorphic to some graph $G' \in \mathcal{G}$.*

**Proof:** This is true of $G(0)$ since all components are singletons. Suppose it is true of $G(k)$. Either rule 1 or rule 2 above is used. In the first case, suppose that $i$ and $j$ are chosen so that $iR(k)j$. Then $G(k+1)$ is the same as $G(k)$ except that $G(k+1)$ contains the additional edge $(i, j)$ joining $S_i(k)$ and $S_j(k)$ together. Since $iR(k)j \Rightarrow \mathcal{A}(s_i(k), s_j(k)) \Rightarrow S_i(k).i \oplus S_j(k).j \in \mathcal{G}$, the new component is isomorphic to a graph in $\mathcal{G}$. In the second case, suppose that $X$ is chosen as a minimal component of $G(k)$. Then $G(k+1)$ has the same components as $G(k)$ except that it does not contain $X$ and it does contain $|X|$ singletons. □

In the next theorem we suppose that $G(k_1)$ contains no copies of the final assembly. The does not reduce the generality of our arguments because final assemblies do not play any part in the execution of a system. Thus, once a final assembly is built, we can remove the nodes in it from consideration.

Before we state the theorem, we have a definition. Let the predicate $P(k)$ be defined by

$$\begin{aligned} P(k) \quad &\Leftrightarrow \quad R(k) = \emptyset \\ &\wedge \quad G(k) \text{ contains at least two nonfinal assemblies.} \end{aligned}$$

Thus, $P(k)$ is equivalent to the condition for rule two in the definition of execution. We also define a new property on $\mathcal{G}$ that is that we require of assembly sequences in addition to Property 1:

**Property 2** *$\{\{1\}, \emptyset\} \in \mathcal{G}$ and for all $G \in \mathcal{G}$ there is a $u \in V(G)$ such that $G.u \oplus \{\{1\}, \emptyset\}.1$ is isomorphic to some graph in $\mathcal{G}$, unless $G$ is the final assembly.*

**Lemma 1** *Suppose Property 2 holds for $\mathcal{G}$. Let $k_1 < k_2$ be such that*

1 *$G(k_1)$ contains no final assemblies;*

2 *$P(k_1)$ is true;*

3 *$P(k_2)$ is true;*

4 *There is no $j$ such that $k_1 < j < k_2$ and $P(j)$ is true.*

*Then the number of components of $G(k_2)$ is less than then number of components of $G(k_1)$.*

**Proof:** Suppose that $G(k_1)$ has $x$ components. To obtain the next state, rule 2 is chosen, by condition 2 in the assumptions of the theorem. Suppose that a minimal assembly of size $y$ is chosen to be broken up. Then $G(k+1)$ has $x-1+y$ components.

In steps $k_1+1, ..., k_2-1$, rule 1 is used exclusively by assumption 4 in the theorem. Also, since the singletons added at step $k$ cannot join with each other, by definition of their ignore sets in rule 2, they must each combine with assemblies descendant from one of the $x-1$ non singletons in $G(k_1+1)$. There is always one available since all non-final assemblies can join with singletons. Thus, at step $k_2$ there are $x-1$ or fewer components in $G(k_2)$. □

**Corollary 1** *Every sequence of states ends with a maximum number of final assembles being formed.*

**Proof:** Let $c(k)$ be the number of components in $G(k)$. By Lemma 1 and the fact that rule 1 always decreases the number of components in the neighbor graph, we can find a sequence of states $k_1, ..., k_r$ such that $c(k_1), ..., c(k_r)$ is a decreasing sequence. At some point, $c(k_i) \leq m/n$ and thus some component of $G(k_i)$ has $n$ elements. Since the only assembly with $n$ elements that is allowed is the final assembly, $G(k)$ has a final assembly in it. We can remove it and start the process again with $m-n$ parts and no non-final assemblies. □.

# 6. CONCLUSION

The ideas in this paper represent only the first steps toward understanding and realizing specifiable, programmable self assembly. Many relatively unexplored and apparently fruitful issues remain. First, although simulations and the proof in Section 5 suggest that the implementation (particular choice of $F_i$) combined with the deadlock avoidance procedure produces controllers that assemble a maximum number of parts safely (without collisions), this must be verified analytically using the tools in Section 5 and tools from non-linear dynamical systems.

Arbitrary graphs (as opposed to trees) require certain embeddings of their subassemblies in order to assemble themselves. For example, suppose we

assemble a graph by first assembling a spanning tree of the graph and then "closing" it. If we require the closing procedure to respect the $d_{nbr}$ distance requirements we have used, then the tree can not *cross over* itself while closing. This means the tree must assemble to an appropriate embedding class — a constraint we do not yet deal with, but plan to address soon.

Many variations on the theme presented here should also be explored: hierarchical assembly with intermediate goal assemblies, three dimensional assembly (which has fewer "closing" problems than in two dimensions), assembly of non-homogeneous parts, assembly of parts with complex dynamics (e.g. nonholonomic), and so on. Finally, we are exploring hardware implementations of these algorithms so that the issues of asynchronous processing, inaccurate sensors and faulty communications may be realistically addressed.

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