Simple Kelvin wave model with vertical modes

The model integrates the zonal momentum equation in the form

$$u_t + g' h_x = \frac{\tau^x}{\rho H} \tag{1}$$

where H is the upper layer depth, with anomalies h, along Kelvin wave characteristics, summing the wind forcing felt by the traveling Kelvin signal. A simplification is due to the proportionality between the u and h field of the Kelvin wave, such that $h_K=(c/g')u_K$, and (1) becomes

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)h_K = \frac{\tau^x}{\rho c} \tag{2}$$

which has the advantage that no factors of H or g′ (which are ambiguous by at least a factor of two) appear, whereas c is a well-determined quantity. This simplification results from identifying the layer depth H with the wind-forced layer. Such an identification is not possible for higher vertical modes. The integration of (2) along characteristics gives the solution at any (x_0,t)

$$h_K(x_0,t) = \frac{1}{\rho c^2}\int_{WB}^{x_0} \tau^x\,(x, t+(x-x_0)/c)\,dx \tag{3}$$

where WB is the western boundary. Integration in x in (3) substitutes for integration in t since along the characteristic, t=x/c. This gives the extra factor of 1/c.

To find solutions over vertical modes, we use the formalism developed in Cane (1984). If the vertical dependence is separable, then

$$\left[u(x,y,z,t),\, v(x,y,z,t),\, \rho^{-1}P(x,y,z,t)\right] = \sum_{n=0}^{\infty}\left[u'_n(x,y,t),\, v'_n(x,y,t),\, gh'_n(x,y,t)\right]\psi_n(z) \tag{4}$$

where the $\psi_n(z)$ are the vertical structure eigenfunctions, with eigenvalues c_n. The $\psi_n(z)$ are normalized so that

$$\int_{-D}^{0}\psi_n^2(z)\,dz = D \tag{5}$$

D is the total ocean depth. (McCreary normalizes $\psi_n(0)=1$). The $\psi_n(0)$ decrease with n, but not monotonically (table 1). The n-th system of shallow water equations is driven by a body force

$$\tau_n(x,y,t) = \frac{1}{D}\int_{-D}^{0}\frac{\partial\tau}{\partial z}\psi_n(z)\,dz \tag{6}$$

where τ_n has units of stress per depth. Assuming the stress is confined to a fully-mixed surface layer, the $\psi_n(z)$ are constant in this layer, and (6) reduces to

$$\tau_n(x,y,t) = \tau(x,y,0,t)\frac{\psi_n(0)}{D} \tag{7}$$

where $\tau(x,y,0,t)$ is the surface wind stress. The usual system of momentum and continuity equations (including the Kelvin wave equations (1)-(3)) is then formally the same for each of the vertical modes n. Note that (3) is the same as Cane's (1994) Kelvin integration (his eqn (11)).

Cane shows that each system n can be identified as the usual 1½ layer system with upper layer depth D' (where D' is analogous to H in (1)) if

$$D' = \frac{D}{\psi_n^2(0)}, \quad and \quad c_n^2 = g\frac{\Delta\rho}{\rho}D' \tag{8}$$

Higher modes are then equivalent to a 1½ layer system with an increasingly deep mean pycnocline D' and smaller density contrast $\Delta\rho/\rho$. With regard to the Kelvin characteristic integration (3), the solution h_K has larger magnitude for the higher vertical modes, since the forcing amplitude reduction associated with the term $\psi_n(0)$ in (7) is dominated by the factor $(1/c^2)$ in (3). The increase in "pycnocline" amplitude is appropriate because of the smaller density contrast for the higher modes. The use of (7) for the forcing term, and the first of (8) for D, results in forcing proportional to $1/D'\psi_n(0)$, compared to (1) where it is proportional to 1/H, but otherwise solutions are identical. When the solutions u_n,v_n,h_n are reconverted using (4), solutions are the same as from (3) but with the vertical structure $\psi_n(z)/\psi_n(0)$. For 1½ layer formulations, u,v,h are taken to be at z=0, so no factors of the ψ_n appear.

To find the sea level expression η of each mode, the solution h_K to (3) is converted via $\eta_K=(\Delta\rho/\rho)h_K$. Note that $\Delta\rho/\rho$ in (8) varies as $(c_n\psi_n(0))^2$ (table 1). It is usually the case that the sea level expression of each mode decreases with mode number (table 1). In particular, there is a sharp falloff in sea level amplitude after mode 2.

It is also possible to find the isopycnal displacement due to each mode, which can then be compared directly to 20°C depth. Anomalies of density are found from pressure anomalies using the hydrostatic equation, then isopycnal displacement is estimated by dividing by the mean vertical density gradient. Note that the vertical structure of density anomalies is proportional to the vertical derivative of the structure functions $\psi_n(z)$, so the higher modes can have quite large values of density anomaly.

Table 1 gives values for various parameters of the modal problem using a mean temperature profile from the Hawaii-Tahiti shuttle (150°W to 158°W), with ocean depth 4500 m.

TABLE 1	1	2	3	4	5
Equivalent depth H_n (cm)	76.0	30.9	11.5	5.9	3.5
Surface amplitude $\psi_n(0)$	4.293	3.904	1.698	1.247	1.825
Wave speed c_n=gH_n (m s^{-1})	2.73	1.74	1.06	0.76	0.59
Time scale $(\beta c_n)^{-1/2}$ (days)	1.47	1.84	2.35	2.78	3.16
Rossby radius $(c_n/\beta)^{1/2}$ (km)	346.0	276.3	215.6	182.7	160.9
ULT analogue D' (m)	244.2	295.3	1560.9	2893.4	1351.1
$\Delta\rho/\rho$ analogue ($\times 10^3$)	3.1115	1.0450	0.0734	0.0204	0.0263

A more general form of equation (3) can be written for each mode n

$$A_n(x_0, z, t) = \alpha_n(z) \int_{WB}^{x_0} \tau^x\left(x, t+(x-x_0)/c_n\right) dx \tag{9}$$

where A_n is the baroclinic mode n Kelvin part of the ocean signal (with corresponding coefficients $\alpha_n(z)$) for any of the quantities upper layer thickness, zonal speed, pressure, density or isopycnal displacement, depending on the form of the coefficients α_n. This generality expresses the fact that all the Kelvin wave variables have the same phase and structure in (x,y,t), differing only by an amplitude factor (and also in their vertical structure). The following table gives the coefficients for the various quantities, with appropriate values, using values for c_n and ψ_n listed in the previous table. The first three lines give the reduced gravity coefficients, and the rest are for modal solutions. Note that the factor $\psi_n(0)/D$ that multiplies the surface wind stress τ in (7) has been absorbed into the coefficients α.

Quantity	Coefficient α	Typical Value (mks)
RG Upper layer thickness	$1/\rho c^2$	1.31×10^{-4}
RG Sea level	$\Delta\rho/(\rho c)^2$	4.07×10^{-7}
RG Zonal current	$1/\rho c H$	2.38×10^{-6}
Sea Level mode 1	$\dfrac{\psi_n(0)^2}{\rho g D}$	4.07×10^{-7}
mode 2		3.37×10^{-7}
mode 3		6.36×10^{-8}
Zonal current mode 1	$\dfrac{\psi_n(0)\,\psi_n(z)}{\rho c D}$	1.46×10^{-6} (at z=0)
mode 2		1.90×10^{-6}
mode 3		5.89×10^{-7}
Pressure mode 1	$\dfrac{\psi_n(0)\,\psi_n(z)}{D}$	4.10×10^{-3} (at z=0)
mode 2		3.39×10^{-3}
mode 3		6.41×10^{-4}
Isopycnal displacement mode 1	$\dfrac{\psi_n(0)}{gD}\,\dfrac{d\psi_n(z)/dz}{d\rho_b/dz}$	6.01×10^{-5} (at z=125m)
mode 2		1.15×10^{-4}
mode 3		4.84×10^{-5}