Soundness and completeness

1. Soundness:
   a. If $\varphi$ is derivable from $\Delta$, then $\varphi$ is a consequence of $\Delta$.
   b. In symbols: If $\Delta \vdash \varphi$, then $\Delta \models \varphi$.
   c. Transposing: Every satisfiable set is consistent.

2. Completeness:
   a. If $\varphi$ is an $\omega$-consequence of $\Delta$, then $\varphi$ is derivable from $\Delta$.
   b. In symbols: If $\Delta \vdash_\omega \varphi$, then $\Delta \vdash \varphi$.
   c. Transposing: Every consistent set is satisfiable in the domain of natural numbers.

3. The Löwenheim-Skolem theorem: Any set of sentences that is satisfiable is satisfiable in the domain of natural numbers. Follows by the syllogism Barbara from (1c) and (2c).

4. The compactness theorem: If every finite subset of a set of sentences is satisfiable, then the set itself is satisfiable.
   a. Suppose that $\Delta$ is not satisfiable.
   b. Then $\Delta$ is not satisfiable in the domain of natural numbers.
   c. So $\Delta$ is inconsistent.
   d. This means that there is a sentence $\phi$ such that $\Delta \vdash (\phi \& \neg \phi)$.
   e. But derivations are always finite in length.
   f. So there is a finite subset of $\Delta$, call it $\Delta'$, such that $\Delta' \vdash (\phi \& \neg \phi)$.
   g. $(\phi \& \neg \phi)$ is not satisfiable.
   h. Therefore, by soundness, $\Delta'$ is not satisfiable.