On the Role of Government Expenditure in a Growing Economy with Endogenous Labor Supply

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Abstract

The role of government expenditure is analyzed in the context of growing economy with endogenous labor supply and with investment adjustment costs. Assuming the dynasty model of representative consumer's utility maximization - i.e., the present value of over time felicity function maximization under the law of motion capital where felicity function depends consumption, leisure and government expenditure, the global stability of the economy is derived showing investment-capital ratio to decrease as increase in capital-labor ratio. Further the increase in government expenditure is seen to increase investment-capital ratio and labor but decrease consumption globally.

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I. Introduction

This paper tries to analyze the role of government expenditure in a growing economy employing the intertemporal optimizing behavior of the representative consumer. Our model is closely related to Brock and Turnovsky (1981), Turnovsky and Fisher (1995), Turnovsky (1995), in the following sense; there exist only one kind of good which is used for consumption, investment or government expenditure, consumer's instantaneous felicity function depends on consumption, leisure and government expenditure in the case of government consumption service, and in the case of public input affecting production on only consumption and leisure. The representative consumer tries to maximize the present value of the felicity function overtime subject to the law of motion of capital. However our model is different from theirs in introducing adjustment costs of investment explicitly. In this sense, we owe Abel and Blanchard (1983)'s a lot to derive the global stability of the economy when such adjustment costs are taken into consideration. Our model is a generalization of their model first in introducing endogenous labor supply (or leisure-labor choice), and second government expenditure which is included either into felicity function in the case of government service or into production in the case of public input.

Third we analyze the (not local but) global effects of government expenditure on per capita consumption, investment-capital ratio and per capita labor supply along transitional path. More specifically we show that the increase in government expenditure decreases consumption, but increases both investment-capital-ratio and labor in both cases of government expenditures. To our best knowledge this result is new.

In the field of macro dynamic model with and without government expenditure Trunovsky (1995)'s book is very comprehensive and an important contribution. Since the pioneering works by Frankel and Razin (1985) most of whose contributions are contained in their book (1996), the role of fiscal policies in the growing economy has been discussed by many authors. Employing finite horizon continuous time model, Blanchard (1985) analyzed the role of fiscal policy. Aschauer (1988) employed discrete time optimizing model of representative consumer whose felicity function depends on consumption, leisure and government expenditure, while the government expenditure affects production as well, and analyzed the effects of government expenditure on consumption, output and interest rate etc. in a variety of situations. Barro (1990) employed continuous time optimizing model where government expenditure affects felicity or production, and analyzed its effects on growth rates and saving rates with empirical analysis.

Using AK model with public expenditure included into production function, Barro and Sala-I-Martin (1992) studied the role of tax policy in various situations. King and Rebelo (1990) explained the differences in growth rates based on the role of government expenditure and tax policy employing endogenous growth model with physical capital and human capital. Deveraux and Love(1995) also analyzed the effects of government expenditure in the two sector endogenous growth model; among others.

In the next section we show our model. First, we analyze the case of government service which affects felicity but not production. The social planner is assumed to maximize the intertemporal utility, i.e., the present value of the sum of the consumer's felicity function over time composed of consumption, leisure and government expenditure under the equilibrium market condition of the good which is used either for consumption, investment or government expenditure, and under the law of motion of capital. There exist investment adjustment costs. Then from the present value Hamiltonian we derive the first order conditions and the transversarity condition, and show the existence and uniqueness of the stationary state.

Next we show the stationary state to be (not local but) a globally stable saddle point (Theorem 1) where investment-capital ratio decreases as capital increases. This is a generalization of Abel and Blanchard (1984). Next we show that consumption increases as capital stock increases employing the same method used for investment-capital ratio. (Theorem 2) Then we analyze the effects of government expenditure g, on investment-capital ratio z, consumption c and labor l. Under fairly weak assumptions (A.1 through A.4), z and l increase while c decreases as g increases (Theorem 3). This is our main contribution of this paper. Especially the technique employed to prove these results can be applied for many similar cases Next we analyze the case of government expenditure, which affects production but not felicity. We show as Theorems 4 and 5 that the characteristics of optimal path of consumption, investment-capital ratio and labor supply (i.e., the conclusions of Theorems 1, 2 and 3) remain unchanged.

II. Government Expenditure in Felicity Functions

First we discuss the social planner's optimum.

II.1 Social Planner's Optimum

Let there exist only one good which is used either for consumption, investment or government expenditure. Good is produced employing capital and labor. Let $y = f(k,l)$ be the labor productivity function where $k = K/N$ and $l = L/N$ being respectively the per capita capital and the per capita labor, K being the capital stock. L being the total labor employed and N being the total hours of the population. N is assumed to be equal to the total population. Then the market equilibrium condition is expressed as

$$f(k,l) = c + i(1+\phi(i/k)) + g \,. \tag{1}$$

Furthermore $c = C/N$, $i = I/N$ and $g = G/N$ are respectively per capita consumption, investment and government expenditure (=government service) where C, I and G are respectively total consumption, investment and government expenditure.

The function ϕ reflects the investment adjustment costs, first introduced by Eisner and Strotz (1963), then Lucas (1967) and Abel and Blanchard (1983) ét.al. ϕ is convex with $\phi(0) = 0$, $\phi > 0$ for $i/k > 0$, $\phi' > 0$, $\phi'' > 0$ (from convexity) and $\phi'(0) = 1$. Intuitively ϕ states that given K, the investment adjustment costs increase more, the more investment. Total hour N is used either for production as labor hour L or for leisure N-L. Then per capita leisure is $1 - l$. Here by construction $0 \le l \le 1$ holds. The productivity function f, is assumed to be concave, homogenous of degree one, $f(\underset{+}{k},\underset{+}{l})$, i.e., $f_k > 0$ and $f_l > 0$, thrice continuously differentiable in (k, l), f is assumed to satisfy the Inada Condition, i.e., $f_k \to 0$ as $k \to +\infty$ and $f_k \to \infty$ as $k \to 0$ given $l > 0$.

Furthermore both productive factors are indispensable for production, i.e., $f(0,l) = f(k,0) = 0$ holds for any $l > 0$ and $k > 0$.

The representative consumer has the felicity function

$$u(c, 1-l, g) = \log c + \alpha \log(1-l) + \beta \log g \tag{2}$$

where $\alpha > 0$ and $\beta > 0$ are constant reflecting the constant elasticity of substitution between any two of three – consumption, leisure and government expenditure. The government expenditure is included into the felicity in our first model, later it is included into the productivity function. Furthermore the felicity function is seen to be separable. Although this looks too much simplified and specified, yet will turn to be inevitable in order to obtain the definite conclusions on the characteristics of the optimal path. The characteristic of this felicity function lies in that consumption, leisure and government expenditure are all indispensable for the consumer in the sense that level of utility decreases to minus infinity if any of these decreases to zero. The law of motion of capital is expressed as

$$\dot{k} = i - nk \tag{3}$$

where $\dot{k}$ is time rate of k, and $n > 0$ is the population growth rate. (In general time rate change of variable x is expressed as $\dot{x}$.) Then the social planner tries to maximize

$$\int_0^\infty u(c, 1-l, g)e^{-(\rho-n)t}dt$$

with respect to *c*, *l*, *k*, and *i* subject to (1) and (3). g is given as parameter which is not controlled by the social planner in our model. Here $t = 0$ is the present (initial) time, $\rho(> n)$ is the intertemporal discount rate of the felicity function. By defining the Hamiltonian,

$$H = [\log c + \alpha \log(1-l) + \beta \log g + \xi(f(k,l) - c - i(1+\phi(i/k)) - g) + \lambda(i - nk)]e^{-(\rho-n)t}$$

we obtain the first order conditions;

$$1/c = \xi \tag{4}$$

$$\alpha/(1-l) = \xi f_l \tag{5}$$

$$\lambda = \xi(1 + \phi + \phi' \cdot z) \tag{6}$$

and

$$\dot{\lambda} = \rho\lambda - \xi(f_k + \phi' \cdot z^2) \tag{7}$$

where $z = i/k$ being investment-capital ratio. Then

$$c = (1-l)f_l/\alpha \tag{8}$$

is obtained from (4) and (5). The transversality condition is

$$\lim_{t\to\infty} \lambda k e^{-(\rho-n)t} = 0.$$

II. 2 Stationary State

First we derive the stationary state. By setting $\dot{k} = 0$, we obtain $\bar{z} = n$ where $\bar{z}$ is the stationary value of *z*. (In general stationary value of variable x is expressed as $\bar{x}$.) Then from (6) and (7) by setting $\dot{\lambda} = 0$, we obtain

$$\rho = (f_k + \bar{\phi}' \cdot \bar{z}^2)/(1 + \bar{\phi} + \bar{\phi}' \cdot \bar{z}) \tag{9}$$

from which $\bar{h} = (\overline{k/l})$, the stationary value of capital labor ratio k/l is obtained from $f_k(k,l) = f_k(k/l, 1)$ where $\bar{\phi} = \phi(n)$ and $\bar{\phi}' = \phi'(n)$. Then by substituting (8) into (1), we obtain

$$f(\bar{h}, 1)l = (1-l)f_l(\bar{h}, 1)/\alpha + n\bar{h}l(1 + \bar{\phi}) + g$$

and hence

$$\bar{l} = \{f_l(\bar{h}, 1)/\alpha + g\}/\{f(\bar{h}, 1) + f_l(\bar{h}, 1)/\alpha - n\bar{h}(1 + \bar{\phi})\}. \tag{10}$$

Here we assume

A.1 $g < f(\bar{h},1) - n\bar{h}(1+\bar{\phi})(\equiv g_M)$

Then we obtain A.1 $\Leftrightarrow \bar{l} < 1 \Leftrightarrow \bar{c} > 0$ from (8). Lastly $\bar{k} = \bar{h}\bar{l}$ follows.

II.3 Global Stability

Next we show the global stability of our economy. Although basically we follow the argument of Abel and Blanchard (1983), our model is more complicated than theirs because labor supply is endogenous in our model. From (1) and (8), we can express *l* and *c* as functions of *k*, *z* and *g*;

$$\begin{bmatrix} f_l & -1 \\ 1/(1-l) - f_{ll}/f_l & 1/c \end{bmatrix} \begin{bmatrix} dl \\ dc \end{bmatrix} = \begin{bmatrix} -(f_k - z(1+\phi))dk + k(1+\phi+\phi' \cdot z)dz + dg \\ (f_{lk}/f_l)dk \end{bmatrix}.$$

Hence

$$l_k = \{-(f_k - z(1+\phi))/c + f_{lk}/f_l\}/D \tag{11}$$

$$l_z = k(1+\phi+\phi' \cdot z)/Dc > 0 \tag{12}$$

$$l_g = 1/Dc > 0 \tag{13}$$

where

$$D = (1+\alpha)/(1-l) - f_{ll}/f_l > 0 \tag{14}$$

and hence

$$l = l(k, \underset{+}{z}, \underset{+}{g}).$$

$$c_k = \{f_{lk} + (1/(1-l) - f_{ll}/f_l)(f_k - z(1+\phi))\}/D, \tag{15}$$

$$c_z = -\{1/(1-l) - f_{ll}/f_l\}k(1+\phi+\phi' \cdot z)/D < 0, \tag{16}$$

$$c_g = -\{1/(1-l) - f_{ll}/f_l\}/D < 0, \tag{17}$$

and hence

$$c = (k, \underset{-}{z}, \underset{-}{g}).$$

Here in view of (8), *c* is also expressed as

$$c = c(\underset{+}{k}, \underset{-}{l}) = c(k, l(k,z,g)).$$

Here we make the following assumption;

A.2 $\sigma > \theta/(1+\alpha^{-1})$

where σ being the elasticity of substitution between capital and labor, i.e., $\sigma = f_k f_l / f_{kl} \cdot f$, and θ being the share of capital income, i.e., $\theta = f_k \cdot k / f (<1)$[1]. A.2 is assumed through the paper.

Then from (3) through (8) and A.1 and A.2, we obtain the following phase diagram of (*k*, *z*) showing the existence of the stable saddle point path $z = z(\underset{-}{k}, g)$. (See

Appendix I.

Fig. 1

Here the dotted lines show the saddle point path $z = z(\underline{k}, g)$ and E is the stationary point. Next we show that the optimal path $z = z(\underline{k}, g)$ converges monotonically to the stationary point E globally. For this we employ the following lemma;

Lemma 1. (Poincaré-Bendixon Theorem (Hsu and Meyer (1968) Section 5-8))

(1) For the two dimentional autonomous differential equation system, the path (trajectory) must become unbounded or converge to a limit cycle or to a point.

(2) If a limit cycle exists, then the Poincaré index N-S is 1 where N denotes the number of nodes, centers and forci enclosed by a limit cycle, and S denotes the number of saddle points enclosed by a limit cycle.

Since the Poincaré index $N - S = -1$, because the stationary point is locally a saddle point, the optimal path must converge to a stationary point from (1). Hence it suffices to show both k and z are bounded. First we show the boundedness of k. From (1) and (3), we obtain

$$\dot{k} = i - nk = f(k,l) - c - i \cdot \phi - g - nk \leq f(k,1) - nk .$$

Fig. 2

Then recalling f satisfies Inada condition, and hence observing $\dot{k} < 0$ for $k > \tilde{k}$ where $\tilde{k} (< +\infty)$ is the value of k such that $f(k,1) = nk$ holds, we obtain that k is bounded. Then from (1) z is also seen to be bounded from above. Hence we obtain from (1) of Lemma 1, the optimal path is globally stable, and since it is at least locally a saddle point path, so is also globally from the continuity of the optimal path with respect to its initial values[2].

Theorem 1

Under A.1 and A.2, the optimal path $z = z(\underline{k}, g)$ is globally a saddle point path.

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[1] A. 2 is satisfied for the Cobb-Douglas production function since $\sigma = 1$ holds.

[2] In Fig. 1, if the initial point is too close either to $\dot{z} = 0$ curve or $\dot{k} = 0$ curve then either $z \to +\infty$ or $z \to 0$ as k approaches to $\bar{k}$. Hence by continuity of the optimal path and its global stability there must exist initial value of $(k, z), (k_0, z_0)$ from which the optimal path converges to a stationary state.

Optimal Path of $c = c(k, g)$

Next we show the solution path of consumption c as function of k and g. Although we can obtain the properties of the optimal path of $c = c(k,g)$ utilyzing the results of the optimal path of $z = z(k,g)$ to some extent, i.e., $c = c(\underset{+}{k}, g)$ holds as far as $f_k > (1+\phi)z$ holds as shown by Abel and Blanchard (1983)[3], the property of $c = c(\underset{+}{k}, g)$ through entire path is not derived without analyzing this in (k, c) plane as shown below.

Fig 3

The derivation of the optimal path $c = c(k,g)$ is carried out by the similar method as $z = z(k,g)$. (See Appendix II for detail.)

As shown in Fig. 3, the optimal path of $c = c(\underset{+}{k}, g)$ is a stable saddle point path shown by the dotted lines toward the stationary point E where k_m and k_M are defined implicitly by $\dot{k} = \dot{c} = 0$, i.e., $f(k, \tilde{l}, (k,g)) = z(k,g)(1+\phi(z(k,g))$ where $l = l(k,z,g)$ $= l(k, z(k,g), g) = \tilde{l}(k,g)$. Noting that stationary point E is locally a saddle point, and k is bounded, and hence c is bounded from (1), we can employ Poincaré-Bendixon Theorem once again, and derive;

Theorem 2

Under A.1 and A.2, the optimal path $c = c(\underset{+}{k}, g)$ is globally a stable saddle point path.

Next we investigate the effects of the change in government expenditure on z and c.

II. 4 Effects of Government Expenditure

Here we show

$$c = c(k, \underset{-}{g}), \quad z = z(k, \underset{+}{g}) \quad \text{and} \quad l = l(k, \underset{+}{g})$$

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3 The iso consumption plus government expenditure curve $c + g = f(k,l) - i(1 + \phi(i/k)) = const$, is shown to be upper sloped as far as $f_k \geq (1+\phi)z$ holds in (k, z) plane. From this and $z = z(\underset{-}{k}, g)$, we obtain $c = c(\underset{+}{k}, g)$ as far as $f_k \geq (1+\phi)z$ holds.

to hold locally, and then generalize these into global properties. First we show $z = z(k, \underset{+}{g})$ to hold globally. From (9), $\bar{h} = (\overline{k/l})$ is seen to be independent of *g*. Then from (10) and (8)

$$g \uparrow \Leftrightarrow \bar{l} \uparrow \Leftrightarrow \bar{c} \downarrow$$

in view of $f_l(\bar{k}, \bar{l}) = f_l(\bar{h})$. Then $\bar{k}$ increases by the same rate as $\bar{l}$. This implies in Fig. 1, $\dot{z} = 0$ curve shifts to the right around the stationary state *E*, and hence $z = z(k, g)$ curve increases upward around *E*, or $z = z(k, \underset{+}{g})$ to hold locally around *E*.

Next we show $c = c(k, \underset{-}{g})$. $\dot{k} = 0$ curve is shown to be

$$c = f(k, l) - nk(1 + \bar{\phi}) - nk - g$$

where $\dot{k} = 0$ if and only if $z = n$. Given *k*, increases in *g* lowers $c - f(k, l)$ which is made possible by increasing *l* and decreasing *c* from (13) and (17). This implies that $\dot{k} = 0$ curve shifts down in Fig. 3 as a results of increases in *g*. Here recalling $\bar{c}$ to decrease and $\bar{k}$ to increase we observe that $c = c(k, g)$ curve shifts down around *E*, implying $c = c(k, \underset{-}{g})$ to hold locally. Lastly we observe $l = l(k, g)$ to hold locally from (8), (18) implies $c = c(k, \underset{-}{l}) = c(k, l(k, g))$, and hence $l = l(k, \underset{+}{g})$. Next we generalize the above results in to the global ones.

From (4), (6), (7), (15) and (16), we obtain

$$A(k,z,l)\dot{z} = B(k,z,l) + E(k,z,l)\dot{k} \tag{18}$$

Where

$$A(k,z,l) = (2\phi' + \phi'' z)/(1 + \phi + \phi' z) + \{1/(1-l) - f_{ll}/f_l\}k(1 + \phi + \phi' z)/cD > 0 \tag{19}$$

$$B(k,z,l) = \dot{\lambda}/\lambda = \rho - (f_k + \phi' z^2)/(1 + \phi + \phi' z) \tag{20}$$

and

$$E(k,z,l) = \{f_{lk} + (1/(1-l) - f_{ll}/f_l)(f_k - z(1+\phi))\}/cD \tag{21}$$

(See Appendix I for derivation of the above equations.)

(3) is expressed as

$$\dot{k} = kz - nk = F(k, z). \tag{3}$$

Then

$$\dot{z} = \{B(k,z,l) + E(k,z,l)(zk - nk)\}/A(k,z,l) = G(k,z,l) \tag{22}$$

$$= G(k, z, l(k, z, g)) = \tilde{G}(k, z, g).$$

Recalling $l = l(k, z, g)$ and $c = c(k, l)$, in view of (8) we can see that the change in *k* and *z* caused by the change in *g* is channeled through change in *l* only. To obtain the effects on *z* and *k* of *g*, we assume

A.3 Given (*k*, z), $\partial \widetilde{G}(k, z, g)/\partial g \neq 0$.

This is equivalent to $\partial G(k, z, l(k, z, g))/\partial l \neq 0$ given (*k*, *z*) in view of $l = (k, z, g)$. Now we show

Theorem 3

Under A.1 through A.3, $z = z(k, \underset{+}{g})$, $c = c(k, \underset{-}{g})$ and $l = l(k, \underset{+}{g})$ holds globally.

Proof

Let the solution path (*k*, *z*) of (3) and (22) converging to *E* as $t \to \infty$ be $k = k(t)$, $z = z(k, g)$. For a contradiction we assume there exist $(\hat{k}, \hat{z})$, $t = t_0$, g_1 and g_2 $(g_1 < g_2)$ such that

$\hat{k} = k(t_0)$, $\hat{z} = z(k(t_0), g_1) = z(k(t_0), g_2)$ with $g_1 < g_2$. That is the two solution paths $(k, z(k, g_1))$ and $(k, z(k, g_2))$ meet at $(\hat{k}, \hat{z})$. Then by the mean value theorem there exists $g_0 \in (g_1, g_2)$ such that $\partial z(\hat{k}, g_0)/\partial g = 0$.

By partially differentiating (22) with respect to *g*, we obtain

$$\partial/\partial t \cdot \partial z(k, g)/\partial g = \partial \widetilde{G}/\partial k \cdot \partial k(t)/\partial g + \partial \widetilde{G}/\partial z \cdot \partial z(k, g)/\partial g + \partial \widetilde{G}/\partial g,$$

which is equivalent to

$$\partial/\partial t \cdot \partial z(k, g)/\partial g = \partial \widetilde{G}/\partial z \cdot \partial z(k, g)/\partial g + \partial \widetilde{G}/\partial g.$$

Now from the above equation, we obtain

$$0 = \partial/\partial t \cdot \partial z(\hat{k}, g_0)/\partial g = \partial \widetilde{G}/\partial g,$$

contradicting A.3.

Hence we obtain the two solution paths never meet. This further implies that if $\partial z(k, g)/\partial g > 0$ near *E* holds, it also holds globally. $c = c(k, \underset{-}{g})$ follows from

$c = c(\underset{-}{k}, l)$, $l = l(k, \underset{+}{z}, \underset{+}{g})$ and $z = z(k, \underset{+}{g})$. $l = l(k, \underset{+}{g})$ follows from $l = l(\underset{+}{k}, \underset{-}{c})$ and $c = c(\underset{-}{k}, g)$. ■

It is interesting to note that under fairly mild condition of A.3, the global effects of g on z is derived. Furthermore to our best knowledge, these effects are new.

Theorem 3 states interesting results. Given capital stock k, while the increase in government expenditure increases accumulation rate of capital, $\dot{k}/k$, it also decreases the consumption level. These results seem opposite to the well known short-run results about the effects of government expenditure on accumulation rate and consumption – i.e., increase in g results in decrease in investment and increase in consumption.

III. Government Expenditure as a Productive Input

Here we investigate the role of government expenditure which affects not consumption but production as a productive input. The market equilibrium condition of good is expressed as

$$f(k, l, g) = c + i(1 + \phi(i/k)) + g \,. \qquad (1)'$$

Here per capita government expenditure g is included in the productivity function f. We assume f is concave, thrice continuously differentiable in (k, l, g) and homogenous of degree one in (k, l). Further $f = 0$ if $k \cdot l \cdot g = 0$ holds. (That is, all of k, l and g are indispensable for production.) $f_g > 0$ and from concavity $f_{gg} < 0$ hold. We assume further f_{lk} and $f_{lg} > 0$. The felicity function of the representative consumer is simplified as

$$u(c, 1-l) = \log c + \alpha \log(1-l) \qquad (2)'$$

where $\alpha > 0$ being constant.

The law of motion of capital is the same as before:

$$\dot{k} = i - nk \qquad (3)$$

The Social Planner's Optimum

The social planner maximizes

$$\int_0^\infty u(c, 1-l) e^{-(\rho - n)t} dt$$

with respect to c, l, k and i subject to (1)' and (3). By defining the Hamiltonian

$$H=\left[\log c+\alpha\log(1-l)+\xi\left\{f(k,l,g)-c-i(1+\phi(i/k)-g\right\}+\lambda(i-nk)\right]e^{-(\rho-n)t},$$

we obtain the first order conditions,

$$1/c=\xi, \tag{4}$$

$$\alpha/(1-l)=\xi f_l, \tag{5}$$

$$\lambda=\xi(1+\phi+\phi' z) \tag{6}$$

and

$$\dot{\lambda}=\rho\lambda-\xi(f_k+\phi'\cdot z^2) \tag{7}$$

and the transversality codition

$$\lim_{t\to\infty}\lambda k e^{-(\rho-n)t}=0.$$

(4) and (5) imply

$$c=(1-l)f_l/\alpha \tag{8}$$

as before.

(1)' and (8) imply

$$f(k,l,g)=(1-l)f_l(k,l,g)/\alpha+kz(1+\phi)+g. \tag{1''}$$

Recalling f is homogenous of degree one in (k, l) and hence $f_k(k,l,g)=f_k(h,1,g)$, we observe from (1)"

$$l=\left\{f_l(h,1,g)/\alpha+g\right\}/\left\{f(h,1,g)+f_l(h,1,g)/\alpha-hz(1+\phi)\right\}.$$

Then $l=\bar{l}$ is derived from this equation.

Existence and Uniqueness of the Stationary State

By letting $\dot{k}=0$ and $\dot{\lambda}=0$, we obtain as before, $\bar{z}=n$, and

$$\rho=(f_k(k,l,g)+\bar{\phi}'\bar{z}^2)/(1+\bar{\phi}+\bar{\phi}/\bar{z}). \tag{9'}$$

(9)' implicitly defines $h(=k/l)=\bar{h}$ with help of $\bar{z}=n$. Lastly $\bar{k}=\bar{h}\cdot\bar{l}$ follows. Here we assume

A. 1' $g<\tilde{g}_M$

is retained where $g=\tilde{g}_M$ is implicitly defined by $g=f(\bar{h},1,g)-n\bar{k}(1+\bar{\phi})$. Fig. 4 illustrates A.1'.

Fig 4

The intersection of curve $f(\bar{h},1,g)$ and straight line $g+n\bar{h}(1+\bar{\phi})$ defines $g=\tilde{g}_M$. Under A. 1' we observe for

$$\bar{l}=\left\{f_l(\bar{h},1,g)/\alpha+g\right\}/\left\{f(\bar{h},1,g)+f_l(\bar{h},1,g)/\alpha-n\bar{h}(1+\phi)\right\} \tag{10'}$$

which is derived from the above equation,

A. 1' $\Leftrightarrow\bar{l}<1\Leftrightarrow\bar{c}>0$.

Next we assume

A.3' $f_g(\bar{k},\bar{l},g)<1$.

That is, at the stationary state, $1-f_g$, the net resource withdrawal effects, which are direct resource costs of a unit of infrastructure government expenditure relative its direct benefit of increasing production is positive[4]. In the different context, as explained fully later in A. 5', $1>f_g$ seems plausible assumption since this allows the existence of positive *c* and *z* if *g* is decreases from its maximum level $\tilde{g}_M$.

Under A. 3', the increase in government expenditure increases capital-labor ratio *k* at the stationary state, i.e., $\partial\bar{k}/\partial g>0$ as shown below.

From (9)' and (1)", we obtain that *l* and *k* are expressed as functions of *g* at the stationary state;

$$\begin{bmatrix} f_{lk} & f_{kk} \\ f_l(1+1/\alpha)-(1-l)f_{ll}/\alpha & f_k-(1-l)f_{lk}/\alpha-n(1+\bar{\phi}) \end{bmatrix}\begin{bmatrix} d\bar{l} \\ d\bar{k} \end{bmatrix}=$$

$$\begin{bmatrix} -f_{lg}dg \\ (1-f_g+(1-l)f_{lg}/\alpha)dg \end{bmatrix}.$$

This implies $\bar{k}=\bar{k}(\bar{z},g)$ and $\bar{l}=\bar{l}(\bar{z},\bar{g})$ with

$$\partial\bar{k}/\partial g=\left\{f_{lk}(1-f_g+(1-l)f_{lg}/\alpha)+f_{lg}(f_l(1+1/\alpha)-(1-l)f_{ll}/\alpha\right\}/\tilde{D}>0 \qquad (24)$$

and

$$\partial\bar{k}/\partial g=-\left\{f_{lk}(f_k-n(1+\bar{\phi}))+f_{lg}(1-f_g)\right\}/\tilde{D}<0 \qquad (25)$$

where

$$\tilde{D}=f_{lk}(f_k-n(1+\phi))-f_{kk}f_l(1+1/\alpha)>0$$

from the homogeneity of degree one of *f* in (*k*, *l*) and $f_k-n(1+\phi)>0$ at the stationary state (To obtain this, we can employ the same arguments as the case of government service affecting felicity function. So, see Appendix I, for this inequality.)

In view of Fig. 4, A. 3' is seen to be equal to $g>\tilde{g}_m$ where $\tilde{g}_m$ is the minimal level of government expenditure such that $1=f_g(\bar{k},\bar{l},g)$ holds. Then from A.3', it is seen that $g\in(\tilde{g}_m,\tilde{g}_M)$. Next we show the global stability of the economy.

Global Stability

Basically we follow the former arguments. From (1)' and (8), *l* and *c* are expressed as

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4 We owe this interpretation to Turnovsky and Fisher (1995).

functions of k, z and g;

$$\begin{bmatrix} f_l & -1 \\ 1/(1-l) - f_{ll}/f_l & 1/c \end{bmatrix} \begin{bmatrix} dl \\ dc \end{bmatrix} = \begin{bmatrix} -(f_k - z(1+\phi)dk + k(1+\phi+\phi' z)dz + (1-f_g)dg \\ (f_{lk}/f_l)dk + (f_{lg}/f_l)dg \end{bmatrix}.$$

Hence

$$l_k = \{-(f_k - z(1+\phi))/c + f_{lk}/f_l\}/D, \qquad (11)'$$

$$l_z = k(1+\phi+\phi' z)/Dc > 0, \qquad (12)'$$

and

$$l_g = \{(1-f_g)/c + f_{lg}/f_l\}/D \qquad (13)'$$

where

$$D = (1+\alpha)/(1-l) - f_{ll}/f_l > 0 \qquad (14)'$$

and hence

$$l = l(k, \underset{+}{z}, g).$$

$$c_k = \{f_{lk} + (1/(1-l) - f_{ll}/f_l)(f_k - z(1+\phi))\}/D, \qquad (15)'$$

$$c_z = -\{1/(1-l) - f_{ll}/f_l\}k(1+\phi+\phi' z)/D < 0, \qquad (16)'$$

and

$$c_g = \{-(1-f_g)(1/(1-l) - f_{ll}/f_l) + f_{lg}\}/D. \qquad (17)'$$

and hence $c = c(k, \underset{-}{z}, g)$. Again, in view of (8) c is expressed also as

$$c = c(\underset{+}{k}, \underset{-}{l}, g) = c(k, l(k,z,g), g).$$

We retain A. 2;

A. 2 $\sigma > \theta/(1+\alpha^{-1})$.

Then from (3) though (8), and A. 1' and A. 2, we obtain the same phase diagram of (k,z), showing the existence of the stable saddle point path $z = z(\underset{-}{k}, g)$, by following exactly the same arguments as in Appendix I. Here the boundedness of k follows from $\dot{k} = f(k,l,g) - c - g - i\cdot\phi - nk \leq f(k,1,\tilde{g}_M) - nk$ and $f(k,1,\tilde{g}_M) - nk < 0$ as k becomes large in view of Inada condition of f. Then we obtain the optimal path of (c,k) with $c = c(\underset{+}{k}, g)$ as a saddle point path again employing the same arguments as the one in Appendix II. In short,

Theorem 4

Under A. 1' and A. 2, the optimal path of (c,k,z) is globally stable saddle point path with $z = z(\underset{-}{k}, g)$ and $c = c(\underset{+}{k}, g)$.

Effects of Government Expenditure

First we recall from (9)', $g \uparrow \leftrightarrow \bar{h} \uparrow$ from $f_k(\bar{h}, 1, g)$ being constant,

??? $\leftrightarrow \bar{l} \downarrow$ and $\bar{k} \uparrow$ from (23) and (24) $\leftrightarrow \bar{c} \uparrow$?????

from (8). (in view of $f_l(\bar{h}, 1, g)$, increases both in $\bar{h}$ and g increases f_l but decreases $\bar{l}$, and hence $\bar{c}$ increases from (8).) ????????????

Next we can show $z = z(k, \underset{+}{g})$ to hold locally around E just as the case of government expenditure in the felicity function.

Now we investigate the global effects of the change in government expenditure on consumption c and investment-capital ratio z. As before we obtain from (4), (6), (7), (15)' and (16)'

$$\tilde{A}(k,z,g)\dot{z} = \tilde{B}(k,z,g) + \tilde{E}(k,z,g)\dot{k} \tag{18'}$$

where

$$\tilde{A}(k,z,g) = A(k,z,l(k,z,g),g) = (2\phi' + \phi'' z)/(1+\phi+\phi' z) + \{1/(1-l) - f_{ll}(k,l,g)/f_l(k,l,g)\}k(1+\phi+\phi' z)/c(k,l,g)\cdot\tilde{D}(k,z,g) \tag{19'}$$

$$\tilde{B}(k,z,g) = B(k,z,l(k,z,g),g) = \dot{\lambda}/\lambda = \rho - (f_k(k,l(k,z,g),g) + \phi' z^2)/(1+\phi+\phi' z) \tag{20'}$$

$$\tilde{E}(k,z,g) = E(k,z,l(k,z,g)) = \{f_{lk}(k,l,g) + (1/(1-l) - f_{ll}(k,l,g)/f_l(k,l,g))(f_k(k,l,g) - z(1+\phi)\}/c(k,l,g)\cdot\tilde{D}(k,z,g) \tag{21'}$$

and

$$\tilde{D}(k,z,g) = D(k,z,l(k,z,g)) = (1+\alpha)/(1-l) - f_{ll}(k,l,g)/f_l(k,l,g).$$

(3) is expressed as

$$\dot{k} = kz - nk. \tag{3}$$

Then employing the system of two ordinary differential equations (3) and

$$\dot{z} = \{\tilde{B}(k,z,g) + \tilde{E}(k,z,g)(zk - nk)\}/\tilde{A}(k,z,g) \equiv \tilde{H}(k,z,g) \tag{22'}$$

we can investigate the effects of change in g. Here we recall if g is contained in $(\tilde{g}_m, \tilde{g}_M)$ with $0 < \tilde{g}_m < \tilde{g}_M < +\infty$ then both z_0 and c_0 are positive.

Then employing the same arguments as the one of government expenditure in felicity function, we can again see that $z = z(k,g)$ and $l = \tilde{l}(k,g) = l(k,z(k,g),g)$ and

$c=\tilde{c}(k,g)=c(k,\tilde{l}(k,g),g)$ are continuously differentiable in (k,g).

Next we assume

A. 4' Given (k,z) $\partial\tilde{H}(k,z,g)/\partial g \neq 0$

A. 5' $f_{lg} < (1-f_g)(1/(1-l)-f_{lg}/f_l)$ $g \in (\tilde{g}_m, \tilde{g}_M)$, and ????

A 5' is equivalent with $c_g < 0$ from (13)7. That is, given *k* and *z*, an increase in lowers *c*.

Recalling $\partial\bar{k}/\partial g > 0$, we obtain

Theorem 5

(1) Under A.1' through A.4', $z = z(k, \underset{+}{g})$ hold globally.

(2) Under A.1' through A.5', $c = c(k, \underset{-}{g})$ and $l = l(k, \underset{+}{g})$ hold globally.

??? If the production function is specified to be of Cobb-Douglas, i.e., $f(k,l,g) = k^{\mu}l^{1-\mu}g^{\tau}$ where $0 < \mu, \tau < 1$.

Then A.5' is reduced to

$$f_g < l + \mu(1-l) \quad (<1) \quad (<1) \quad ???$$

i.e., the marginal product of government expenditure f_g is less than $l+\mu(1-l)$. On necessary condition for A.5' to hold is $f_g < 1$. Recalling the market equilibrium condition (1)', in case of $c = 0$, it is reduced to $f(k,l,g) = i(1+\phi c^i/k)) + g$. Given *k*, *l* and *i*, this equation gives the maximum level of *g*. If $f_g < 1$ holds, then by reducing the level of *g* from this maximum level, it is possible to obtain positive *c* and *z*, a natural plausible condition.

Proof

(See Appendix III.)

IV. Concluding Remarks

This paper analyzes the role of government expenditure in the dynamic optimization model with investment adjustment costs for both cases of government consumption service and public input. The felicity function is of general type so that consumption, leisure and government expenditure are included as an arguments in the function in the former case of government consumption service, but in the latter case of public import, only consumption and leisure are included into felicity function, although it is at the same time, specified to be additive and logarithmic in each argument for both cases. We can immediately generalize our model into the case where the government expenditure affects both consumption and production.

Also we note that these does not exist an optimal level of government expenditure g which solves social planners' maximization problem in either case of $g - g$ in the felicity function or g in the production function. In fact if such g ever exists, it causes time ???(dynamic) inconsistency.

Once thing?? we did not analyze here is the effects of anticipated change in g in the middle of transitional period. Although our results can be readily applied to the unanticipated change in g – social planners solves maximization problem twice (once in the beginning and next at the time of change in g), in general the solution path (c, k, z) of the anticipated change are expected to shift at the ime of change, we leave this to as our next task.

Our stress is on the global nature of the optimal path of investment capital ratio z and consumption c as functions of capital stock and government expenditure. Especially the global effects of government expenditure on these variables are derived from rather weak assumption A. 3 or A. 4 – i.e., the sgn of the partial derivative of RHS of the differential equation of $\dot{z}$ with respect to g does not change.

Although it is common to derive the negative eigen values of the coefficient matrix of the system of ordinary equations by linearlizing around the stationary state, and then to predict the effects of government expenditure g by partially differentiating these eigen values with respect to g, this conventional method is confined only to the local analysis even if we can manage to derive the signs explicitly. (In our model the partial differentiation of eigen values seems extremely tedious and any definite sign seem unobtainable.) Hopefully our predictive method of the effects of government expenditure can be readily applicable in many similar models.

Appendix I

From (6) and (7), we obtain

$$\dot{\lambda}/\lambda = \rho - (f_k + \phi' \cdot z^2)/(1+\phi+\phi' \cdot z) . \quad \text{(A-1)}$$

From (6), it follows that

$$\dot{\lambda}/\lambda = \dot{\xi}/\xi + (2\phi' + \phi'' \cdot z)(1+\phi+\phi' z)^{-1}\dot{z} . \quad \text{(A-2)}$$

From (4), (14), and (15), we observe

$$\dot{\xi}/\xi = -\dot{c}/c = -(c_k \dot{k} + c_z \dot{z})/c . \quad \text{(A-3)}$$

Hence from (A-2) and (A-3)

$$\left[(2\phi' + \phi'' z)/(1+\phi+\phi' z) + \{1/(1-l) - f_{ll}/f_l\}k(1+\phi+\phi' z)/cD\right]\dot{z}$$
$$= \dot{\lambda}/\lambda + \{f_{lk} + (1/(1-l) - f_{ll}/f_l)(f_k - z(1+\phi))\}(cD)^{-1}\dot{k} . \quad \text{(A-4)}$$

Let

$$A(k,z,l) = (2\phi' + \phi'' z)/(1+\phi+\phi' z) + \{1/(1-l) - f_{ll}/f_l\}k(1+\phi+\phi' z)/cD > 0 \quad \text{(A-5)}$$

$$B(k,z,l) = \dot{\lambda}/\lambda = \rho - (f_k + \phi' z^2)/(1+\phi+\phi' z) , \quad \text{(A-6)}$$

and

$$E(k,z,l) = \{f_{lk} + (1/(1-l) - f_{ll}/f_l)(f_k - z(1+\phi))\}/cD . \quad \text{(A-7)}$$

Then (A-4) or

$$A(k,z,l)\dot{z} = B(k,z,l) + E(k,z,l)\dot{k} \quad \text{(A-8)}$$

and

$$\dot{k} = zk - nk \quad (3)$$

constitute the system of ordinary differential equations in (*k*, *z*) in view of $l = l(k,z,g)$.

Here we investigate the slope of $f_k - z(1+\phi) = 0$ curve.

The slope of $f_k - z(1+\phi) = 0$ curve

By total differentiation of $f_k - z(1+\phi) = 0$, we obtain

$$(f_{kk} + f_{kl} l_k)dk = (-f_{kl} l_z + 1 + \phi + \phi' z)dz .$$

Here from (10) along $f_k - z(1+\phi) = 0$,

$$\operatorname{sgn}(f_{kk} + f_{kl} l_k) = \operatorname{sgn}(f_{kk} f_l D + f_{kl}^2)$$
$$= \operatorname{sgn}(f_{kk} f_l (f_l/c + 1/(1-l) - f_{ll}/f_l) + f_{kl}^2) \quad \text{(from (13))} \quad \text{(A-9)}$$
$$= \operatorname{sgn}(f_{kk} f_l (f_l/c + 1/(1-l)) < 0$$

from $f_{kk} f_{ll} - f_{kl}^2 = 0$.

Next, from (11),

$$\operatorname{sgn}(-f_{kl} l_z + 1 + \phi + \phi' z) = \operatorname{sgn}(-f_{kl} k + Dc)$$

$$= \operatorname{sgn}(-f_{kl} k + f_l + c/(1-l) - c f_{ll}/f_l) \quad \text{(from (13))}$$

$= \text{sgn}(-f_{kl}k + f_l + c/(1-l) + cf_{kl}k/lf_l)$ (from $(f_{kl}k + f_{ll}l = 0)$

$= \text{sgn}\left[(f_{kl}k/f_l l)(c - f_l l) + f_l + f_l/\alpha\right]$ (from (8))

$= \text{sgn}\left[-(f_{kl}k/f_l l)g + (1+\alpha^{-1})f_l\right]$

(along $f_k - z(1+\phi) = 0$, $-f_l l + c + g = 0$ follows from (1))

$= \text{sgn}\left[-\sigma^{-1}(f_k k/f \cdot l)g + (1+\alpha^{-1})f_l\right]$

(from the definition of $\sigma = f_k f_l / f_{kl} f$)

$= \text{sgn}\left[-\sigma^{-1}\theta g/l + (1+\alpha^{-1})\alpha g/\{(1+\alpha)l - 1\}\right]$ (from $\theta = f_k k/f$,

and $f_l = \alpha g/((1+\alpha)l - 1)$ derived from $f_l l = c + g$ and

$c = (1-l)f_l/\alpha$ (8))

$= \text{sgn}\left[-\alpha^{-1}\theta + (1+\alpha^{-1})\alpha l/\{(1+\alpha)l - 1\}\right]$.

Here observing for $\varphi(l) = l/\{(1+\alpha)l - 1\} > 0$ ($\varphi(l) > 0$ follows from $f_l = \alpha g/((1+\alpha)l - 1) > 0$) and $\varphi'(l) = -1/\{(1+\alpha)l - 1\}^2 < 0$ holds and hence $\alpha l/\{(1+\alpha)l - 1\} > \alpha/\alpha = 1$. We can conclude $\text{sgn}\left[-\sigma^{-1}\theta + (1+\alpha^{-1})\alpha l/\{(1+\alpha)l - 1\}\right] > 0$ if $\sigma^{-1}\theta + (1+\alpha^{-1})\alpha l/\{(1+\alpha)l - 1\} > -\sigma^{-1}\theta + 1 + \alpha^{-1} > 0$ or $\sigma > \theta/(1+\alpha^{-1})$, i.e., A.2 holds. Hence the slope of the curve $f_k - z(1+\phi) = 0$, $dz/dk\big|_{f_k - z(1+\phi) = 0} < 0$ under A. 2.

F. Intersection of $\dot{\lambda} = 0$ curve and $f_k - z(1+\phi) = 0$ curve

Fig. A. 1

Next we investigate the intersection of $\dot{\lambda} = 0$ curve and $f_k - z(1+\phi) = 0$ curve, from (A-6).

$$\dot{\lambda} = 0 \Leftrightarrow \rho(1+\phi+\phi' z) = f_k + \phi' z^2 .$$

Hence at their intersection

$$\rho(1+\phi+\phi' z) = f_k + \phi' z^2 = z(1+\phi) + \phi' z^2 \tag{A-10}$$

or

$$(\rho - z)(1+\phi+\phi' z) = 0$$

must follow. This shows that two curves intersect at $\rho = z$. Denoting F to be the point of intersection, we observe $\dot{\lambda} = 0$ curve is positively sloped at F. In fact, by totally differentiating $\rho(1+\phi+\phi' z) = f_k + \phi' z^2$, we obtain

$$\{\rho(2\phi' + \phi'' z) - f_{kl} l_z - (\phi'' z^2 + 2\phi' z)\} dz = (f_{kk} + f_{kl} l_k) dk .$$

Here $f_{kk} + f_{kl} l_k < 0$ from (A-9). At *F*, we observe

$$\rho(2\phi'+\phi'' z) - f_{kl} l_z - (\phi'' z^2 + 2\phi' z)$$
$$= \rho(2\phi'+\phi'' z) - f_{kl} k(1+\phi+\phi' z)/Dc - (\phi'' z^2 + 2\phi' z)$$
$$= - f_{kl} k(1+\phi+\phi' z)/Dc < 0 \quad from \quad \rho = z.$$

Hence the slope of $\dot{\lambda} = 0$ curve is positive at *F*, showing that the two curves intersect only at *F*. Although the slope of $\dot{\lambda} = 0$ in general is indefinite, recalling that this curve intersects with $z = n(\dot{k} = 0)$ horizontal line at the stationary point *E* with the slope never becoming flat because $dz/dk = (f_{kk} + f_{kl} l_k)/\{(\rho - z)(2\phi' + \phi'' z) - f_{kl} k(1+ \phi + \phi' z)\} \neq 0$, never intersect with $f_k - z(1+\phi) = 0$ curve other than at point *F*, and hence the curve $\dot{\lambda} = 0$ is drawn as shown in Fig. A. 1, we obtain that the positive orthant of (*k*, *z*) plane is divided into seven regions (region I through region VII as drawn in Fig. A. 1) by three lines $\dot{k} = 0$, $\dot{\lambda} = 0$ and $f_k - z(1+\phi) = 0$.

Derivation of $\dot{z} = 0$ curve

Table A.1

	I	II	III	IV	V	VI	VII
$\dot{\lambda}$	+	–	–	+	–	+	+
$\dot{k}$	+	+	+	+	–	–	–
$f_k - z(1+\phi)$	–	–	+	+	+	+	–
$\dot{z}$	?	–	?	+	–	?	+

Now we derive $\dot{z} = 0$ curve. Table A. 1 shows how sgns$\dot{\lambda}$, $\dot{k}$, $f_k - z(1+\phi)$ and hence $\dot{z}$ are determined in seven regions, $\dot{\lambda}$ is positive (negative) on the right (left) hand side of this curve, so $\dot{\lambda}$ is positive in I, IV, VI and VII, and negative II, III and V. $\dot{k}$ is positive (negative) above (below) the horizontal curve $(z = n)$, and hence $\dot{k}$ is positive in I, II, III and IV, and negative in V, VI and VII. Lastly $f_k - z(1+\phi)$ is positive (negative) on the left (right) and below (above) this curve. Hence $f_k - z(1+\phi)$ is positive in III, IV, V and VI, and negative in I, II and VII. In equation (A-8), recalling $A(k,z,l) > 0$ always and $E(k,z,l) > 0$ if and only if $f_k - z(1+\phi) > 0$, we obtain $\operatorname{sgn} \dot{z}$ in each region from (A-8) as shown in Table A. 1. Since $\dot{z} > 0$ in IV and VII, and $\dot{z} < 0$ in II and V, and $\dot{z} = 0$ curve must pass through the stationary point *E*, $\dot{z} = 0$ curve is seen to path through regions III and VI, with $\dot{z} > 0$ ($\dot{z} < 0$) above (below) $\dot{z} = 0$ curve as shown Fig. A. 1.

Appendix II

First we derive l and z as functions of c and k from (1) and (8). (8) implies

$$l = l(\underset{+}{k}, \underset{-}{c})$$

with

$$l_k = (1-l) f_{lk} / (f_l - (1-l) f_{ll}) > 0 \tag{A-11}$$

and

$$l_c = \alpha / ((1-l) f_{ll} - f_l) < 0 . \tag{A-12}$$

Then from (1), (A-11) and (A-12),

$$z = z(k, \underset{-}{c}, \underset{-}{g})$$

with

$$z_k = (f_k + f_l l_k - z(1+\phi)) / k(1+\phi+\phi' z) , \tag{A-13}$$

$$z_c = (f_l l_c - 1) / k(1+\phi+\phi' z) < 0 , \tag{A-14}$$

and

$$z_g = -1 / k(1+\phi+\phi' z) < 0 . \tag{A-15}$$

Employing (A-2) and (A-3), we obtain

$$\begin{aligned} \dot{c}/c = -\dot{\xi}/\xi &= -\dot{\lambda}/\lambda + (2\phi' + \phi'' z)(1+\phi+\phi' z)^{-1} \dot{z} \\ &= -\dot{\lambda}/\lambda + (2\phi' + \phi'' z)(1+\phi+\phi' z)^{-1} (z_k \dot{k} + z_c \dot{c}) , \end{aligned}$$

and hence from (A-13) and (A-14)

$$\begin{aligned} \left\{ 1/c - (2\phi' + \phi'' z)(1+\phi+\phi' z)^{-1} z_c \right\} \dot{c} = -\dot{\lambda}/\lambda + (2\phi' + \phi'' z)(f_k + f_l l_k \\ - z(1+\phi)) k^{-1} (1+\phi+\phi' z)^{-2} \dot{k} . \end{aligned} \tag{A-16}$$

Here

$$\dot{\lambda}/\lambda = B(k, z(k, c), l(k, c)) = \rho - (f_k + \phi' z^2)/(1+\phi+\phi' z) . \tag{A-6}$$

Then (A-16) and

$$\dot{k} = (z(k, c) - n) k \tag{3}$$

constitute the system of ordinary differential equations in (k, c) – in view of $z = z(k, c)$.

The slope of $f_k - z(1+\phi) = 0$ curve in (*k*, *c*) plane

By totally differentiating $f_k - z(1+\phi) = 0$**,** we obtain

$$\left\{ f_{kk} + f_{kl} l_k - (1+\phi+\phi' z) z_k \right\} dk = \left\{ - f_{kl} l_c + (1+\phi+\phi' z) z_c \right\} dc .$$

Here from (A-11) and (A-13) we obtain

$$\begin{aligned} &\operatorname{sgn}\left\{ f_{kk} + f_{kl} l_k - (1+\phi+\phi' z) z_k \right\} \\ &= \operatorname{sgn}\left\{ f_{kk} + f_{kl} (1-l) f_{kl} / (f_l - (1-l) f_{ll}) - (f_k + f_l l_k - z(1+\phi))/k \right\} \\ &= \operatorname{sgn}\left\{ f_{kk} + (1-l) f_{kl}{}^2 / (f_l - (1-l) f_{ll}) - (f_l (1-l) f_{lk} / k (f_l - (1-l) f_{ll}) \right\} \end{aligned}$$

$$\text{(from } f_k - z(1+\phi) = 0\text{)} \tag{A-17}$$

$$= \text{sgn}\{kf_{kk}(f_l - (1-l)f_{ll}) + k(1-l)f_{kl}^{\,2} - f_l(1-l)f_{lk})\}$$

$$= \text{sgn}(kf_{kk} - (1-l)f_{lk}) \quad \text{(from } f_{kk}f_{ll} - f_{kl}^{\,2} = 0\text{)}$$

$$= \text{sgn}(-f_{lk}) \quad \text{(from } kf_{kk} + lf_{lk} = 0\text{)} < 0 \quad .$$

Next, we observe from (A-12) and (A-14)

$$\text{sgn}\{-f_{kl}l_c + (1+\phi+\phi' z)z_c\}$$

$$= \text{sgn}\{(-f_{kl}\alpha/((1-l)f_{ll} - f_l) + (f_l l_c - 1)/k\}$$

$$= \text{sgn}[-f_{kl}\alpha/((1-l)f_{ll} - f_l) + \{f_l\alpha/((1-l)f_{ll} - f_l) - 1)\}/k]$$

$$= \text{sgn}\{kf_{kl}\alpha - f_l\alpha - f_l + (1-l)f_{ll}\} \quad \text{(from } (1-l)f_{ll} - f_l < 0\text{)}$$

$$= \text{sgn}\{(\alpha k - (1-l)k/l)f_{kl} - (\alpha+1)f_l\} \quad \text{(from } f_{ll} = -f_{lk}k/l\text{)} \tag{A-18}$$

$$= \text{sgn}\{(\alpha k - (1-l)k/l)\sigma^{-1}f_k f_l/f - (\alpha+1)f_l\}$$

$$\text{(from } \sigma = f_k f_l / f_{kl} f\text{)}$$

$$= \text{sgn}\{(\alpha - (1-l)/l)\sigma^{-1}\theta - (\alpha+1)\} < 0 \quad \text{(from } \theta = f_k k/f\text{)}$$

if $(\alpha - (1-l)/l)\sigma^{-1}\theta < \alpha + 1$. Noting $(\alpha - (1-l)/l)\sigma^{-1}\theta < \alpha\sigma^{-1}\theta$, we obtain (A-18) holds if $\alpha\sigma^{-1}\theta < \alpha + 1$, or

A. 2 $\sigma > \theta/(1+\alpha^{-1})$. Hence the slope of $f_k - z(1+\phi) = 0$ in (*k*, *c*) plane

$dc/dk\big|_{f_k - z(1+\phi)=0} > 0$ under A. 2.

Fig. A. 2

Derivation of $\dot{k} = 0$ (i.e., $z = n$) curve

Now we investigate $\dot{k} = 0$ (i.e., $z = n$) curve in (*k*, *c*) plane. First we consider its intersection with the horizontal axis (i.e., $c = 0$). Then (1) is reduced to $f(k,l) = nk(1+\bar{\phi}) + g$.

Furthermore (8) implies $l = 1$. In fact, although $c = 0$ also holds if $f_l = 0$, i.e., $k/l \to 0$ or $k \to 0$, $k \to 0$ would imply $f(k,l) \to 0$, contradicting $z = n > 0$.

Then

$$f(k,1) = nk(1+\bar{\phi}) + g$$

Hence $c = 0 \Leftrightarrow l = 1$, and $c > 0 \Leftrightarrow 0 < l < 1$. (A-19)

must follow. Here from A. 3, there exist two *k*'s, k_m and k_M such that (A-19) hold.

Recalling $\dot{k} = 0 \Leftrightarrow z = z(k,c) = n$,

$$z_k = (f_k + f_l l_k - z(1+\phi))/k(1+\phi+\phi' z) \tag{A-13}$$

and

$$z_c < 0, \tag{A-14}$$

we observe $dc/dk|_{\dot{k}=0} > 0$ if $f_k - z(1+\phi) > 0$ (i.e., on the left and upper side of $f_k - z(1+\phi) = 0$ curve from (A-17)) holds. Then $\dot{k} = 0$ curve is derived as shown in Fig. A. 2. Here $\dot{k} = 0$ curve intersects with $f_k - z(1+\phi) = 0$ when $dc/dk|_{\dot{k}=0} > 0$ holds.

The Derivation of $\dot{\lambda}/\lambda = 0$ curve

The slope of $\dot{\lambda}/\lambda = 0$ curve is obtained by totally differentiating $\rho(1+\phi+\phi' z) - (f_k + \phi' z^2) = 0$ obtained from (A-6);

$$\{- f_{kk} - f_{kl} l_k + (\rho - z)(2\phi' + \phi'' z) z_k\} dk + \{- f_{kl} l_c + (\rho - z)(2\phi' + \phi'' z) z_c\} dc = 0 \tag{A-20}$$

We observe $\rho(1+\phi+\phi' z) - (f_k + \phi' z^2) = 0$ and $f_k - z(1+\phi) = 0$ holds if and only if $\rho = z$, i.e., $\dot{\lambda}/\lambda = 0$ curve and $f_k - z(1+\phi) = 0$ curve intersect at the point where $\rho = z$ holds. As shown in Fig. A. 2, $\rho = z$ curve is a dotted line below $z = n$ curve. And these three curves intersect at point *F*. Next we observe that although the slope of $\dot{\lambda}/\lambda$ is not definite, it is negative at *F*. In fact (A-20) is reduced to

$$-(f_{kk} + f_{kl} l_k) dk - f_{kl} l_c dc = 0$$

at *F*. Here $l_c < 0$ from (A-12), and from (A-10)

$$\mathrm{sgn}(f_{kk} + f_{kl} l_k) = \mathrm{sgn}\{f_{kk} + f_{kl}(1-l) f_{kl} / (f_l - (1-l) f_{ll})\}$$

$$= \mathrm{sgn}\{f_{kk}(f_l - (1-l) f_{ll}) + f_{kl}(1-l) f_{kl}\} \quad (\text{from} \quad f_{kk} f_{ll} - f_{kl}{}^2 = 0)$$

$$= \mathrm{sgn}(f_{kk} f_l) < 0.$$

Derivation of $\dot{c} = 0$ curve

E is the stationary point at which both $\dot{k} = 0$ and $\dot{\lambda}/\lambda = 0$ curves, (and hence $\dot{c} = 0$ curve from (A-16)) intersect. k_λ is determined by $\rho = (f_{kk}(k,1) + \phi' z^2)/(1+\phi+\phi' z)$ (i.e., $\dot{\lambda}/\lambda = 0$) and $f(k,1) = g + z(1+\phi)k$, and k_N is by $f_k(k,1) - (1+\phi)z = 0$ and $f(k,1) = g + z(1+\phi)k$. Then the positive orthant of (*k*, *c*) plane is divided into seven regions I through VII, by the three curves $\dot{k}=0$, $\dot{\lambda}/\lambda=0$ and $f_k - z(1+\phi)=0$. As in Appendix I, we now investigate the sgns of these curves in respective regions as shown in Table A. 2.

Table A. 2

	I	II	III	IV	V	VI	VII
$\dot{\lambda}$	+	+	–	+	–	+	–
$\dot{k}$	–	–	–	+	+	+	+
$f_k - z(1+\phi)$	–	+	+	+	+	–	–
$\dot{c}$	?	–	?	?	+	?	?

Now in (A-16), observing that the coefficient of $\dot{c}$ is positive from (A-14), and the coefficient of $\dot{k}$ is also positive if $f_k - z(1+\phi)>0$ from (A-11) and (A-16), we can see that $\operatorname{sgn}\dot{c}<0$ in II and $\operatorname{sgn}\dot{c}>0$ in V, while $\operatorname{sgn}\dot{c}$ is indeterminate in other regions. Finally recalling that $\dot{c}=0$ curve must path through *E*, we can derive $\dot{c}=0$ as shown in Fig. A. 2.

Figures

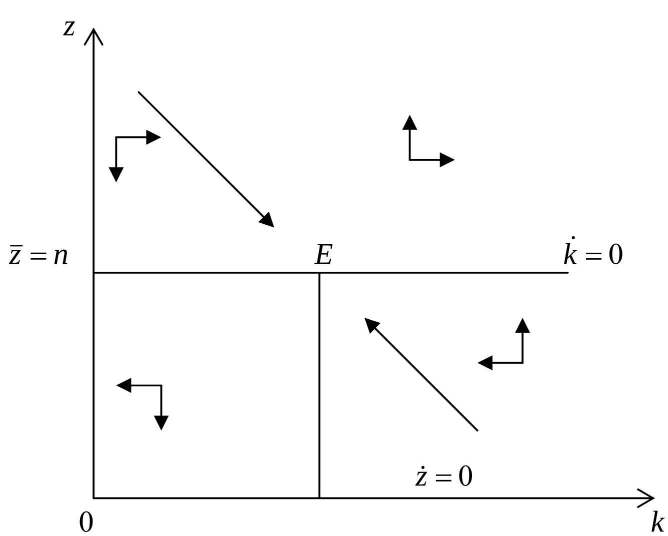

Fig. 1 Phase diagram of (*k*, *z*) and saddle point path of *z*

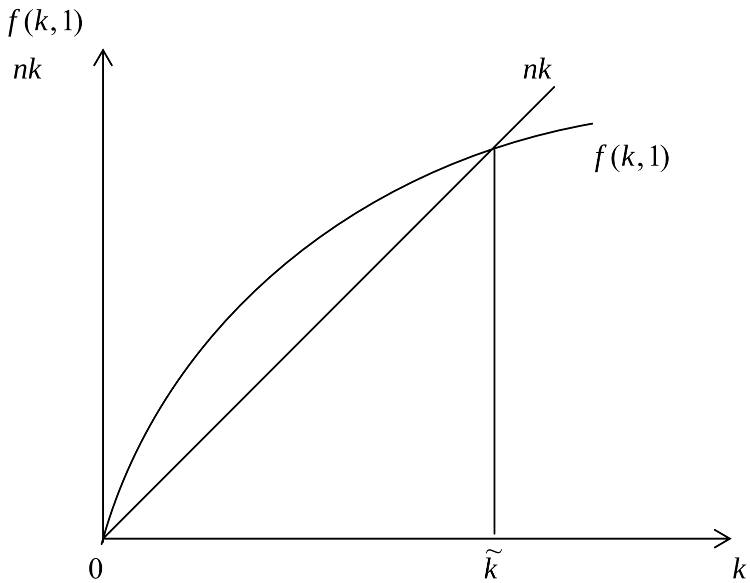

Fig. 2

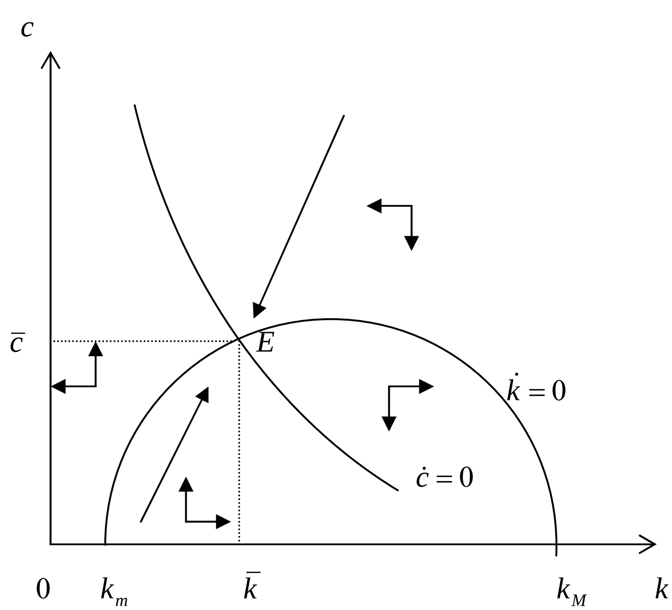

Fig. 3 Phase diagram of (k, c) and saddle point path of c

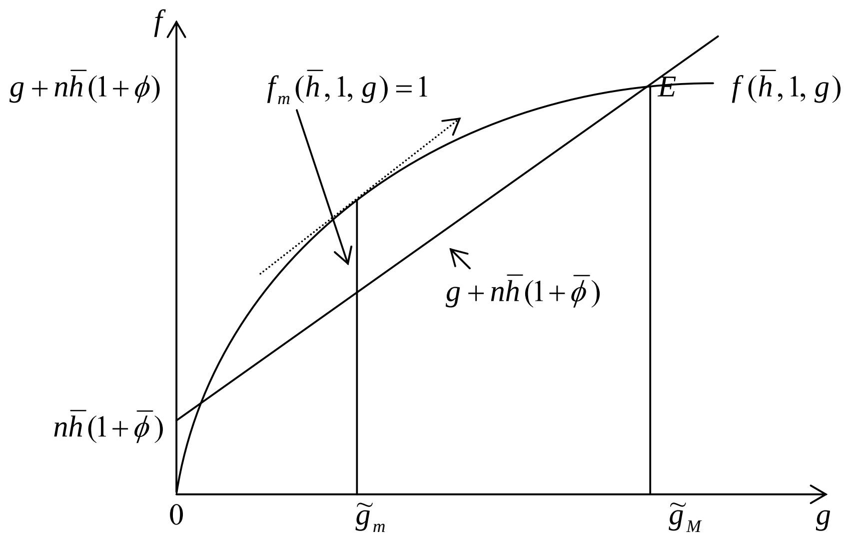

Fig. 4

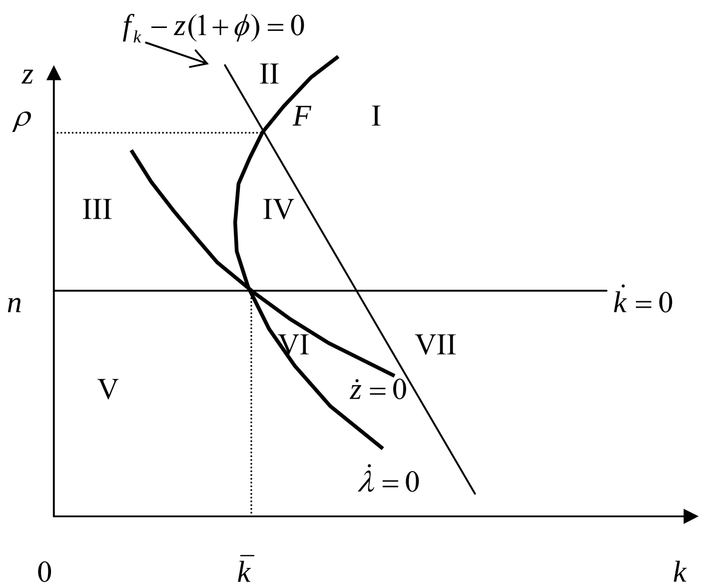

Fig. A. 1

$$f_k - (1+\phi)z = 0$$

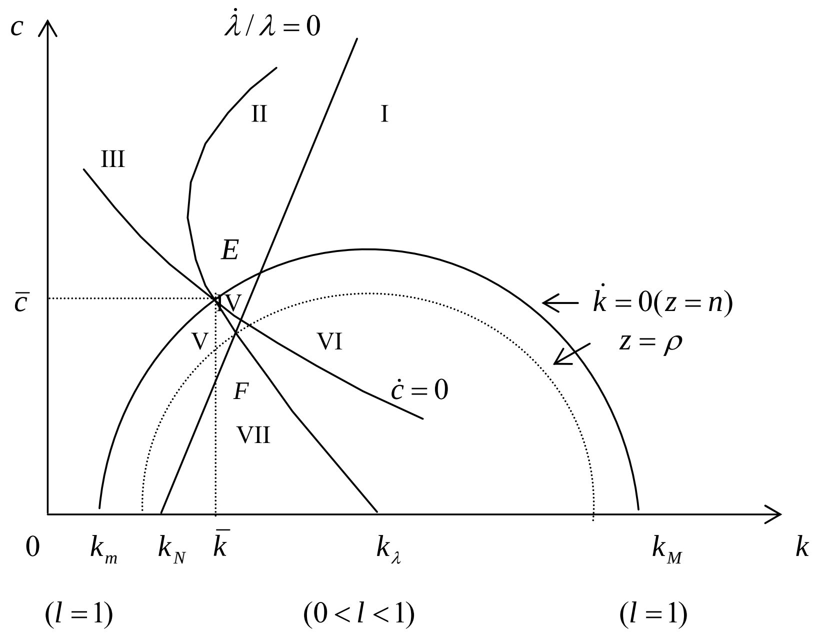

Fig. A. 2

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Appendix I

Strict Concavity of the value function *W(k, k*)* ,

Proof[8] Strict concavity of c_w, c and c^* in (k, k^*).

Let

$$W(k_0, k_0^*, g, g^*) = \max_{\dot{k},\dot{k}^*,c,c^*,i,i^*} \int [u(c,g) + \beta u(c^*, g^*)] e^{-(\rho-n)t} dt \qquad \text{(A-1)}$$

and

$$W(k_0', k_0^{*\prime}, g, g^*) = \max_{\dot{k}',\dot{k}^{*\prime},c',c^{*\prime},i',i^{*\prime}} \int [u(c',g) + \beta u(c^{*\prime}, g^*)] e^{-(\rho-n)t} dt \qquad \text{(A-1)*}$$

and

$$\{(\dot{k}, \dot{k}^*, i, i^* c, c^*)\}_{t=0}^{\infty} = \arg\max \int [u(c,g) + \beta u(c^*, g^*)] e^{-(\rho-n)t} dt \qquad \text{(A-2)}$$

$$\{(\dot{k}', \dot{k}^{*\prime}, i', i^{*\prime} c', c^{*\prime})\}_{t=0}^{\infty} = \arg\max \int [u(c',g) + \beta u(c^{*\prime}, g^*)] e^{-(\rho-n)t} dt \qquad \text{(A-2)*}$$

Further let

$$J(k, k^*, \dot{k}, \dot{k}^*) = \max_{c,c^*} [u(c,g) + \beta u(c^*, g^*)] = u(c,g) + \beta u(c^*, g^*) \qquad \text{(A-3)}$$

and

$$J(k', k^{*\prime}, \dot{k}', \dot{k}^{*\prime}) = \max_{c',c^{*\prime}} [u(c',g) + \beta u(c^{*\prime}, g^*)] = u(c',g) + \beta u(c^{*\prime}, g^*) . \qquad \text{(A-3)*}$$

Here

$$f(k) + f(k^*) = c + c^* + i + i^* + g + g^* \qquad \text{(A-4)}$$

and

$$f(k') + f(k^{*\prime}) = c' + c^{*\prime} + i' + i^{*\prime} + g + g^* \qquad \text{(A-4)*}$$

hold by definition. We want to show

$$J(k_\lambda, k_\lambda^*, \dot{k}_\lambda, \dot{k}_\lambda^*) > \lambda J(k, k^*, \dot{k}, \dot{k}^*) + (1-\lambda) J(k', k^{*\prime}, \dot{k}', \dot{k}^{*\prime}) \qquad \text{(A-5)}$$

for any λ with $0 < \lambda < 1$ where $(k, k^*) \neq (k', k^{*\prime})$, $k_\lambda = \lambda k + (1-\lambda) k'$, $k_\lambda^* = \lambda k^* + (1-\lambda) k^{*\prime}$, $\dot{k}_\lambda = \lambda \dot{k} + (1-\lambda) \dot{k}'$ and $\dot{k}_\lambda^* = \lambda \dot{k}^* + (1-\lambda) \dot{k}^{*\prime}$, and thereby the strict concavity of c and c^* as functions of k and k^*. From (A-4) and (A-4)*, we observe when $(k, k^*) \neq (k', k^{*\prime})$ holds,

$$\lambda f(k) + (1-\lambda) f(k') + \lambda f(k^*) + (1-\lambda) f(k^{*\prime}) < f(k_\lambda) + f(k_\lambda^*)$$

and hence

$$c_\lambda + c_\lambda^* + i_\lambda + i_\lambda^* + g + g^* < f(k_\lambda) + f(k_\lambda^*) . \qquad \text{(A-6)}$$

(The subscript λ denotes the linear combination of the variables x and x' with

---

8 We follow Brock and Scheinkman (1977) for the following arguments.

$x_\lambda = \lambda x + (1-\lambda)x'$. Furthermore from (1) and (1)*, we obtain

$$\dot{k}_\lambda = \lambda\phi(z)k + (1-\lambda)\phi(z')k' - (n+\delta)k_\lambda \tag{A-7}$$

and

$$\dot{k}_\lambda * = \lambda\phi(z*)k * + (1-\lambda)\phi(z*')k*' - (n+\delta)k_\lambda * \tag{A-7)*}$$

Let $\mu = \lambda k / k_\lambda$ and $1-\mu = (1-\lambda)k'/k_\lambda$, and $\mu* = \lambda k*/k_\lambda *$ and $1-\mu* = (1-\lambda)k*'/k_\lambda *$. Then from (A-7) and (A-7)* we obtain

$$\dot{k}_\lambda = k_\lambda(\mu\phi(z) + (1-\mu)\phi(z')) - (n+\delta)k_\lambda \le k_\lambda\phi(z_\mu) - (n+\delta)k_\lambda$$

and

$$\dot{k}_\lambda * \le k_\lambda * \phi(z_\mu *) - (n+\delta)k_\lambda *$$

from the concavity of ϕ, where $z_\mu = \mu z + (1-\mu)z'$ and $z_\mu * = \mu * z * + (1-\mu*)z*'$. Let z^λ and $z*^\lambda$ be the values of z and z* respectively such that

$$\dot{k}_\lambda = \phi(z^\lambda)k_\lambda - (n+\delta)k_\lambda \tag{A-8}$$

and

$$\dot{k}_\lambda * = \phi(z*^\lambda)k_\lambda * - (n+\delta)k_\lambda * \tag{A-8)*}$$

hold.

Here we note by construction

$$z^\lambda \le z_\mu \text{ and } z*^\lambda \le z_\mu * \tag{A-9}$$

and by definition $z^\lambda = z(k_\lambda, k_\lambda *)$ and $z^\lambda * = z*(k_\lambda, k_\lambda *)$ holds from (A-8) and (A-8)*. (In general *z* and *z** are seen to depend on *k* and *k**. Furthermore (A-8) and (A-8)* state that z^λ and $z*^\lambda$ can be expressed as functions of k_λ and $k_\lambda *$.) Furthermore, we observe $i^\lambda = k_\lambda \cdot z^\lambda \le k_\lambda \cdot z_\mu = k_\lambda(\mu z + (1-\mu)z') = \lambda kz + (1-\lambda)k'z' = \lambda i + (1-\lambda)i' = i_\lambda$ and similarly $i*^\lambda \le i_\lambda *$. Then we can take c_w^λ with $c_w^\lambda \ge c_\lambda + c_\lambda *$ such that

$$c_w^\lambda + i^\lambda + i*^\lambda + g + g* = f(k_\lambda) + f(k_\lambda *). \tag{A-10}$$

Here we note $c_w^\lambda = c^\lambda + c*^\lambda > c_\lambda + c_\lambda *$ if $(k, k*) \ne (k', k*')$ (which is obtained from $c^\lambda * / c^\lambda = c_\lambda * / c_\lambda$ in view of (4) and (4)*[9]) holds.

---

[9] (4) and (4)* show that given g and g*, c and c* are proportionate independent of k. Hence $c* = \gamma c$,

Hence we note

$\lambda(u(c,g)+\beta u(c^*,g^*))+(1-\lambda)(u(c',g)+\beta u(c^{*\prime},g^*))\leq u(c_\lambda,g)+\beta u(c_\lambda^*,g^*)$

$< u(c^\lambda,g)+\beta u(c^{*\lambda},g^*)\leq J(k_\lambda,k_\lambda^*,\dot{k}_\lambda,\dot{k}_\lambda^*)$, and hence (A-5) holds. Since $\dot{k}_\lambda$ and $\dot{k}_\lambda^*$ depend on k_λ and k_λ^* from (A-8), (A-8)*, and from $z^\lambda = z(k_\lambda,k_\lambda^*)$ and $z^{*\lambda}=z(k_\lambda,k_\lambda^*)$, $i^\lambda = k_\lambda z^\lambda$ and $i^{*\lambda}=k_\lambda^* z^{*\lambda}$, we observe that from (A-10) c_w^λ, c^λ and $c^{*\lambda}$ depend also on k_λ and k_λ^*, with

$$c_w^\lambda = c_w(k_\lambda,k_\lambda^*) > c_\lambda + c_\lambda^* = \lambda(c+c^*)+(1-\lambda)(c'+c^{*\prime})$$
$$=\lambda c_w(k,k^*)+(1-\lambda)c_w(k',k^{*\prime}),$$
$$c^\lambda = c(k_\lambda,k_\lambda^*) > c_\lambda = \lambda c+(1-\lambda)c' = \lambda c(k,k^*)+(1-\lambda)c(k',k^{*\prime}),$$

and

$$c^{*\lambda} = c^*(k_\lambda,k_\lambda^*) > c_\lambda^* = \lambda c^*+(1-\lambda)c^{*\prime} = \lambda c^*(k,k^*)+(1-\lambda)c^*(k',k^{*\prime})$$

showing the strict concavity of c_w, *c* and *c* in (*k*, *k**). Here we note (A.5) implies

$$\int J(k_\lambda,k_\lambda^*,\dot{k}_\lambda,\dot{k}_\lambda^*)e^{-(\rho-n)t}dt \geq \lambda\int J(k,k^*,\dot{k},\dot{k}^*)e^{-(\rho-n)t}dt$$
$$+(1-\lambda)\int J(k',k^{*\prime},\dot{k}',\dot{k}^*)e^{-(\rho-n)t}dt.$$

If $(k,k^*)\neq(k',k^{*\prime})$ holds for some time with $0<t<t_0$, then the strict inequality holds for the above, showing

$$W(k_{0\lambda},k_{0\lambda}^*)\geq \max_{\dot{k}_\lambda,\dot{k}_\lambda^*,c^\lambda,c^{*\lambda},i^\lambda,i^{*\lambda}}\int J(k,k^*,\dot{k},\dot{k}^*)e^{-(\rho-n)t}dt > \lambda W(k_0,k_0^*)+(1-\lambda)W(k_0',k_0^{*\prime}) \tag{A-11}$$

i.e., the strict concavity of *W* in (*k*, *k**). ■

Appendix II

Solution Path of *k*, *k**, *z* and *z** - local representation

II-(1)

By totally differentiating (5) and (5)* and rearranging, we obtain

$$\dot{z}=(\dot{\lambda}/\lambda-\dot{\mu}/\mu)\phi'/\phi'', \tag{A-12}$$

$c^{*\prime}=\gamma c'$ and $c^\lambda=\gamma c^{*\lambda}$ hold. This implies $c^{\lambda *}/c^\lambda = c_\lambda^*/c_\lambda=\gamma$.

and

$$\dot{z}^* = (\dot{\lambda}/\lambda - \dot{\mu}^*/\mu^*)\phi^{*\prime}/\phi^{*\prime\prime}, \tag{A-12)*}$$

where $\phi^{*\prime} = \phi'(z^*)$ and $\phi^{*\prime\prime} = \phi''(z^*)$.

Next by totally differentiating (4) and (4)*, we observe

$$\dot{\lambda}/\lambda = -\bar{\sigma}\,\dot{c}/c = -\bar{\sigma}\,\dot{c}^*/c^* = -\bar{\sigma}\,\dot{c}_w/c_w \tag{A-13}$$

where $\bar{\sigma} = 1 - \alpha(1-\sigma) > 0$.

Total differentiation of (2) leads to

$$\dot{c}_w = (f' - z)\dot{k} + (f^{*\prime} - z^*)\dot{k}^* - \dot{z}k - \dot{z}^* k^* \tag{A-14}$$

where $f^{*\prime} = f'(k^*)$.

By substituting (A-14) into (A-13) (thereby deleting $\dot{c}_w$) and then substituting (A-13) into (A-12) and (A-12)* (thereby deleting $\dot{\lambda}/\lambda$), and rearranging, we obtain

$$\begin{bmatrix} 1-\bar{\sigma}k\phi'/c_w\phi'' & -\bar{\sigma}k^*\phi'/c_w\phi'' \\ -\bar{\sigma}k\phi^{*\prime}/c_w\phi^{*\prime\prime} & 1-\bar{\sigma}k^*\phi^{*\prime}/c_w\phi^{*\prime\prime} \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{z}^* \end{bmatrix}$$

$$= \begin{bmatrix} \left[-\dfrac{\bar{\sigma}}{c_w}\left\{(f'-z)\dot{k} + (f^{*\prime}-z^*)\dot{k}^*\right\} - \dot{\mu}/\mu \right]\phi'/\phi'' \\ \left[-\dfrac{\bar{\sigma}}{c_w}\left\{(f'-z)\dot{k} + (f^{*\prime}-z^*)\dot{k}^*\right\} - \dot{\mu}^*/\mu^* \right]\phi^{*\prime}/\phi^{*\prime\prime} \end{bmatrix}. \tag{A-15}$$

Here let A, the determinant of coefficient matrix of (A-15) be evaluated at E (the stationary state) .

$$A = 1 - \frac{\bar{\sigma}}{c_w}(k\phi'/\phi'' + k^*\phi^{*\prime}/\phi^{*\prime\prime}) = (1 - 2\bar{\sigma}c_w^{-1}\phi' k\phi''^{-1}) > 1. \tag{A-16}$$

Let $\dot{k}$, $\dot{k}^*$, $\dot{\mu}$ and $\dot{\mu}^*$ be represented near E so that

$$\dot{k} = \bar{\phi}'\bar{k}(z - \bar{z}) \tag{A-17}$$

$$\dot{k}^* = \bar{\phi}^{*\prime}\bar{k}^*(z^* - \bar{z}) \tag{A-17)*}$$

$$\dot{\mu} = \bar{\phi}''(\bar{z} - \bar{f}')\bar{\mu}(z - \bar{z}) - \bar{\phi}'\bar{f}''\bar{\mu}(k - \bar{k}) \tag{A-18}$$

$$\dot{\mu}^* = \bar{\phi}''(\bar{z} - \bar{f}')\bar{\mu}(z^* - \bar{z}) - \bar{\phi}'\bar{f}''\bar{\mu}(k^* - \bar{k}) \tag{A-18)*}$$

from (1), (1)* (6) and (6)*. (Henceforth all coefficients are evaluated at E, so $\phi' = \phi^{*\prime}$, $\phi'' = \phi^{*\prime\prime}$, $f' = f^{*\prime}$, $z = z^*$, and $k = k^*$ etc. hold.) From (A-15), we obtain

$$A\dot{z} = \left[-\frac{\bar{\sigma}}{c_w}\left\{(f'-z)\dot{k} + (f^{*\prime}-z^*)\dot{k}^*\right\} - \dot{\mu}/\mu \right]\frac{\phi'}{\phi''}\left(1 - \frac{\bar{\sigma}k^*\phi^{*\prime}}{c_w\phi^{*\prime\prime}}\right)$$

$$+\left[-\frac{\bar{\sigma}}{c_w}\left\{(f'-z)\dot{k} + (f^{*\prime}-z^*)\dot{k}^*\right\} - \dot{\mu}^*/\mu^* \right]\frac{\phi^{*\prime}}{\phi^{*\prime\prime}}\frac{\bar{\sigma}k^*\phi'}{c_w\phi''}$$

$$= -\frac{\bar{\sigma}}{c_w}\{(f'-z)\dot{k}+(f^{*\prime}-z^*)\dot{k}^*\}\frac{\phi'}{\phi''}-\frac{\dot{\mu}}{\mu}\frac{\phi'}{\phi''}\left(1-\frac{\bar{\sigma}k^*\phi^{*\prime}}{c_w\phi^{*\prime\prime}}\right)-\frac{\dot{\mu}^*}{\mu^*}\frac{\phi^{*\prime}}{\phi^{*\prime\prime}}\bar{\sigma}\frac{k^*\phi'}{c_w\phi''} \quad \text{(A-19)}$$

and

$$A\dot{z}^* = \left[-\frac{\bar{\sigma}}{c_w}\{(f'-z)\dot{k}+(f^{*\prime}-z^*)\dot{k}^*\}-\frac{\dot{\mu}^*}{\mu^*}\right]\frac{\phi^{*\prime}}{\phi^{*\prime\prime}}\left(1-\frac{\bar{\sigma}k\phi'}{c_w\phi''}\right)$$

$$+\left[-\frac{\bar{\sigma}}{c_w}\{(f'-z)\dot{k}+(f^{*\prime}-z^*)\dot{k}^*\}-\frac{\dot{\mu}}{\mu}\right]\frac{\phi'}{\phi''}\bar{\sigma}\frac{k\phi^{*\prime}}{c_w\phi^{*\prime\prime}}$$

$$= -\frac{\bar{\sigma}}{c_w}\{(f'-z)\dot{k}+(f^{*\prime}-z^*)\dot{k}^*\}\frac{\phi^{*\prime}}{\phi^{*\prime\prime}}-\frac{\dot{\mu}^*}{\mu^*}\frac{\phi^{*\prime}}{\phi^{*\prime\prime}}\left(1-\bar{\sigma}\frac{k\phi'}{c_w\phi''}\right)-\frac{\dot{\mu}}{\mu}\frac{\phi'}{\phi''}\bar{\sigma}\frac{k\phi^{*\prime}}{c_w\phi^{*\prime\prime}}. \quad \text{(A-19)*}$$

Let

$$A\dot{z} = a_{11}(z-\bar{z})+a_{12}(z^*-\bar{z})+a_{13}(k-\bar{k})+a_{14}(k^*-\bar{k}^*),$$

and

$$A\dot{z}^* = a_{21}(z-\bar{z})+a_{22}(z^*-\bar{z}^*)+a_{23}(k-\bar{k})+a_{24}(k^*-\bar{k}^*).$$

Then by substituting (A-17) through (A-18)* in to (A-19) and (A-19)*, we observe

$$a_{11} = -\bar{\sigma}c_w^{-1}(f'-z)\phi'\phi''^{-1}\phi'k-\phi'\phi''^{-1}(1-\bar{\sigma}k^*\phi^{*\prime}c_w^{-1}\phi^{*\prime\prime-1})\phi''(z-f')$$

$$= -\phi'\phi''^{-1}(f'-z)(2\bar{\sigma}c_w^{-1}\phi'k-\phi'') = \phi'\phi''^{-1}(f'-z)\phi''A = \phi'(f'-z)A,$$

$$a_{12} = -\bar{\sigma}c_w^{-1}(f^{*\prime}-z^*)\phi'\phi''^{-1}\phi^{*\prime}k^*-\phi^{*\prime}\phi^{*\prime\prime-1}\bar{\sigma}k^*\phi'c_w^{-1}\phi''^{-1}\phi^{*\prime\prime}(z^*-f^*)=0,$$

$$a_{13} = \phi'\phi''^{-1}(1-\bar{\sigma}k^*\phi^{*\prime}c_w^{-1}\phi^{*\prime\prime-1})\phi'f'' \quad ,$$

$$a_{14} = \phi^{*\prime}\phi^{*\prime\prime-1}\bar{\sigma}k^*\phi'c_w^{-1}\phi''^{-1}\phi'f'' \quad ,$$

$$a_{21} = -\bar{\sigma}c_w^{-1}(f'-z)\phi^{*\prime}\phi^{*\prime\prime-1}\phi'k-\phi'\phi''^{-1}\bar{\sigma}k\phi^{*\prime}c_w^{-1}\phi^{*\prime\prime-1}\phi''(z-f')=0 \quad ,$$

$$a_{22} = -\bar{\sigma}c_w^{-1}(f^{*\prime}-z^*)\phi^{*\prime}\phi^{*\prime\prime-1}\phi^{*\prime}k^*-\phi^{*\prime}\phi^{*\prime\prime-1}(1-\bar{\sigma}k\phi'c_w^{-1}\phi''^{-1})\phi''(z^*-f^*)$$

$$= -\phi^{*\prime}\phi^{*\prime\prime-1}(f'-z)(2\bar{\sigma}c_w^{-1}\phi'k-\phi'') = \phi^{*\prime}(f'-z)A \quad ,$$

$$a_{23} = \phi'\phi''^{-1}\bar{\sigma}k\phi^{*\prime}c_w^{-1}\phi^{*\prime\prime-1}\phi'f'',$$

$$a_{24} = \phi^{*\prime}\phi^{*\prime\prime-1}(1-\bar{\sigma}k\phi'c_w^{-1}\phi''^{-1})\phi^{*\prime}f^{*\prime\prime}.$$

Recalling

$$\rho-n = \phi'(f'-z) \quad (9)$$

we observe that $f'-z>0$ at E holds. Now let

$$a_{11}A^{-1} = a_{22}A^{-1} = \phi'(f'-z) = \rho-n = a_1 > 0$$

$$a_{13}A^{-1} = a_{24}A^{-1} = a_3 > 0$$

$$a_{14}A^{-1} = a_{23}A^{-1} = a_4 < 0$$

$$\phi' k = \phi^{*'} k^* = a_5 > 0 .$$

This shows (24).

Appendix III

Let $(\nu_{i1}, \nu_{i2}, 1, \nu_{i3})'$, $i = 1,2$, satisfy

$$\begin{pmatrix} a_1 - \omega_i & 0 & a_3 & a_4 \\ 0 & a_1 - \omega_i & a_4 & a_3 \\ a_5 & 0 & -\omega_i & 0 \\ 0 & a_5 & 0 & -\omega_i \end{pmatrix} \begin{pmatrix} \nu_{i1} \\ \nu_{i2} \\ 1 \\ \nu_{i3} \end{pmatrix} = 0. \qquad \text{(A-20)}$$

Then we obtain

$$\left.\begin{array}{l} z - \bar{z} = A\nu_{11} e^{\omega_1 t} + B\nu_{21} e^{\omega_{21} t}, \\ z^* - \bar{z} = A\nu_{12} e^{\omega_1 t} + B\nu_{22} e^{\omega_{21} t}, \\ k - \bar{k} = Ae^{\omega_1 t} + Be^{\omega_2 t}, \\ \text{and} \\ k^* - \bar{k} = A\nu_{13} e^{\omega_1 t} + B\nu_{23} e^{\omega_2 t}, \end{array}\right\} \qquad \text{(A-21)}$$

where A and B are determined by the initial values and stationary values of k and k*. From (A-20), we obtain

$$(a_1 - \omega_i)\nu_{i1} + a_3 + a_4 \nu_{i3} = 0 \qquad \text{(i)}$$

$$(a_1 - \omega_i)\nu_{i2} + a_4 + a_3 \nu_{i3} = 0 \qquad \text{(ii)}$$

$$a_5 \nu_{i1} - \omega_i = 0, \qquad \text{(iii)}$$

and

$$a_5 \nu_{i2} - \omega_i \nu_{i3} = 0 . \qquad \text{(iv)}$$

From (iii) and (iv),

$$\nu_{i1} = \omega_i / a_5 = \nu_{i2} / \nu_{i3},$$

and hence $\nu_{i2} = \nu_{i3}\nu_{i1}$ are obtained. Substituting this into (ii), we obtain

$$(a_1 - \omega_i)\nu_{i3}\nu_{i1} + a_4 + a_3 \nu_{i3} = 0,$$

or

$$(a_1 - \omega_i)\nu_{i1} + a_4 / \nu_{i3} + a_3 = 0 . \qquad \text{(v)}$$

From (i) and (v), $\nu_{i3}^2 = 1$ is derived. We recall ω_1 and ω_2 (which are expressed as (25) and (26).) satisfy respectively $\omega_1^2 - a_1\omega_1 - a_5(a_3 + a_4) = 0$ and $\omega_2^2 - a_1\omega_2 - a_5(a_3 - a_4) = 0$.

(1) $\nu_{i3} = 1$. Then $\nu_{i1} = \nu_{i2}$, and from (i) and $\nu_{i1} = \omega_i / a_5$, $(a_1 - \omega_i)\omega_i / a_5 + a_3 + a_4 = 0$ is derived. This is satisfied if $i = 1$.

Similarly

(2) $\nu_{i3} = -1$. Then $\nu_{i1} = -\nu_{i2}$, and from (i) and $\nu_{i1} = \omega_i / a_5$ $(a_1 - \omega_i)\omega_i / a_5 + a_3 - a_4 = 0$. This is satisfied if $i = 2$.

In short we obtain $\nu_{11} = \nu_{12} = \omega_1 / a_5$, $\nu_{13} = 1$, $\nu_{21} = \omega_2 / a_5$, $\nu_{22} = -\nu_{21} = -\omega_2 / a_5$, $\nu_{23} = -1$. Then (A-21) is expressed as

$$z - \bar{z} = (\omega_1 / a_5) A e^{\omega_1 t} + (\omega_2 / a_5) B e^{\omega_2 t},$$

$$z^* - \bar{z} = (\omega_1 / a_5) A e^{\omega_1 t} - (\omega_2 / a_5) B e^{\omega_2 t}$$

$$k - \bar{k} = A e^{\omega_1 t} + B e^{\omega_2 t}$$

and

$$k^* - \bar{k} = A e^{\omega_1 t} - B e^{\omega_2 t}$$

where A and B must satisfy

$$k_0 - \bar{k} = A + B \text{, and } k_0{}^* - \bar{k} = A - B,$$

or

$$A = (k_0 + k_0{}^* - \bar{k}_w)/2 \quad \text{and} \quad B = (k_0 - k_0{}^*)/2.$$

Substituting *A* and *B* into the above, we obtain (27) through (31)*.

Appendix IV

Derivation of (33)

From (11) and (11)*, we obtain

$$-\dot{\lambda}/\lambda = -(u_{cc}/u_c)\dot{c} = \bar{\sigma}\,\dot{c}/c = \bar{\sigma}\,\dot{c}_W / c_W \qquad \text{(A-22)}$$

From (32), we observe

$$\dot{c}_W = \omega_1 (c_W - \bar{c}_W). \qquad \text{(A-23)}$$

Hence from (12), (A-22) and (A-23)

$$r = \rho - \dot{\lambda}/\lambda = \rho + \bar{\sigma}\omega_1 (c_W - \bar{c}_W)/c_W = \rho + \bar{\sigma}\omega_1 (c_W - \bar{c}_W)/\bar{c}_W. \qquad \text{(A-24)}$$

(Here we note $d(c_W - \bar{c}_W) c_W{}^{-1} / dc_W = \bar{c}_W / c_W{}^2$, and hence linearlization around the stationary state implies $(c_W - \bar{c}_W)/c_W = (c_W - \bar{c}_W)/\bar{c}_W$).

Next (10) is linearlized around the stationary state;

$$
\begin{aligned}
\dot{a} &= (r-n)a + f(k) - c - zk - g \\
&= \bar{a}(r-\rho) + (\rho-n)(a-\bar{a}) + (\bar{f}' - \bar{z})(k-\bar{k}) - \bar{k}(z-\bar{z}) - (1+\gamma)^{-1}(c_W - \bar{c}_W) \\
&= \bar{a}\bar{\sigma}\omega_1(c_W - \bar{c}_W)/\bar{c}_W + (\rho-n)(a-\bar{a}) + (\bar{f}' - \bar{z})(k-\bar{k}) \\
&\quad - \bar{k}(z-\bar{z}) - (1+\gamma)^{-1}(c_W - \bar{c}_W)
\end{aligned}
$$

from (A-24).

Employing (27),(28),(32), and $\bar{f}' - \bar{z} - (\omega_i/a_5)\bar{k} = (\rho - n - \omega_i)\overline{\phi'}^{-1}$ (obtained from (9) and $a_5 = \overline{\phi'} \cdot \bar{k}$), the above is rewritten as

$$\dot{a} = Ae^{\omega t} + Be^{\omega_2 t} + (\rho-n)(a-\bar{a}) \tag{A-25}$$

where

$$A = (\overline{\sigma a}\,\omega_1/\bar{c}_W - (1+\gamma)^{-1} + 2^{-1})(\rho - n - \omega_1)\overline{\phi'}^{-1}(k_{0W} - \bar{k}_W),$$

and

$$B = (\rho - n - \omega_2)(k_0 - k_0{*})\overline{\phi'}^{-1}2^{-1}.$$

Now let

$$a = \alpha_1 e^{\omega t} + \alpha_2 e^{\omega_2 t} + \xi e^{(\rho-n)t} + \bar{a} \tag{A-26}$$

where α_1, α_2 and ξ depend on initial values and stationary values of k and k*, $\bar{c}_W$, $\omega_i, i = 1,2$, and g and g*. Then by totally differentiating (A-26), we obtain

$$
\begin{aligned}
\dot{a} &= \alpha_1\omega_1 e^{\omega t} + \alpha_2\omega_2 e^{\omega_2 t} + \xi(\rho-n)e^{(\rho-n)t} &\text{(A-27)} \\
&= \alpha_1(\omega_1 - \rho + n)e^{\omega t} + \alpha_2(\omega_2 - \rho + n)e^{\omega_2 t} + (\rho-n)(a-\bar{a}).
\end{aligned}
$$

Then by comparing (A-25) and (A-27), we obtain

$$\alpha_1 = A/(\omega_1 - \rho + n) = -(\overline{\sigma a}\,\omega_1/\bar{c}_W - (1+\gamma)^{-1} + 2^{-1})\overline{\phi'}^{-1}(k_{0W} - \bar{k}_W) \tag{A-28}$$

and

$$\alpha_2 = B/(\omega_2 - \rho + n) = -(k_0 - k_0{*})\overline{\phi'}^{-1}2^{-1}. \tag{A-29}$$

In (A-26), by letting t = 0, we obtain

$$\xi = a_0 - \alpha_1 - \alpha_2 - \bar{a}\,. \tag{A-30}$$

By multiplying $e^{-(\rho-n)t}$ on both sides of (A-26), we observe

$$ae^{-(\rho-n)t} = \alpha_1 e^{(\omega_1 - \rho + n)t} + \alpha_2 e^{(\omega_2 - \rho + n)t} + \xi + \bar{a}e^{-(\rho-n)t}\,.$$

From NPG (17) and, $a \to \bar{a}$ and $r \to \rho$ as $t \to \infty$, we can conclude $\xi = 0$, or

$$\bar{a} = a_0 - \alpha_1 - \alpha_2\,. \tag{A-31}$$

Next by substituting (A-28) and (A-29) into the above and by rearranging we obtain

$$
\begin{aligned}
\bar{a} = \{&a_0\overline{\phi'} - (g{*} - g)2^{-1}(2\bar{f} - 2\bar{z}\bar{k} - g_W)^{-1}(k_{0W} - \bar{k}_W) + (k_0 - k_0{*})2^{-1}\}\Big/\Big[(k_{0W} - \bar{k}_W) \\
&\left\{-\bar{\sigma}\,\omega_1/\bar{c}_W + (\rho-n)(2\bar{f} - 2\bar{z}\bar{k} - g_W)^{-1}\right\} + \overline{\phi'}\Big]
\end{aligned}
\tag{33}
$$

noting

$$(1+\gamma)^{-1} = \left\{\bar{a} + (\bar{f} - \bar{k}\bar{z} - g)(\rho-n)^{-1}\right\}(2\bar{f} - 2\bar{z}\bar{k} - g_W)^{-1}(\rho-n)$$

and

$$(\bar{f} - \bar{k}\bar{z} - g)/(2\bar{f} - 2\bar{k}\bar{z} - g_W) - 1/2 = (g{*} - g)2^{-1}(2\bar{f} - 2\bar{k}\bar{z} - g_W)^{-1}.$$

Figures

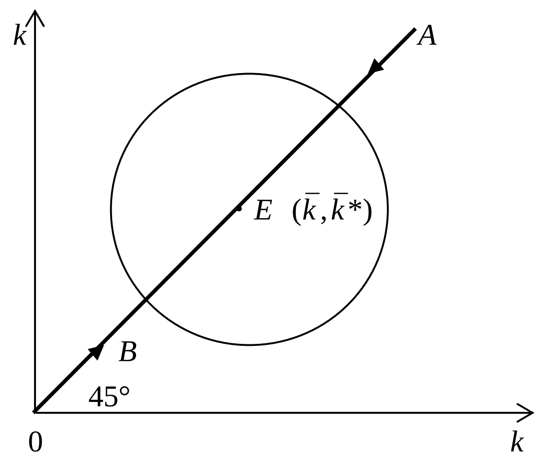

Fig. 1 Monotone convergence of the path (k,k*)

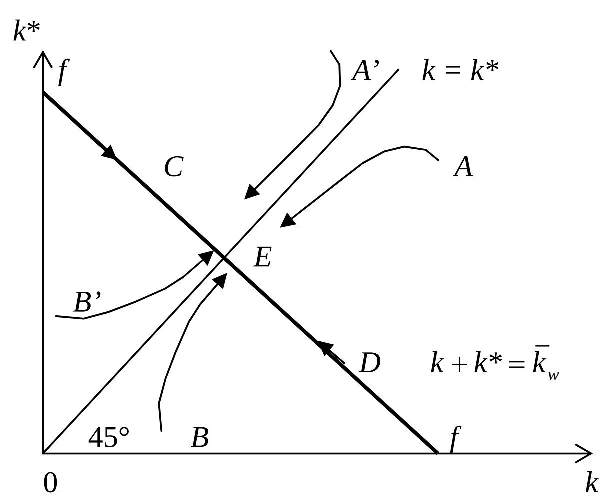

Fig. 2 Properties of the Stationary State $(\bar{k}, \bar{k}^*)$

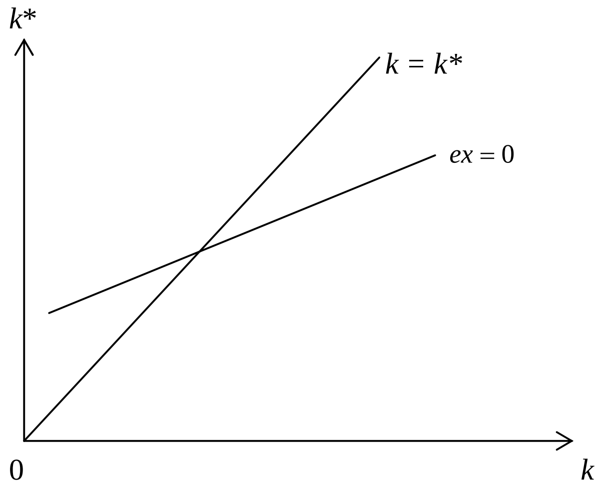

Fig. 3.I

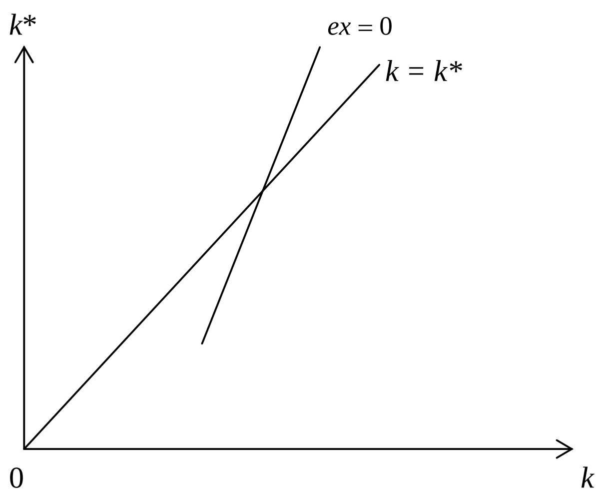

Fig. 3.II

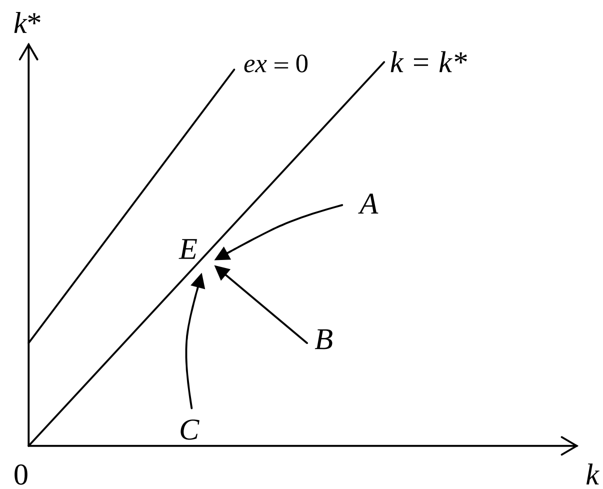

Fig. 3.III $\bar{a} < 0 (\overline{ex} > 0)$

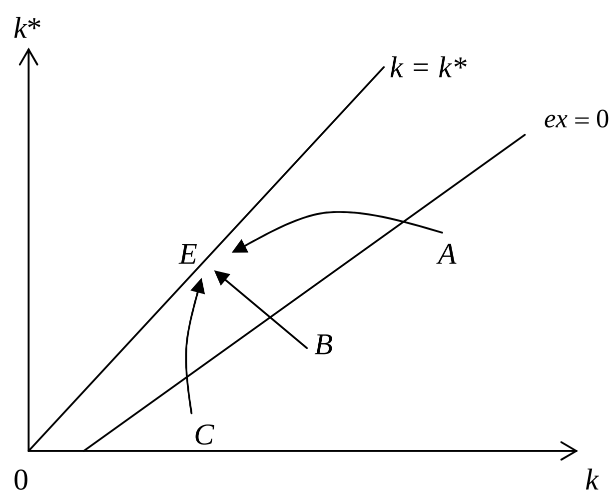

Fig. 3.IV $\bar{a} > 0 (\overline{ex} < 0)$

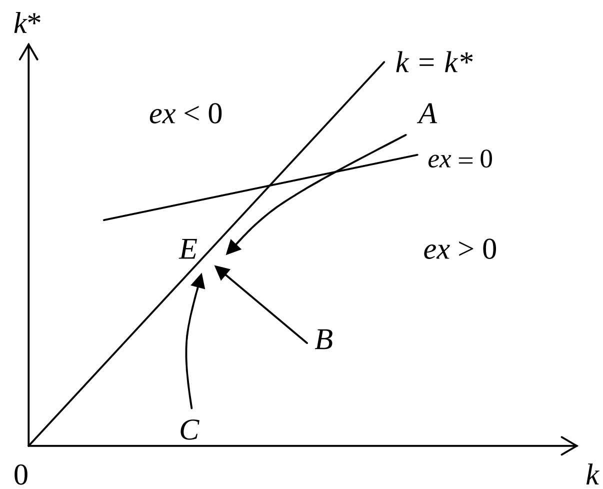

Fig. 3.I.a $(\overline{a} < 0, i.e., \overline{ex} > 0)$

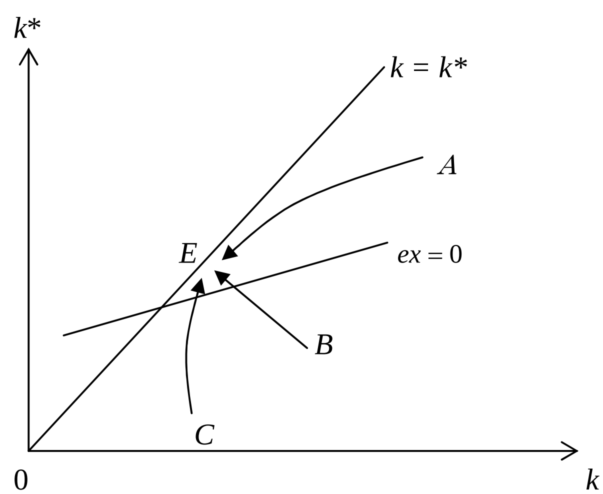

Fig. 3.I.b $(\overline{a} > 0, i.e., \overline{ex} < 0)$

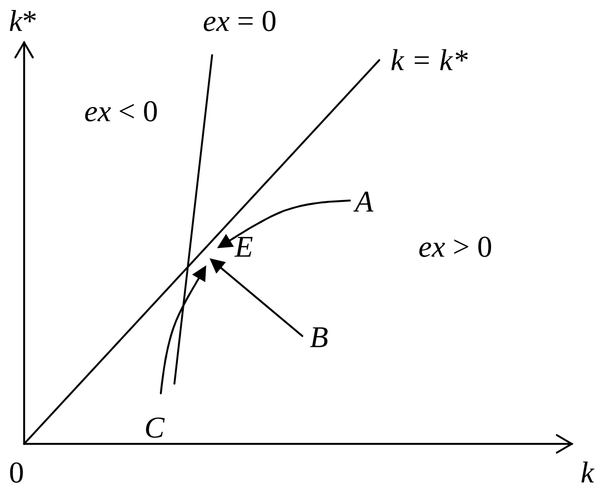

Fig. 3.II.a $(\bar{a} < 0, i.e., \overline{ex} > 0)$

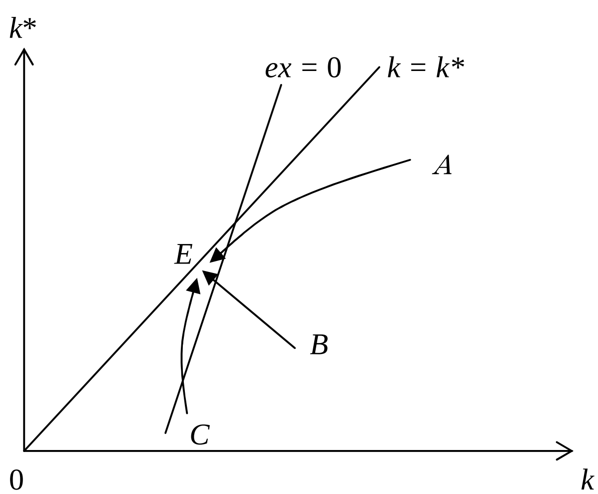

Fig. 3.II.b $(\bar{a} > 0, i.e., \overline{ex} < 0)$

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