Instrumental variables estimation of a flexible nonlinear model

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Abstract

The applicability of Hamilton's (2001) flexible nonlinear model for estimating simultaneous equations model or errors in variables model is placed under restraint due to the existence of endogenous explanatory variables. This paper proposes IVFM (instrumental variables estimation of a flexible nonlinear model) for solving the case of endogenous explanatory variables using a standard estimation method. The findings of this paper are as follows: this paper theoretically solves a flexible nonlinear model with the endogenous explanatory variables by using instrumental variables; and also empirically proves the applicability of IVFM for simultaneous equations model or error in variables model. As we applied the proposed model to Campbell and Mankiw's (1989) consumption function, we found that the relationship is linear between the log difference of per-capita disposable income and the log difference of per-capita consumption on non-durable goods and services.

Keywords: a flexible nonlinear model, IVFM (instrumental variables estimation of a flexible nonlinear model), income, consumption

JEL classification: C13; C32

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1. Introduction

One set of studies on parametric nonlinear models include (i) time-varying parameters (Sims, 1993); (ii) threshold autoregressions (Tong, 1983, Tsay, 1989); (iii) regime-switching (Hamilton, 1994, chapter 22); and (iv) smooth transition autoregressions (Granger, Terasvirta and Anderson, 1993). These parametric nonlinear models have demerits while focusing on which parametric model to use and deciding in what way the data might be nonlinear. Most studies on nonparametric models are classified into (i) kernels (Härdle, 1990), (ii) series expansions (Gallant and Nychka, 1987), (iii) wavelets (Donoho, et al., 1995), (iv) nearest neighbor (Yakowitz, 1987), and (v) smoothing splines (Eubank, 1988). The studies of nonparametric models also sacrifice many of benefits of parametric models. Thus Hamilton (2001) proposed the Random Field regression approach that belongs to the class of flexible nonlinear models.

Most papers analyzing Hamilton's flexible nonlinear model cannot estimate the case that regressors are correlated with disturbances. In other words, areas which have not been analyzed in the studies on flexible nonlinear models are not only simultaneous equations model but also errors in variables model. In this respect, this paper introduces a new model called the instrumental variables estimation of a flexible nonlinear model (hereafter IVFM) that solves the problems resulting from the presence of endogenous explanatory variables.

In order to get a consistent estimation of the parameters in two-steps, the IVFM can utilize the two methods. One is a "standard" estimation method and the other is an "alternative estimation method proposed by Kim (2004a, 2004b) and Radchenko and Tsurumi (2004). Here we adopt the "standard" estimation method in which instrument variables are estimated first by a least squares regression or maximum likelihood estimation, and then the nonlinear regression is estimated by MLE after instrument variables are inserted directly instead of endogenous explanatory variables. It has been known that "standard" and "alternative" estimation methods are mathematically identical in the case of parametric nonlinear regression models. However, two estimation methods are not mathematically identical in the special case of Hamilton's flexible nonlinear model (Yoon and Goo, 2006) because the random field part is correlated with the disturbance term.

The findings of this paper are as follows. This paper theoretically solves a flexible nonlinear model with the endogenous explanatory variables by using a "standard" estimation method. The benefit of the "standard" estimation method is that the estimation procedure is very simple and standard estimation converges very well because the "standard" estimation method that doesn't need an appropriate transformation of the error terms. In addition, this paper empirically proves the applicability of IVFM for simultaneous equation or error in variables model. As we applied the proposed IVFM to Campbell and Mankiw's (1989) consumption function, the relationship is found to be linear between the log difference of per-capita disposable income and the log difference of per-capita consumption on non-durable goods and services.

This paper is divided into 5 sections. Section 2 reviews the Hamilton's flexible nonlinear model with endogenous explanatory variables, and then introduces IVFM by using the "standard" estimation method. Section 3 tests the nonlinearly of Campbell and Mankiw's consumption model by using IVFM and then summarizes the empirical results. Section 5 concludes this paper.

2. Inference based on a flexible nonlinear model with instrumental variables

The following nonlinear model is considered to deal with the flexible nonlinear inference in the presence of endogenous explanatory variables.

$$y_1 = \mu(Y_1, X_1) + \varepsilon_1 \tag{1}$$

$$= Y_1\beta_1 + X_1\gamma_1 + \lambda m\ (g \odot (Y_1, X_1)) + \varepsilon_1 \tag{2}$$

$$Y_1 = X_1\Pi_1 + X_2\Pi_2 + v_2 \tag{3}$$

In equation (1), y_1 is (T x 1), the functional form μ is unknown and considered as the outcome of a random process, the endogenous explanatory variables $Y_1 = [y_2, \cdots, y_m]$ is [T x (m-1)] thus Y_1 is correlated with ε_1, exogenous explanatory variables $X_1 = [x_1, \cdots, x_{k1}]$ is (T x k1), and error terms ε_1 is (T x 1). In equation (2), coefficient β_1 and γ_1 are [(m-1) x 1]and (k1 x 1) respectively, λ is scalar and reflects the units of the dependent variables. $m\ (g \odot (Y_1, X_1))$ denotes the realization of Gaussian random field with a mean of zero by Hamilton (2001)[1]. Population parameter g is (k x 1) vector and a zero value for the *i*th element of g implies that the conditional expectation is linear in (Y_1, X_1). $\odot$ indicates element-by-element multiplication. In equation (3), Π_1 is [k1 x (m-1)], another exogenous explanatory variables $X_2 = [x_{k1+1}, \cdots, x_k]$ is [T x (k-k1)], Π_2 is [(k-k1) x (m-1)], and v_2 is [T x (m-1)].

[1] Please refer to P540~P541 in the paper of Hamilton(2001)

The covariance matrix of $[\varepsilon_1, \nu_2] \sim N(0, \Omega)$ is given by (4).

$$\Omega = \begin{bmatrix} \sigma_{1,1} & \delta' \\ \delta & \Omega_2 \end{bmatrix} \tag{4}$$

where Ω is (m x m) matrix and a positive definite, $\sigma_{1,1} = (C_0 + \sigma_{\varepsilon 1}^2 I_T)$,

$C_0 = \lambda^2 H(g) = [\lambda^2 H_k(h_{ij})]_{i,j=1,\dots,T}$ where the function $H_k(\cdot)$ is described by Hamilton[2],

$\delta = Cov(\varepsilon_1, Y_1)$, $\Omega_2 = Var(\nu_2)$.

In order to provide the instrumental variables estimation within the framework of Hamilton's (2001) maximum likelihood estimation, we derived the following two-step procedures.

Step 1 : Estimate equation (3) using an OLS or MLE and get $\hat{Y}_1$ as $\hat{Y}_1 = X_1 \hat{\Pi}_1 - X_2 \hat{\Pi}_2$.

Step 2 : By employing Hamilton's maximum log likelihood function, estimate equation (2) based on $\hat{Y}_1$ obtained from the first step.

$$\begin{aligned} L &= (2\pi)^{-T/2} (\sigma_1^2)^{-T/2} \, | W_1 |^{-1/2} \\ &\quad \cdot \exp[-\frac{1}{2\sigma_1^2} (y_1 - \hat{Y}_1 \beta_1 - X_1 \gamma_1)' W_1^{-1} (y_1 - \hat{Y}_1 \beta_1 - X_1 \gamma_1)] \end{aligned} \tag{5}$$

where $W_1 \equiv \varsigma_1^2 H(g) + I_T$, $\varsigma_1^2 = \lambda^2 / \sigma_{\varepsilon 1}^2$, $\varsigma_1^2 H(g) = \varsigma_1^2 [H(h_{kl})]_{k,l=1,\dots,T}$,

$h_{kl} = (1/2) \{ [\ g \odot ((\hat{Y}_1, X_1)_k - (\hat{Y}_1, X_1)_l) \ ]' [\ g \odot ((\hat{Y}_1, X_1)_k - (\hat{Y}_1, X_1)_l) \ ] \}^{1/2}$,

$\sigma_1^2 = \sigma_{1,1} - \delta' \Omega_2^{-1} \delta$ and $\hat{Y}_1 = X_1 \hat{\Pi}_1 - X_2 \hat{\Pi}_2$.

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[2] Please refer to Theorem 2.2 in P541 or Table1 in P542 in the paper of Hamilton (2001)

Taking logs and rearranging equation (5), we derive the following log likelihood estimation.

$$-\frac{T}{2}\ln(2\pi)-\frac{T}{2}\ln\sigma_1^2-\frac{1}{2}\ln|W_1|-\frac{1}{2\sigma_1^2}(y_1-\hat{Y}_1\beta_1-X_1\gamma_1)'W_1^{-1}(y_1-\hat{Y}_1\beta_1-X_1\gamma_1) \qquad (6)$$

Given (g,ς_1) in equation (6), the values of $(\beta_1,\gamma_1),\sigma_1^2$ that maximizes equation (6) can be calculated analytically as

$$\begin{bmatrix}\beta_1\\ \gamma_1\end{bmatrix}=(X'W_1^{-1}X)^{-1}X'W_1^{-1}y_1, \qquad (7)$$

$$\hat{\sigma}_1^2=(y_1-\hat{Y}_1\beta_1-X_1\gamma_1)'W_1^{-1}(y_1-\hat{Y}_1\beta_1-X_1\gamma_1)/T \qquad (8)$$

Equation (8) allows us to concentrate the log likelihood in (6) as

$$L=-\frac{T}{2}\ln(2\pi)-\frac{T}{2}\ln\sigma_1^2-\frac{1}{2}\ln|W_1|-\frac{T}{2} \qquad (9)$$

Maximizing (9) gives the MLE $(\hat{g},\hat{\varsigma}_1)$ and thus from (7) and (8) $(\hat{\beta}_1,\hat{\gamma}_1),\hat{\sigma}_1^2$ are derived.

3. Empirical results

Let's consider Campbell and Mankiw's (1989) consumption model such as equation (10)

$$\Delta C_t = \alpha + \beta \Delta Y_t + \varepsilon_t, \quad (10)$$

where Y_t is the log of per-capita disposable income, C_t is the log of per-capita consumption on non-durable goods and services, and ΔY_t and ε_t are correlated. Δ represent the first difference of a variable, for example, ΔY_t is the log difference of per-capita disposable income.

Following Campbell and Mankiw (1989), the vector of instrumental variables employed is given by $[\Delta Y_{t-2}, \Delta Y_{t-3}, \Delta Y_{t-4}; \Delta C_{t-2}, \Delta C_{t-3}, \Delta C_{t-4}; \Delta i_{t-2}, \Delta i_{t-3}, \Delta i_{t-4}]$, where Δi_t is the first difference of the three-month T-bill rate.

There is a disagreement about ΔY_t and ΔC_t is linear or not. Campbell and Mankiw (1989) assume that the relationship is linear and constant. While Kim (2004b) adopted the LIML Markov Switching estimation procedure with the same variables as in Campbell and Mankiw (1989) for quarterly data covering 1953:QII ~ 2003:QII, it was found that in the 1970's and 1980's, the measure of sensitivity was the highest and statistically significant while it was not statistically significant in the rest of sample and, Kim (2004b) didn't make a hypothesis test. Taking these different views into consideration, we try to find out whether the relationship is really linear or not using a more general model called a parametric approach to a flexible nonlinear model.

To find out the nonlinearity of the relationship, we extend the equations (10) in the following way which adopts two-step IVFM.

$$\Delta C_t = \alpha + \beta \Delta Y_t + \lambda m(g \odot \Delta Y_t) + e_t \quad (11)$$

Step 1 : Estimate equation (3) using an OLS and get $\Delta \hat{Y}_t$ as $\Delta \hat{Y}_t = Z_t (Z_t' Z_t)^{-1} Z_t' \Delta Y_t$

where $Z_t = [\Delta Y_{t-2}, \Delta Y_{t-3}, \Delta Y_{t-4}; \Delta C_{t-2}, \Delta C_{t-3}, \Delta C_{t-4}; \Delta i_{t-2}, \Delta i_{t-3}, \Delta i_{t-4}]$

Step 2 : By employing Hamilton's maximum likelihood function, estimate equation (11) based on $\Delta \hat{Y}_t$ obtained from the first step.

$$-\frac{T}{2}\ln(2\pi)-\frac{T}{2}\ln\sigma_1^2-\frac{1}{2}\ln|W|-\frac{1}{2\sigma_1^2}(\Delta C_t-\beta\,\Delta\hat{Y}_t-\alpha)^{'}W^{-1}(\Delta C_t-\beta\,\Delta\hat{Y}_t-\alpha) \qquad (12)$$

where, $W \equiv \varsigma^2 H(g)+I_T\,, \varsigma=\lambda/\sigma_e\,,\ C_0=\varsigma^2 H(g)=[\varsigma^2 H(h_{kl})]_{k,l=1,\dots,T}$,

$$\sigma_{1,1}=(C_0+\sigma_e^2 I_T),\ \ \sigma_1^2=\sigma_{1,1}-\delta^{'}\Omega_2^{-1}\delta$$

Equation (13) reports estimation results for the proposed IVFM.

$$\Delta C_t = 0.415+0.250\,\Delta\hat{Y}_t+0.374\quad[1.066m(0.608\,\Delta\hat{Y}_t)+e_t] \qquad (13)$$

(0.265) (0.0845) (0.021) (0.439) (0.135)

where $m(0.608\,\Delta\hat{Y}_t)$ represents the value at $g=0.608$ of an unobserved realization of a random field. The estimate $\varsigma=1.066$ and $\sigma=0.374$ characterize the relation between *m*(.) and the conditional mean function $\mu(x)$. Numbers in parentheses in (13) are the usual MLE asymptotic standard errors based on second derivatives of equation (12).

Although the coefficients $g=0.608$ and $\varsigma=1.066$ seems to be statistically significant, the linearity is shown by LM $\chi^2(1)$ test statistic of 0.0139 which cannot reject the null hypothesis of linearity at the 5% level. [3]

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3 Hamilton (2001) suggest that If v^2 > 3.84, reject the null hypothesis of linearity at the 5% level

Figure 1. The relationship between consumption C_t and Income Y_t

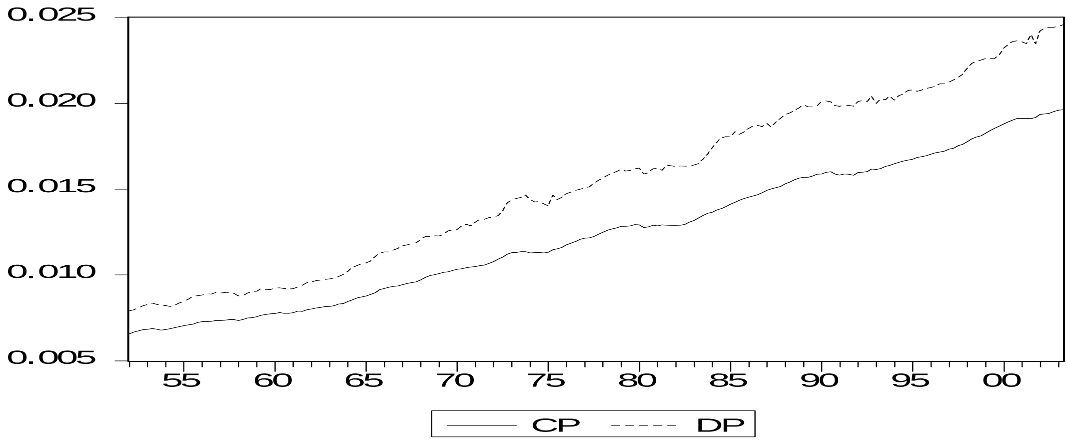
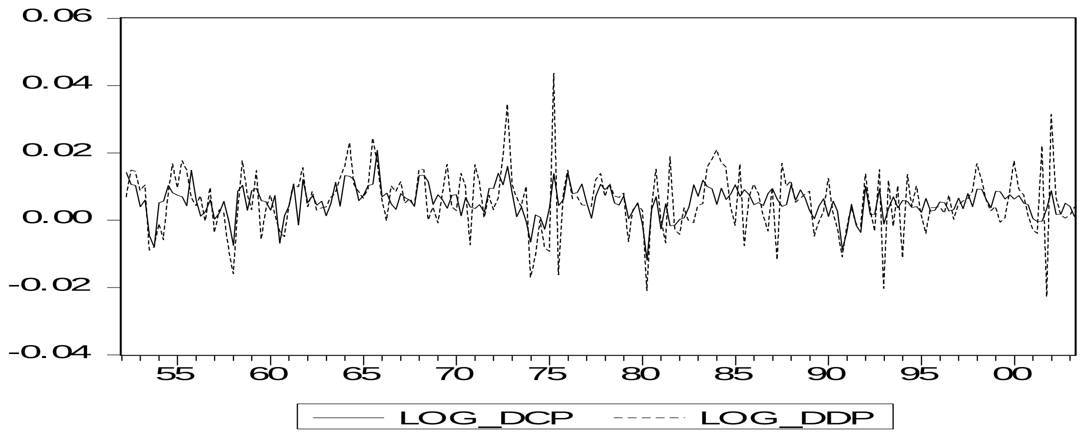


Figure 2. The relationship between consumption ΔC_t and Income ΔY_t



4. Conclusion

The application of Hamilton's flexible nonlinear model is placed under restraint because of the presence of endogenous explanatory variables. Thus this paper proposed a new model called IVFM that was well derived theoretically and tested empirically.

The findings of this paper are as follows. First, this paper theoretically solves the endogenous explanatory variables by IVFM. The benefits claimed for the IVFM are the following: (i) IVFM gets the consistent estimation of the parameters in the two-step and also it converges very well, (ii) its procedure is very simple. IVFM employed the standard estimation method which has two-step procedure by an instrumental variables treatment in the flexible nonlinear model instead of transformation of error terms. Second this paper tests empirically the applicability of IVFM for simultaneous equation or error in variables models. The proposed IVFM is applied to Campbell and Mankiw's (1989) consumption function. The empirical results suggest that the relationship between the log difference of per-capita disposable income and the log difference of per-capita consumption on non-durable goods and services is linear, which is the same result with Campbell and Mankiw (1989).

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